MA231 Vector Analysis

2010, term 1
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Example Sheet 4: Hints and partial solutions

A1 (b) If $f = u(x,y) + iv(x,y)$ then $u =$ so that $v_y = u_x = 0$ and $v_x = -u_y = 0$. Hence v is a constant. (c) $u_{xx} = (v_y)_x = v_{xy}$ and $u_{yy} = (-v_x)_y = -v_{xy}$.

A2 (a) $2i/3$. (b) 1.

A3 (a) (i) $2\pi i\mathrm{e}$. (ii) π. (iii) 0. (b) $\frac{2\pi i\mathrm{e}}{(n-1)!}$.

A4 (i) 0, (ii) π, (iii) $-\pi$, (iv) 0, (v) π.

A5 Integrate around the semicircle $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_1(t) = R\mathrm{e}^{it}$ for $t \in [0,\pi]$ and $\gamma_2(t) = t$ for $t \in [-R,R]$. (a) $|z^2 - 2z + 2| \geq |z^2| - |2z| - 2 = R^2 - 2R - 2$ for $z \in \gamma_1$ by the triangle inequality. Use this to show $|f(z)| \leq 1/(R^2 - 2R - 2)$ for $z \in \gamma_1$ and hence, using the estimation lemma, that $\int_{\gamma_1} f\,\mathrm{d}z \to 0$ as $R \to \infty$. $\int_{\gamma_2} f(z)\,\mathrm{d}z \to \int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2-2x+2}\,\mathrm{d}x + i\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x^2-2x+2}\,\mathrm{d}x$. $z^2 - 2z + 2 = (z-(1+i))(z-(i-i))$ so that $f(z)$ is holomorphic on and inside γ except at $z = 1+i$. Hence by Cauchy's integral formula $\int_\gamma = 2\pi i \left.\frac{\mathrm{e}^{\pi}iz}{z-(1-i)}\right\|_{z=1+i} = -\pi\mathrm{e}^{-\pi}$. Conclude that $\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2-2x+2}\,\mathrm{d}x = -\pi\mathrm{e}^{-\pi}$ and $\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x^2-2x+2}\,\mathrm{d}x = 0$.

B1 (a) Applying the CR equations to $f(x+iy) = u(y) + iv(x)$ gives $v'(x) = -u'(y)$ for all $x,y \in \mathbb{R}$. Thus u' and v' are constant and thus f is of the form $f(z) = ay + b + i(cx+d)$ for some $a,b,c,d \in \mathbb{R}$. Using the condition from the CR equations again gives the claim. (b) f is differentiable at the origin and nowhere else. g is differentiable on the circline around the origin with radius 1. Both functions are nowhere holomorphic. (c) Only $a = b = -1$ guarantees that the function is complex differentiable on $\mathbb{C}$. $f_{-1,-1}(z) = \mathrm{e}^{iz}$.

B2 (a) parametrise the three sides of the triangle separately. From 1 to i: $\int_{\gamma_1} f = i$, from i to -1: $\int_{\gamma_2} f = i$, from -1 to 1: $\int_{\gamma_3} f = 0$. Thus the result is $2i$. This can be also derived from example 15.6 in the lecture where it was shown that $\int_\gamma \overline{z}\,\mathrm{d}z = 2i(\textbf{area enclosed})$. (b) Imitate the proof for $\int_{\partial B(0,\epsilon)} f(z)/z = 2\pi i f(0)$ from the lecture. (c) (i) $2\pi i(i^3) = -2\pi + i6\pi$ by Cauchy's integral formula. (ii) Use Cauchy's representation for the coefficient $c_2 = f^{(2)}(1)/2$ in the power series of $f(z) = \mathrm{e}^{z^2}$ about $z = 1$.

B3 (a) the integrals are zero . . . (i) by the Cauchy integral formula, (ii) by Cauchy's theorem, and (iii) by the fundamental theorem of calculus ($-(z-2)^{-2}/2$ is a primitive). (b) Integrals are zero for $b < \pi/2$ respectively $a < 1$. For $b > \pi/2$ resp. $a > 1$ one gets $\int_C \frac{\mathrm{e}^z}{(z-i\pi/2)^2}\,\mathrm{d}z = -2\pi$ and $\int_C \frac{z^3-4z^2+\sin z}{(z-1)^3}\,\mathrm{d}z = -i\pi(2+\sin 1)$.

B4 Imitate the calculation from the lecture. Result: $\int_{-\infty}^{\infty} \sin^2(x)/x^2\,\mathrm{d}x = \pi$.

C1 $v(x,y) = \sqrt{|xy|}$ vanishes along the axes so has zero partial derivatives at the origin. The Cauchy-Riemann equations do hold at the origin. However $\lim_{r\to 0} \frac{f(r\mathrm{e}^{i\theta})}{r\mathrm{e}^{i\theta}} = \frac{i\sqrt{|\cos\theta\sin\theta}}{\cos\theta + i\sin\theta}$ which varies as θ varies so that $\lim_{z\to 0} \frac{f(z)-f(0)}{z}$ does not exist.

C2 If $f = u(x,y) + iv(x,y)$ then $u^2 + v^2 = 0$. Differentiate to find $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Combine these with the Cauchy-Riemann equations to show that u and v are constants.

C2 $f'(z)$ is also holomorphic on C and so from question A4 we know that $f'(z)$ is a linear function and hence that f is a quadratic, namely $f(z) = f(0) + zf'(0) + z^2(f''(0)/2)$. The hypothesis implies that $f'(0) = 0$. Also by applying Cauchy's integral formula to $f'(z)$ we have that $|f''(0)| \leq \left\| \frac{1}{2\pi i}\int_{\partial B(0,1)} \frac{f'(z)}{z^2}\mathrm{d}z \right\| \leq 1$.

C3 (a) Use the quotient rule for differentiating. (b) By the uniqueness theorem $(1+z)^k$ must have the same power series inside $|z| < 1$ as the real power series known via the binomial theorem on $\mathbb{R}$.

C4 (a) $|c_k| = \left\| \frac{1}{2\pi i} \int_{\partial B(0,R)} \frac{f(z)}{z^{k+1}} \, \mathrm{d}z \right\| \leq \frac{1}{2\pi} \frac{M}{R^{k+1}} 2\pi R = \frac{M}{R^k}$. Apply this with $M = 1 + R$ and let $R \to \infty$ to see that $c_k = 0$ whenever $k > 1$. (b) Apply the bound from part (a) with $M = A + BR^L$ and let $R \to \infty$ to see that $c_k = 0$ whenever $k > L$.

C5 Zeros at $\pm i, \ \pm 2i$. The usual semicircle contour therefore has 2 singularities inside it. Bound the integral around the top of the semicircle using $\frac{z^2}{(z^2+1)(z^2+4)} \leq \frac{R^2}{(R^2-1)(R^2-4)}$ when $|z| = R$ is larger than 2. The final integral has value $\pi/3$.

C6 $\int_0^\infty \cos^2 t \, \mathrm{d}t = \sqrt{\pi/8}$. The hard part is to bound $\int_{\gamma_2} f(z) \, \mathrm{d}z$. By the estimation lemma this is bounded by $\int_0^R \mathrm{e}^{-R^2 \cos(2t)} R \, \mathrm{d}t$. Now use the fact that $\cos(2t) \geq 1 - (4t/\pi)$ for $t \in [0, \pi/4]$ (draw the two functions on this interval). This implies that $\int_{\gamma_2} f(z) \, \mathrm{d}z \leq R \int_0^R \mathrm{e}^{-R^2} \mathrm{e}^{-4R^2 t/\pi} \, \mathrm{d}t$ which can be exactly calculated.