Extreme Behaviour

How big can things get?

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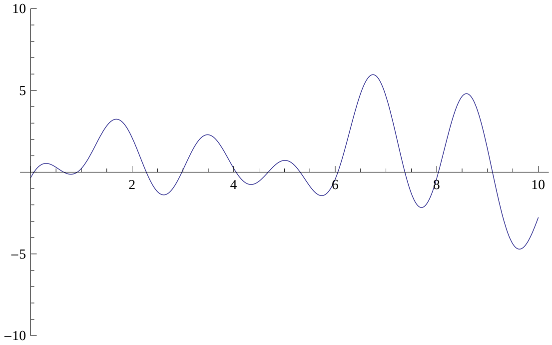
Motivation: The Riemann zeta function

The Riemann zeta function is of fundamental importance in number theory.

- Understand the distribution of prime numbers.
- Prototypical L-function (Dirichlet, elliptic curve, modular form, etc).
- An interesting and challenging function to understand in its own right.
- Universality (approximates any holomorphic function arbitrarily well).
- Can be used as a “experimental” test-bed for certain physical systems.
- One of the most important open problems in modern mathematics concerns it.
- It’s popular!

The Riemann zeta function



A real version of the Riemann zeta function,
plotted for $10^4 \leq t \leq 10^4 + 10$

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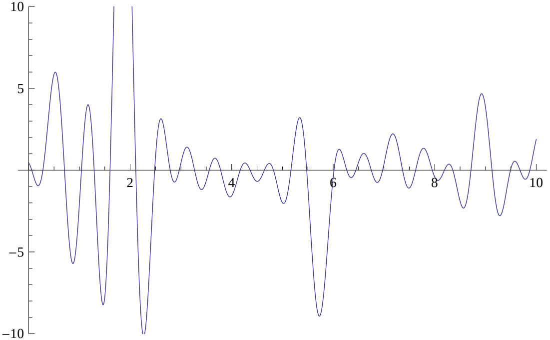


A real version of the Riemann zeta function, plotted for $10^{10} \leq t \leq 10^{10} + 10$

Extreme values of zeta

Conjecture (Farmer, Gonek, Hughes)

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log T \log\log T}\right)$$

Bounds on extreme values of zeta

Theorem (Littlewood; Ramachandra and Sankaranarayanan; Soundararajan; Chandee and Soundararajan)

Under RH, there exists a C such that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left(C\frac{\log T}{\log\log T}\right)\right)$$

Theorem (Montgomery; Balasubramanian and Ramachandra; Balasubramanian; Soundararajan)

There exists a C' such that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \Omega\left(\exp\left(C'\sqrt{\frac{\log T}{\log\log T}}\right)\right)$$

Characteristic polynomials

Keating and Snaith modelled the Riemann zeta function with

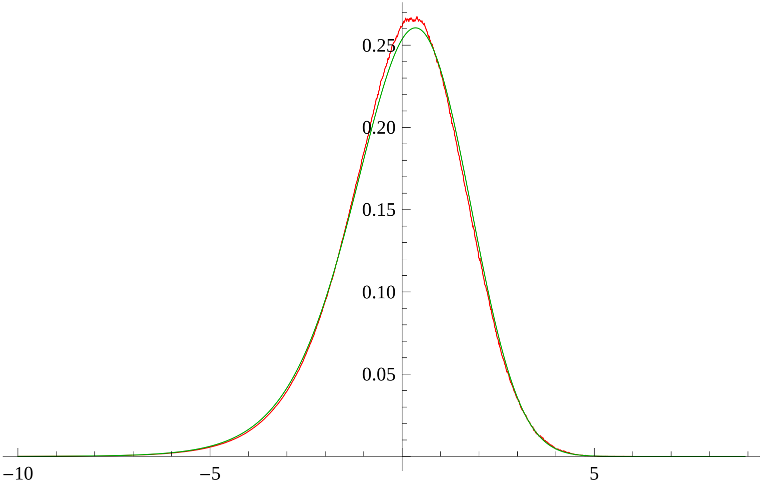
$$Z_{U_N}(\theta) := \det(I_N - U_N e^{-\mathrm{i}\theta}) = \prod_{n=1}^{N} (1 - e^{\mathrm{i}(\theta_n - \theta)})$$

where U_N is an $N \times N$ unitary matrix chosen with Haar measure.

The matrix size N is connected to the height up the critical line T via

$$N = \log \frac{T}{2\pi}$$

Characteristic polynomials



Graph of the value distribution of $\log|\zeta(\frac{1}{2}+\mathrm{i}t)|$ around the 10^{20}th zero (red), against the probability density of $\log|Z_{U_N}(0)|$ with $N = 42$ (green).

An Euler-Hadamard hybrid

Theorem (Gonek, Hughes, Keating)

A simplified form of our theorem is:

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t;X)Z(t;X) + errors$$

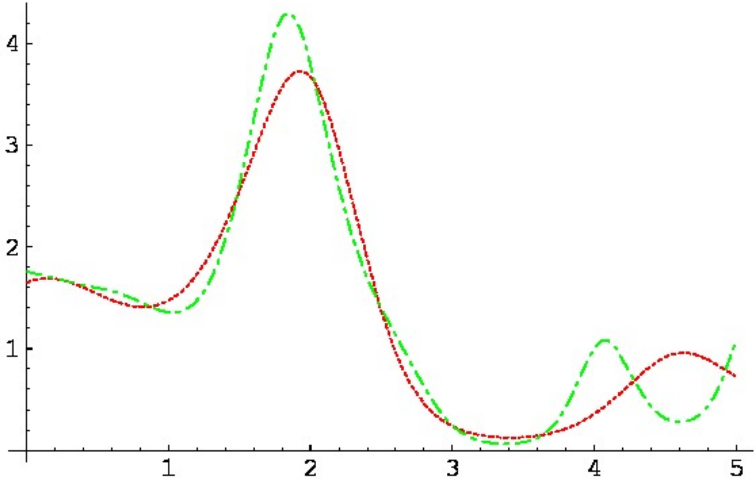
where

$$P(t;X) = \prod_{p \le X} \left(1 - \frac{1}{p^{\frac{1}{2}+\mathrm{i}t}}\right)^{-1}$$

and

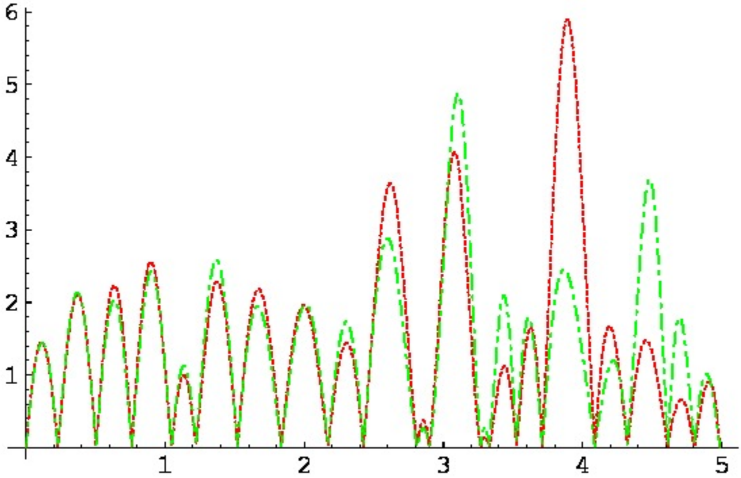
$$Z(t;X) = \exp\left(\sum_{\gamma_n} \mathrm{Ci}(|t-\gamma_n| \log X)\right)$$

An Euler-Hadamard hybrid: Primes only



Graph of $|P(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$,
with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

An Euler-Hadamard hybrid: Zeros only



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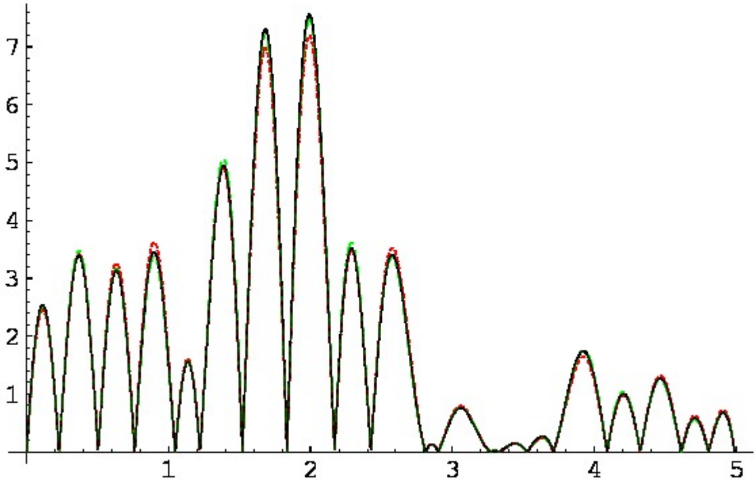
An Euler-Hadamard hybrid: Primes and zeros



Graph of $|\zeta(\frac{1}{2} + \mathrm{i}(t + t_0))|$ (black) and $|P(t + t_0; X)Z(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

RMT model for extreme values of zeta

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Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix.
Find the smallest $K = K(M, N)$ such that choosing M independent characteristic polynomials of size N, almost certainly none of them will be bigger than K.

RMT model for extreme values of zeta

Note that

$$\mathbb{P}\left\{\max_{1\leq j\leq M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)|\leq K\right\}=\mathbb{P}\left\{\max_{\theta}|Z_{U_N}(\theta)|\leq K\right\}^M$$

Theorem

Let $0<\beta<2$. *If* $M=\exp(N^\beta)$, *and if*

$$K=\exp\left(\sqrt{\left(1-\tfrac{1}{2}\beta+\varepsilon\right)\log M\log N}\right)$$

then

$$\mathbb{P}\left\{\max_{1\leq j\leq M}\max_{\theta}|Z_{U_N^{(j)}}(\theta)|\leq K\right\}\to 1$$

as $N\to\infty$ *for all* $\varepsilon>0$, *but for no* $\varepsilon<0$.

RMT model for extreme values of zeta

Recall

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t; X) Z(t; X) + \text{errors}$$

We showed that $Z(t; X)$ can be modelled by characteristic polynomials of size

$$N = \frac{\log T}{e^{\gamma} \log X}$$

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Therefore the previous theorem suggests

Conjecture

If $X = \log T$, *then*

$$\max_{t \in [0,T]} |Z(t; X)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log T \log\log T} \right)$$

RMT model for extreme values of zeta

Theorem

By the PNT, if $X = \log T$ *then for any* $t \in [0, T]$*,*

$$P(t; X) = O\left(\exp\left(C\frac{\sqrt{\log T}}{\log\log T}\right)\right)$$

Thus one is led to the max values conjecture

Conjecture

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

Distribution of the max of characteristic polynomials

Fyodorov, Hiary and Keating have recently studied the distribution of the maximum of a characteristic polynomial of a random unitary matrix via freezing transitions in certain disordered landscapes with logarithmic correlations. This mixture of rigorous and heuristic calculation led to:

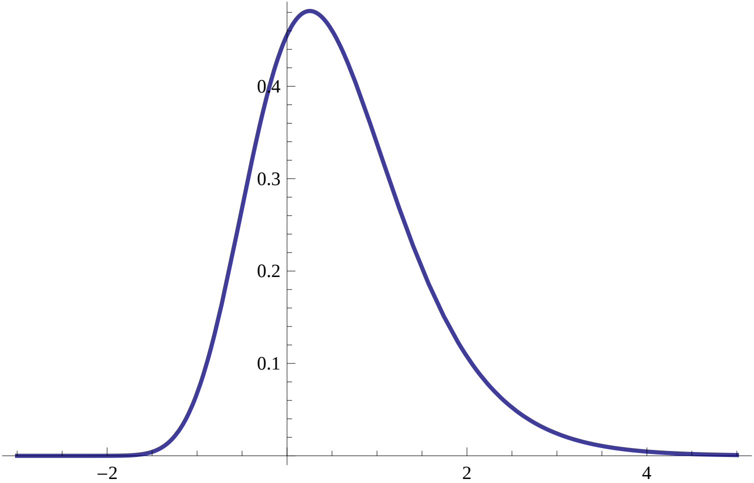
Conjecture (Fyodorov, Hiary and Keating)

For large N,

$$\log \max_{\theta} |Z_{U_N}(\theta)| \sim \log N - \frac{3}{4} \log \log N + Y$$

where Y has the density $\mathbb{P}\{Y \in \mathrm{d}y\} = 4e^{-2y} K_0(2e^{-y})\mathrm{d}y$

Distribution of the max of characteristic polynomials



The probability density of Y

This led them to conjecture that

$$\max_{T \le t \le T+2\pi} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| \sim \exp\left(\log\log\left(\frac{T}{2\pi}\right) - \frac{3}{4}\log\log\log\left(\frac{T}{2\pi}\right) + Y\right)$$

with Y having (approximately) the same distribution as before.

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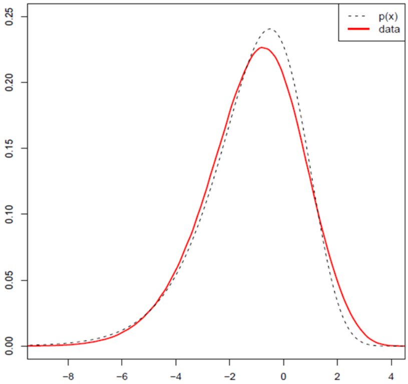
with Y having (approximately) the same distribution as before. This conjecture was backed up by a different argument of Harper, using random Euler products.

Distribution of the max of characteristic polynomials



Distribution of $-2\log\max_{t\in[T,T+2\pi]}|\zeta(\frac{1}{2}+\mathrm{i}t)|$ (after rescaling to get the empirical variance to agree) based on 2.5×10^8 zeros near $T=10^{28}$. Graph by Ghaith Hiary, taken from Fyodorov-Keating.

Distribution of the max of characteristic polynomials

Note that

$$\mathbb{P}\{Y \geq K\} \approx 2Ke^{-2K}$$

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However, one can show that if $K/\log N \to \infty$ but $K \ll N^{\epsilon}$ then

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$$\mathbb{P}\left\{\max_\theta \log |Z_{U_N}(\theta)| \geq K\right\} = \exp\left(-\frac{K^2}{\log N}(1+o(1))\right)$$

Thus there must be a critical K (of the order $\log N$) where the probability that $\max_\theta |Z_U(\theta)| \approx K$ changes from looking like linear exponential decay to quadratic exponential decay.

Moments of the local maxima

Theorem (Conrey and Ghosh)

As $T \to \infty$

$$\frac{1}{N(T)} \sum_{t_n \le T} \left|\zeta(\tfrac{1}{2} + \mathrm{i}t_n)\right|^2 \sim \frac{e^2 - 5}{2} \log T$$

where t_n *are the points of local maxima of* $|\zeta(\frac{1}{2} + \mathrm{i}t)|$*.*

This should be compared with Hardy and Littlewood's result

$$\frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^2 \mathrm{d}t \sim \log T$$

Moments of the local maxima

Recently Winn succeeding in proving a random matrix version of this result (in disguised form)

Theorem (Winn)

As $N \to \infty$

$$\mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left|Z_{U_N}(\phi_n)\right|^{2k}\right] \sim C(k)\,\mathbb{E}\left[\left|Z_{U_N}(0)\right|^{2k}\right]$$

where ϕ_n are the points of local maxima of $\left|Z_{U_N}(\theta)\right|$, and where $C(k)$ can be given explicitly as a combinatorial sum involving Pochhammer symbols on partitions.

In particular,

$$\mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left|Z_{U_N}(\phi_n)\right|^{2}\right] \sim \frac{e^2-5}{2}N$$

Summary

- We looked at the rate of growth of the Riemann zeta function on the critical line.
- The Riemann zeta function can be written in terms of a product of its zeros times a product over all primes.
- The product over zeros can be modelled by $Z_{U_N}(\theta)$.
- One argument required knowing the large deviations of $\max_\theta |Z_{U_N}(\theta)|$.
- The distribution of $\max_\theta |Z_{U_N}(\theta)|$ is now known.
- Its moderate deviations are still under study.
- The moments of the local maxima are not much bigger than ordinary moments.
- Extreme behaviour is rare!

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