MA231 Vector Analysis
Example Sheet 2

2010, term 1
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Hand in solutions to questions B1, B2, B3 and B4 by 3pm Monday of week 6.

A1 **Calculating divergences**

Calculate the divergence $\operatorname{div} v = \nabla \cdot v$ for the following vector fields $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ with:

$$(a)\;\; v(x,y,z) = (x^2, xy, xz) \quad (b)\;\; v(x,y,z) = (\cos(xy), \sqrt{1+x^2y^2z^2}, zy\sin(xy))$$

$$(c)\;\; v(x,y,z) = \nabla f(x,y,z) \quad \text{where } f\colon \mathbb{R}^3 \to \mathbb{R}, (x,y,z) \mapsto f(x,y,z) = xy\mathrm{e}^z.$$

A2 **Examples of the divergence theorem**

For each of the following vector fields $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ calculate the flux integral $\int_{\mathcal{S}} v \cdot \hat{N}\,\mathrm{d}S$ out of the sphere $\mathcal{S}$ given by $x^2+y^2+z^2 = R^2$. Calculate them first as surface integrals and then confirm that your answer agrees with the volume integral given by the divergence theorem.

$$(a)\;\; v(x,y,z) = (x,y,z) \quad (b)\;\; v(x,y,z) = (-y,x,0) \quad (c)\;\; v(x,y,z) = (-x,y,z).$$

A3 **Using the divergence theorem**

Let $f\colon \mathbb{R}^3 \to \mathbb{R}, (x,y,z) \mapsto f(x,y,z) = x^4+y^4+z^4$. Use the divergence theorem to calculate the outward flux of ∇f through the following surfaces:

(a) The boundary of the cube $0 \le x,y,z \le 1$, $\quad (b)$ The sphere $x^2+y^2+z^2 = R^2$.

A4 **Calculating curls**

For each of the following vector fields $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ calculate the curl $\nabla \times v$:

$$(a)\;\; v(x,y,z) = (-y,x,1) \qquad (b)\;\; v(x,y,z) = (xy+z, \frac{1}{2}x^2+2yz, y^2+x)$$

$$(c)\;\; v(x,y,z) = (xz, yz, 0)$$

A5 **Identities**

(a) For $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ show that $\nabla \cdot (\nabla \times v) = 0$.

(b) For $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ show that $\nabla \times (fv) = f\,\nabla \times v + \nabla f \times v$.

A6 **Stokes's theorem in the plane**

Let $v\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto v(x,y) = (0,x)$. Calculate curl$(v)$. Apply Stokes's theorem to v on the region Ω bounded by the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$. Hence prove that the area of Ω is $\pi\alpha\beta$.

A7 **Stokes's theorem in $\mathbb{R}^3$**

Sketch the surface $\mathcal{S}$ given by the portion of a paraboloid $z = 2 - x^2 - y^2$ where $z \ge 0$. Using an inward facing normal vector field, explain how to parameterise the boundary $\partial\mathcal{S}$ so that the unit tangent vector $\hat{T}$ and the unit normal vector $\hat{N}$ are correctly oriented for Stokes' theorem. For the vector field $v\colon \mathbb{R}^3 \to \mathbb{R}^3, (x,y,z) \mapsto v(x,y,z) = (y,z,x)$ calculate both the surface flux $\int_{\mathcal{S}} \nabla \times v \cdot \hat{N}\,\mathrm{d}S$ and the tangential line integral $\int_{\partial\mathcal{S}} v \cdot \hat{T}\,\mathrm{d}s$ and verify that Stokes's Theorem holds.

B1 **The flux integral**

(a) Calculate the flux of the vector field $f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (z, x, -3y^2 z)$ across the surface of the cylinder $\{(x, y, z) \in \mathbb{R}^3 \big| x^2 + y^2 = 16, z \in [0, 5]\}$.

(b) Calculate the flux of the vector field

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (4xz, -y^2, yz)$$

across the surface of the unit cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$. (the unit cube is the set $\{(x, y, z) \in \mathbb{R}^3 \bigm| 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$)

B2 **The divergence theorem**

Use the divergence theorem to calculate the following flux integrals.

(a) The outward flux of the two dimensional vector field

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = (x/2 + y\sqrt{x^2 + y^2}, y/2 - x\sqrt{x^2 + y^2})$$

through the boundary of the ball $\Omega = \{(x, y) \in \mathbb{R}^2 \bigm| x^2 + y^2 \leq R^2\} \subset \mathbb{R}^2, R > 0$.

(b) The outward flux of the vector field

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto f(x, y, z) = (2x, -y, 3z)$$

through the boundary of the pyramid Ω bounded by the planes $x + 2y + 3z = 6$, $x = 0$, $y = 0$ and $z = 0$. (Hint: you may quote a formula for the volume of a pyramid).

B3 **Identities and harmonic functions**

(a) For three dimensional vector fields $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ show that

$$\nabla\cdot(u \times v) = v\cdot(\nabla \times u) - u\cdot(\nabla \times v).$$

(b) For three dimensional scalar fields $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $g\colon \mathbb{R}^3 \to \mathbb{R}$ show that

$$\Delta(fg) = f\,\Delta g + 2\nabla f\cdot\nabla g + g\,\Delta f.$$

(c) For the scalar field $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ and the vector field $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ show that

$$\nabla\cdot(\varphi u) = (\nabla\varphi)\cdot u + \varphi(\nabla\cdot u).$$

(d) Give a sketch of the proof of the following statement: Let $f\colon D \to \mathbb{R}, D \subset \mathbb{R}^3$, be **harmonic** (that is $\Delta f(x) = 0$ for all $x \in D$). Then for any closed ball $\overline{B(a, r)} \subset D$ (recall $\overline{B(a, r)} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \|x - a\| \leq r\}$) having radius $r > 0$ and origin $a \in D$ with surface $\mathcal{S} = \partial B(a, r) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \|x - a\| = r\}$ it holds that the value of the function at the origin of the ball is the mean value of the function over the surface of that ball, i.e.

$$f(a) = \frac{1}{4\pi r^2} \int_{\mathcal{S}} f.$$

B4 **Integration by parts formulae**

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a vector field $v\colon \mathbb{R}^n \to \mathbb{R}^n$ show that

$$\nabla\cdot(fv) = \nabla f\cdot v + f\,\nabla\cdot\boldsymbol{v}.$$

Deduce the integration by parts formula, for a region $\Omega \subseteq \mathbb{R}^3$,

$$\int_\Omega f\,\nabla\cdot v\,\mathrm{d}V = -\int_\Omega \nabla f\cdot v\,\mathrm{d}V + \int_{\partial\Omega} f\,v\cdot\hat{N}\,\mathrm{d}S.$$

Write out the formula in the special case where $v = \nabla g$ for some $g\colon \mathbb{R}^3 \to \mathbb{R}$. Deduce that

$$\int_\Omega f\Delta g\,\mathrm{d}V = \int_\Omega g\Delta f\,\mathrm{d}V + \int_{\partial\Omega} f\,\nabla g\cdot\hat{N}\,\mathrm{d}S - \int_{\partial\Omega} g\,\nabla f\cdot\hat{N}\,\mathrm{d}S.$$

This final identity is called Green's identity, named after George Green who was the son of a Nottinghamshire baker and a self taught mathematician. See question C4 for an application of this identity.

C1 **Area and volume of the n-dimensional sphere and ball**

Consider the ball $B(0,R) = \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$ in $\mathbb{R}^n$ with its boundary, the sphere $\partial B(0,R) = \big\{x \in \mathbb{R}^n \bigm| \|x\| = R\big\}$. Apply the divergence theorem with the vector field $v\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto f(x) = x$, to conclude

$$n \text{ Volume}(B(0,R)) = R \text{ Surface Area } (\partial B(0,R)).$$

Check that this indeed is true in dimensions $n = 2, 3$.

C2 **Identities**

For the vector field $v\colon \mathbb{R}^3 \to \mathbb{R}^3, (x,y,z) \mapsto v(x,y,z) = (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z))$ define Δv by $\Delta v = (\Delta v_1, \Delta v_2, \Delta v_3)$. This is used in the equations for viscous fluid flow. Show that

$$\nabla \times (\nabla \times v) = \nabla(\nabla \cdot v) - \Delta v.$$

Deduce that if v is divergence free and also curl free then $\Delta v = 0$.

C3 **Vector versions of the Stokes and the divergence theorems**

The following disguised versions of these theorems are vector identities — so one way is to try and prove them one co-ordinate at a time.

(a) Show, for a properly oriented surface $\mathcal{S}, \partial\mathcal{S}, \hat{N}, \hat{T}$ in $\mathbb{R}^3$ and $f\colon \mathbb{R}^3 \to \mathbb{R}$, that

$$\int_{\mathcal{S}} \nabla f \times \hat{N} \,\mathrm{d}S = -\int_{\partial\mathcal{S}} f\hat{T} \,\mathrm{d}s.$$

(Hint: You might start by considering the vector field $v = (f, 0, 0)$ in the usual statements of Stokes's theorem.)

(b) Show for an outward unit normal $\hat{N}$

$$\int_{\Omega} \nabla \times v \,\mathrm{d}V = \int_{\partial\Omega} v \times \hat{N} \,\mathrm{d}S.$$

C4 **An application of Green's identity**

The resonant frequencies of small oscillations of a drum with shape $\Omega \subset \mathbb{R}^2$ are given by the eigenvalues λ corresponding to eigenfunctions $f(x)$ solving

$$-\Delta f(x) = \lambda f(x) \text{ for } x \in \Omega \text{ and } f(x) = 0 \text{ for } x \in \partial\Omega.$$

Suppose $\lambda_1 \neq \lambda_2$ are two eigenvalues corresponding to two eigenfunctions $f_1(x)$ and $f_2(x)$. Apply Green's identity to show that $\int_\Omega f_1 f_2 \,\mathrm{d}A = 0$. (The argument works in any dimension. For dimension $n = 1$ this yields the familiar result that $\int_0^{2\pi} \sin(kx)\sin(lx)\,\mathrm{d}x = 0$ when $k \neq l$.)