

Interacting Stochastic Processes

Problem sheet 3

1. Read the hand-out on phase separation and condensation in the ZRP (Section 3.4 of the notes) carefully, then do question 2.

2. Exam question from last year.

Consider a zero-range process on the lattice $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ with jump probabilities $p(x, y)$ and space-homogeneous jump rates $g_x(k) = g(k)$ for all $x \in \Lambda_L$.

(a) Give the state space X and write down the generator of this process.

Under which conditions on p and g is the process non-degenerate?

(b) Give a general formula for a family of stationary product measures of this process, explaining all the quantities in the formula.

Assuming that $p(x, y) = q(x - y)$ is translation invariant and the process is non-degenerate, show that the stationary measures are homogeneous, i.e. marginals do not depend on the lattice site x .

From now on, consider the jump rates $g(k) = \left(\frac{k+1}{2k}\right)^b$ for $k > 0$ and $g(0) = 0$ with $b > 0$ and translation invariant jump probabilities $p(x, y) = q(x - y)$.

(c) Show that the stationary current across each bond (x, y) is given by $\phi(q(x - y) - q(y - x))$, where $\phi \geq 0$ is the fugacity.

(d) Compute the stationary product measures of the process.

What is the radius of convergence of the partition function $z(\phi)$?

(e) Under which condition on the parameter b is the critical density $\rho_c < \infty$?

Sketch a phase diagram of this process and sketch a typical stationary density profile for each of the phase regions for a large finite system.

3. Exam question from last year.

(a) Consider the contact process on the finite lattice $\Lambda = \mathbb{Z}/8\mathbb{Z}$ with 8 sites and periodic boundary conditions.

Provide a graphical construction of the contact process with initial configuration $\eta = (0, 1, 1, 0, 1, 0, 0, 0)$ and the final configuration $\zeta = (0, 0, 0, 1, 0, 1, 1, 1)$.

Your construction should contain at least 3 of each basic event, which are recovery (marked by $*$), infection to the right (marked by \rightarrow) and infection to the left (marked by \leftarrow).

Now consider a modified version of the contact process in one dimension with $\Lambda = \mathbb{Z}$, given by the jump rates

$$101 \xrightarrow{\lambda} 111 \quad \text{and} \quad 1 \xrightarrow{1} 0.$$

So infection is only possible if both neighbours are infected.

(b) Write down the generator of this process.

(c) Give the mean field rate equation for the particle density, analyze it for stable and unstable stationary solutions and give a mean field prediction for the phase diagram.

(d) Consider the process on the finite lattice $\Lambda = \mathbb{Z}/8\mathbb{Z}$ with 8 sites and periodic boundary conditions. Give an initial condition $\eta = (\eta(1), \dots, \eta(8))$ with 4 infected sites (1) and 4 healthy sites (0), such that the number of infected sites has to decrease monotonically to 0 (no increase possible).

Let T be the time at which the process reaches the absorbing empty state starting from η . Show that

$$\mathbb{P}(T \leq t) = (1 - e^{-t})^4.$$

4. Exam question from last year.

(a) Define a basic coupling for the totally asymmetric simple exclusion process (TASEP) with $p = 1$ and $q = 0$ on the state space $X = \{0, 1\}^{\mathbb{Z}}$, by giving the jump rates of the coupled process. Use this to show that the TASEP is attractive.

(b) Is the TASEP on the finite lattice $\{1, \dots, L\}$ with open boundary conditions

$$c(1, \eta) = \alpha(1 - \eta(1)) \quad \text{and} \quad c(L, \eta) = \beta \eta(L)$$

attractive?

Justify your answer by adapting your argument from (a).

5. Consider the linear voter model on an arbitrary lattice Λ with generator

$$\mathcal{L}f(\eta) = \sum_{x, y \in \Lambda} p(x, y) \left(\eta(x)(1 - \eta(y)) + (1 - \eta(x))\eta(y) \right) (f(\eta^x) - f(\eta)),$$

where $p(x, y)$ are irreducible transition rates on Λ .

Use the duality function

$$D(\eta, A) = \prod_{x \in A} \eta(x) = \begin{cases} 1 & , \text{ if } \eta \equiv 1 \text{ on } A \\ 0 & , \text{ otherwise} \end{cases}$$

for finite $A \subseteq \Lambda$, to show that the linear voter model is dual to a system of coalescing random walks.

What are the transition rates of the random walks?

Why do we assume A to be finite? Under which conditions can this be relaxed?