Multispecies Virial Expansion

Stephen James Tate[1]
s.j.tate@warwick.ac.uk
joint work with: Sabine Jansen[2]
Dmitrios Tsagkarogiannis[3] Daniel Ueltschi[1]

[1]University of Warwick

[2]Leiden University

[3]University of Crete

May 7th 2013

Outline

Single Species Expansions

- Context of a Classical Interacting Gas

Single Species Expansions

- Context of a Classical Interacting Gas
  - We have the pair potential $\phi(x_i - x_j)$
  - The Grand Canonical Partition Function
    $\Xi(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda} \cdots \int_{\Lambda} \exp\left(-\beta \sum_{1 \le i < j \le n} \phi(x_i - x_j)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n$, where
    z is the fugacity parameter
  - In the Thermodynamic Limit $|\Lambda| \to \infty$, we have the pressure
    $\beta P = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \Xi(z)$
  - Expansion for pressure P in terms of fugacity z is the cluster expansion
  - We have $\rho = z \frac{\partial}{\partial z} P$, the density
  - The virial development of the Equation of State is the power series
    $P = \sum_{n=1}^{\infty} c_n \rho^n$ called the virial expansion.

Single Species Expansions

- Context of a Classical Interacting Gas
- Cluster and Virial Expansions are reasonably well understood

Single Species Expansions

- Context of a Classical Interacting Gas
- Cluster and Virial Expansions are reasonably well understood
- Represent cluster coefficients as weighted connected graphs ...

Single Species Expansions

- Context of a Classical Interacting Gas
- Cluster and Virial Expansions are reasonably well understood
- Represent cluster coefficients as weighted connected graphs . . .
- . . . and virial coefficients as weighted 2-connected graphs or irreducible integrals

Single Species Expansions

- Context of a Classical Interacting Gas
- Cluster and Virial Expansions are reasonably well understood
- Represent cluster coefficients as weighted connected graphs ...
- ... and virial coefficients as weighted 2-connected graphs or irreducible integrals
- Main work done in the 1930's and 40's by Mayer
  J. E. Mayer, M. G. Mayer, Statistical Mechanics New York, John Wiley and Sons Inc. (1940)
- But what happens if we have a mixture of different particles?

Finitely Many Species - Early Ideas/Formulae

- Imagining why we would want to generalise to many different types of particles is easy

Finitely Many Species - Early Ideas/Formulae

- Imagining why we would want to generalise to many different types of particles is easy
- Considering how to implement such a model was studied by Fuchs **K. Fuchs, The Statistical Mechanics of Many Component Gases, Proc. R. Soc. Lond. A., 179, (1942)**

Finitely Many Species - Early Ideas/Formulae

- Imagining why we would want to generalise to many different types of particles is easy
- Considering how to implement such a model was studied by Fuchs **K. Fuchs, The Statistical Mechanics of Many Component Gases, Proc. R. Soc. Lond. A., 179, (1942)**
- Initial difficulty: going from a single type particle to two different types gives 3 degrees of freedom (one for each of the 'single types' and one for the mixture)

Finitely Many Species - Early Ideas/Formulae

- Imagining why we would want to generalise to many different types of particles is easy
- Considering how to implement such a model was studied by Fuchs **K. Fuchs, The Statistical Mechanics of Many Component Gases, Proc. R. Soc. Lond. A., 179, (1942)**
- Initial difficulty: going from a single type particle to two different types gives 3 degrees of freedom (one for each of the 'single types' and one for the mixture)
- Paper implicitly uses Lagrange-Good Inversion and Tree-like relationships

Finitely Many Species - Early Ideas/Formulae

- Imagining why we would want to generalise to many different types of particles is easy
- Considering how to implement such a model was studied by Fuchs **K. Fuchs, The Statistical Mechanics of Many Component Gases, Proc. R. Soc. Lond. A., 179, (1942)**
- Initial difficulty: going from a single type particle to two different types gives 3 degrees of freedom (one for each of the 'single types' and one for the mixture)
- Paper implicitly uses Lagrange-Good Inversion and Tree-like relationships
- Notion of generalised Radii of convergence (Borel)

Combinatorial Tools

Approaching the Multispecies Cluster Expansion, we come armed with tools developed in Combinatorics:

- The Lagrange-Good Inversion
  I. J. Good, The generalisation of Lagrange's expansion and the enumeration of trees, Proc. Cambridge Philos. Soc., 61, 499-517 (1965) provides the way in which we can invert power series of form:

$$\rho(z) = z + \sum_{n\geq 2} n b_n z^n \tag{1}$$

Combinatorial Tools

Approaching the Multispecies Cluster Expansion, we come armed with tools developed in Combinatorics:

- The Lagrange-Good Inversion
  I. J. Good, The generalisation of Lagrange's expansion and the enumeration of trees, Proc. Cambridge Philos. Soc., 61, 499-517 (1965) provides the way in which we can invert power series of form:
  $$\rho(z) = z + \sum_{n\geq 2} n b_n z^n \tag{1}$$
- The Dissymmetry Theorem for Connected Graphs (and also trees)
  F. Bergeron, G. Labelle, P. Leroux, Combinatorial Species and Tree-like Structures, Encyclopaedia of Mathematics and its Applications, Vol. 67, Cambridge University Press, Cambridge, U.K. (1998)

Combinatorial Tools

- The notion of coloured graphs and an extension of the Dissymmetry Theorem - Application of this to the multivariate virial expansion **W. G. Faris, Biconnected graphs and the multivariate virial expansion, Markov Proc. Rel. Fields 18, 357–386 (2012)**

Combinatorial Tools

- The notion of coloured graphs and an extension of the Dissymmetry Theorem - Application of this to the multivariate virial expansion **W. G. Faris, Biconnected graphs and the multivariate virial expansion, Markov Proc. Rel. Fields 18, 357–386 (2012)**
- There is a lack of attention on the convergence of such expansions - only as formal power series

Statistical Mechanics

- The context of the work is on the multispecies generalisation of the paper by Poghosyan and Ueltschi
  Poghosyan, S. and Ueltschi, D., Abstract cluster expansion with applications to statistical mechanical systems, Journal of Mathematical Physics, 50, 5, (2009)

Statistical Mechanics

- The context of the work is on the multispecies generalisation of the paper by Poghosyan and Ueltschi
  Poghosyan, S. and Ueltschi, D., Abstract cluster expansion with applications to statistical mechanical systems, Journal of Mathematical Physics, 50, 5, (2009)
- We begin with a collection of fugacity parameters $\{z_i\}_{i\in\mathbb{N}}$ with z_i being the activity of the species i

Statistical Mechanics

- The context of the work is on the multispecies generalisation of the paper by Poghosyan and Ueltschi
Poghosyan, S. and Ueltschi, D., Abstract cluster expansion with applications to statistical mechanical systems, Journal of Mathematical Physics, 50, 5, (2009)
- We begin with a collection of fugacity parameters $\{z_i\}_{i\in\mathbb{N}}$ with z_i being the activity of the species i
- We assume the achievement of a 'cluster expansion' for the pressure and understand conditions to achieve a convergent virial expansion

Statistical Mechanics

- The context of the work is on the multispecies generalisation of the paper by Poghosyan and Ueltschi
  Poghosyan, S. and Ueltschi, D., Abstract cluster expansion with applications to statistical mechanical systems, Journal of Mathematical Physics, 50, 5, (2009)
- We begin with a collection of fugacity parameters $\{z_i\}_{i\in\mathbb{N}}$ with z_i being the activity of the species i
- We assume the achievement of a 'cluster expansion' for the pressure and understand conditions to achieve a convergent virial expansion
- We start from the 'formal' power series representation (Cluster Expansion):

$$P(\mathbf{z}) = \sum_{\mathbf{n}} b(\mathbf{n})\mathbf{z}^{\mathbf{n}} \tag{CE}$$

Statistical Mechanics

- $$P(\mathbf{z}) = \sum_{\mathbf{n}} b(\mathbf{n}) \mathbf{z}^{\mathbf{n}} \qquad \text{(CE)}$$

Statistical Mechanics

- $$P(\mathbf{z}) = \sum_{\mathbf{n}} b(\mathbf{n})\mathbf{z}^{\mathbf{n}} \tag{CE}$$
- We may formally define:

$$\rho_k := z_k \frac{\partial}{\partial z_k} P \tag{R1}$$

or via the power series:

$$\rho_k := \sum_{\mathbf{n}} n_k b(\mathbf{n})\mathbf{z}^{\mathbf{n}} \tag{R2}$$

Statistical Mechanics

- $$P(\mathbf{z}) = \sum_{\mathbf{n}} b(\mathbf{n})\mathbf{z}^{\mathbf{n}} \qquad \text{(CE)}$$
- We may formally define:

$$\rho_k := z_k \frac{\partial}{\partial z_k} P \qquad \text{(R1)}$$

or via the power series:

$$\rho_k := \sum_{\mathbf{n}} n_k b(\mathbf{n})\mathbf{z}^{\mathbf{n}} \qquad \text{(R2)}$$

- We wish to invert (R2), substitute for z in (CE) to obtain:

$$P(\rho) = \sum_{\mathbf{n}} c(\mathbf{n})\rho^{\mathbf{n}} \qquad \text{(VE)}$$

Convergence Conditions

Theorem (Jansen, T., Tsagkarogiannis, Ueltschi)

Assume that there exist $0 < r_i < R_i$ *and* $a_i \geq 0$, $i \in \mathbb{N}$, *such that*

- $p(\mathbf{z})$ *converges absolutely in the polydisc* $D = \{\mathbf{z} \in \mathbb{C}^{\mathbb{N}} \mid \forall i \in \mathbb{N} : |z_i| < R_i\}$.
- $\left|\log \frac{\partial p}{\partial z_i}(\mathbf{z})\right| < a_i$ *for all* $i \geq 1$ *and all* $\mathbf{z} \in D$.
- $\sum_{i \geq 1} \sqrt{\frac{r_i}{R_i}} < \infty$ *and* $\sum_{i \geq 1} \frac{r_i a_i^2}{R_i} < \infty$.

Then there exists a constant $C < \infty$ *(which depends on the* r_i, R_i, a_i, *but not on* $\mathbf{n}$*) such that*

$$|c(\mathbf{n})| \leq C \sup_{\mathbf{z} \in D} |p(\mathbf{z})| \prod_{i \geq 1} \left(\frac{e^{a_i}}{r_i} \right)^{n_i}. \tag{C1}$$

Convergence Conditions

The estimate for $c(\mathbf{n})$ guarantees convergence of the series $\sum_{\mathbf{n}} c(\mathbf{n})\boldsymbol{\rho}^{\mathbf{n}}$ for all $\boldsymbol{\rho}$ in the polydisc

$$D' = \left\{ \boldsymbol{\rho} \in \mathbb{C}^{\mathbb{N}} \mid \forall i \in \mathbb{N} : \ |\rho_i| < r_i e^{-a_i}, \ \sum_{i \in \mathbb{N}} |\rho_i| \frac{e^{a_i}}{r_i} < \infty \right\}.$$

Lagrange-Good Inversion

Theorem

Let $\mathbf{z}(\boldsymbol{\rho})$ *be a summable collection of power series and* $\mathbf{G}(\boldsymbol{\rho})$ *be a collection of formal power series, such that* $\forall i \in \mathbb{N}$

$$z_i(\boldsymbol{\rho}) = \rho_i G_i(\mathbf{z}(\boldsymbol{\rho})) \tag{LI1}$$

Let $J = \{i \in \mathbf{N} \mid n_i \neq 0\}$ *and* $\mathbf{n} \geq \mathbf{k}$*, then we have that:*

$$[\boldsymbol{\rho}^{\mathbf{n}}]\mathbf{z}(\boldsymbol{\rho})^{\mathbf{k}} = [\mathbf{z}^{\mathbf{n}-\mathbf{k}}]\left|\delta_{i,j} G_i(\mathbf{z})^{n_i} - z_j \frac{\partial G_i}{\partial z_j} G_i(\mathbf{z})^{n_i - 1}\right|_{i,j \in J} \tag{2}$$

Lagrange-Good Inversion

Theorem

Let $\mathbf{z}(\boldsymbol{\rho})$ be a summable collection of power series and $\mathbf{G}(\boldsymbol{\rho})$ be a collection of formal power series, such that $\forall i \in \mathbb{N}$

$$z_i(\boldsymbol{\rho}) = \rho_i G_i(\mathbf{z}(\boldsymbol{\rho})) \tag{LI1}$$

Let $J = \{i \in \mathbf{N} \mid n_i \neq 0\}$ and $\mathbf{n} \geq \mathbf{k}$, then we have that:

$$[\boldsymbol{\rho}^{\mathbf{n}}]\mathbf{z}(\boldsymbol{\rho})^{\mathbf{k}} = [\mathbf{z}^{\mathbf{n}-\mathbf{k}}]\left|\delta_{i,j} G_i(\mathbf{z})^{n_i} - z_j \frac{\partial G_i}{\partial z_j} G_i(\mathbf{z})^{n_i-1}\right|_{i,j\in J} \tag{2}$$

Recall that:

$$\rho_i(\mathbf{z}) := z_i \frac{\partial P}{\partial z_i} \tag{R}$$

Lagrange-Good Inversion

Theorem

Let $\mathbf{z}(\boldsymbol{\rho})$ be a summable collection of power series and $\mathbf{G}(\boldsymbol{\rho})$ be a collection of formal power series, such that $\forall i \in \mathbb{N}$

$$z_i(\boldsymbol{\rho}) = \rho_i G_i(\mathbf{z}(\boldsymbol{\rho})) \tag{LI1}$$

Let $J = \{i \in \mathbf{N} \mid n_i \neq 0\}$ and $\mathbf{n} \geq \mathbf{k}$, then we have that:

$$[\boldsymbol{\rho}^{\mathbf{n}}]\mathbf{z}(\boldsymbol{\rho})^{\mathbf{k}} = [\mathbf{z}^{\mathbf{n}-\mathbf{k}}]\left|\delta_{i,j} G_i(\mathbf{z})^{n_i} - z_j \frac{\partial G_i}{\partial z_j} G_i(\mathbf{z})^{n_i-1}\right|_{i,j\in J} \tag{2}$$

Recall that:

$$\rho_i(\mathbf{z}) := z_i \frac{\partial P}{\partial z_i} \tag{R}$$

So we have that $G_i = \frac{1}{\frac{\partial P}{\partial z_i}}$

THE UNIVERSITY OF WARWICK

Lagrange-Good Inversion

Theorem

Let $\mathbf{z}(\boldsymbol{\rho})$ be a summable collection of power series and $\mathbf{G}(\boldsymbol{\rho})$ be a collection of formal power series, such that $\forall i \in \mathbb{N}$

$$z_i(\boldsymbol{\rho}) = \rho_i G_i(\mathbf{z}(\boldsymbol{\rho})) \tag{LI1}$$

Let $J = \{i \in \mathbf{N} \mid n_i \neq 0\}$ and $\mathbf{n} \geq \mathbf{k}$, then we have that:

$$[\boldsymbol{\rho}^{\mathbf{n}}]\mathbf{z}(\boldsymbol{\rho})^{\mathbf{k}} = [\mathbf{z}^{\mathbf{n}-\mathbf{k}}]\left|\delta_{i,j} G_i(\mathbf{z})^{n_i} - z_j \frac{\partial G_i}{\partial z_j} G_i(\mathbf{z})^{n_i-1}\right|_{i,j\in J} \tag{2}$$

This gives us the Lagrange Inversion Formula:

$$[\boldsymbol{\rho}^{\mathbf{n}}]P(\boldsymbol{\rho}) = [\mathbf{z}^{\mathbf{n}}]P(\mathbf{z})\left|\delta_{i,j}\left(\frac{1}{\frac{\partial P}{\partial z_i}}\right)^{n_i} - z_j \frac{\partial}{\partial z_j}\left(\frac{1}{\frac{\partial P}{\partial z_i}}\right)\left(\frac{1}{\frac{\partial P}{\partial z_i}}\right)^{n_i-1}\right|_{i,j\in J} \tag{LI3}$$

Lagrange-Good Inversion

This gives us the Lagrange Inversion Formula:

$$[\boldsymbol{\rho}^{\mathbf{n}}]P(\boldsymbol{\rho}) = [\mathbf{z}^{\mathbf{n}}]P(\mathbf{z})\left|\delta_{i,j}\left(\frac{1}{\frac{\partial P}{\partial z_i}}\right)^{n_i} - z_j\frac{\partial}{\partial z_j}\left(\frac{1}{\frac{\partial P}{\partial z_i}}\right)\left(\frac{1}{\frac{\partial P}{\partial z_i}}\right)^{n_i-1}\right|_{i,j\in J} \tag{LI3}$$

We rearrange this to:

$$[\boldsymbol{\rho}^{\mathbf{n}}]P(\boldsymbol{\rho}) = [\mathbf{z}^{\mathbf{n}}]P(\mathbf{z})\frac{1}{\left(\frac{\partial P}{\partial \mathbf{z}}\right)^{\mathbf{n}}}\left|\delta_{i,j} + z_j\frac{\partial}{\partial z_j}\ln\frac{\partial P}{\partial z_i}\right|_{i,j\in J} \tag{LI4}$$

Lagrange-Good Inversion

We rearrange this to:

$$[\boldsymbol{\rho}^{\mathbf{n}}]P(\boldsymbol{\rho}) = [\mathbf{z}^{\mathbf{n}}]P(\mathbf{z})\frac{1}{\left(\frac{\partial P}{\partial \mathbf{z}}\right)^{\mathbf{n}}}\left|\delta_{i,j} + z_j \frac{\partial}{\partial z_j} \ln \frac{\partial P}{\partial z_i}\right|_{i,j\in J} \tag{LI4}$$

Recall the bound we have:

$$|c(\mathbf{n})| \leq C \sup_{\mathbf{z}\in D} |P(\mathbf{z})| \prod_{i\geq 1}\left(\frac{e^{a_i}}{r_i}\right)^{n_i}. \tag{C1}$$

Lagrange-Good Inversion

We rearrange this to:

$$[\boldsymbol{\rho}^{\mathbf{n}}]P(\boldsymbol{\rho}) = [\mathbf{z}^{\mathbf{n}}]P(\mathbf{z})\frac{1}{\left(\frac{\partial P}{\partial \mathbf{z}}\right)^{\mathbf{n}}}\left|\delta_{i,j} + z_j \frac{\partial}{\partial z_j} \ln \frac{\partial P}{\partial z_i}\right|_{i,j\in J} \tag{LI4}$$

Recall the bound we have:

$$|c(\mathbf{n})| \leq C \sup_{\mathbf{z}\in D} |P(\mathbf{z})| \prod_{i\geq 1}\left(\frac{e^{a_i}}{r_i}\right)^{n_i}. \tag{C1}$$

We can therefore see where the bound comes from - the C as uniform bound on determinant, the final product from bounds on the derivative in the assumption

What are the Virial Coefficients?

- We now have some good ideas on the convergence of such series, but would like to understand whether we have the same interpretation of the virial coefficients as 2-connected (irreducible) graphs

What are the Virial Coefficients?

- We now have some good ideas on the convergence of such series, but would like to understand whether we have the same interpretation of the virial coefficients as 2-connected (irreducible) graphs
- Requires notion of coloured graphs - each node has a colour, which represents the species of the particle with the given label

What are the Virial Coefficients?

- We now have some good ideas on the convergence of such series, but would like to understand whether we have the same interpretation of the virial coefficients as 2-connected (irreducible) graphs
- Requires notion of coloured graphs - each node has a colour, which represents the species of the particle with the given label
- The dissymmetry theorem can be generalised to this case, but we still have issues in considering block-multiplicative weight-functions

What are the Virial Coefficients?

- We now have some good ideas on the convergence of such series, but would like to understand whether we have the same interpretation of the virial coefficients as 2-connected (irreducible) graphs
- Requires notion of coloured graphs - each node has a colour, which represents the species of the particle with the given label
- The dissymmetry theorem can be generalised to this case, but we still have issues in considering block-multiplicative weight-functions
- The dissymmetry theorem gives us a relationship between connected and two-connected graphs via rooted connected graphs

The Dissymmetry Theorem

The dissymmetry theorem for connected graphs gives us the relation:

$$\mathcal{C} + \mathcal{B}^{\bullet}(\mathcal{C}^{\bullet}) = \mathcal{C}^{\bullet} + \mathcal{B}(\mathcal{C}^{\bullet}) \tag{D1}$$

There is a combinatorial interpretation of what each term means:

The Dissymmetry Theorem

The dissymmetry theorem for connected graphs gives us the relation:

$$\mathcal{C} + \mathcal{B}^{\bullet}(\mathcal{C}^{\bullet}) = \mathcal{C}^{\bullet} + \mathcal{B}(\mathcal{C}^{\bullet}) \tag{D1}$$

There is a combinatorial interpretation of what each term means:

- The superscript $\bullet$ denotes a rooted structure
- The composition of structures eg $\mathcal{B}'(\mathcal{C})$ indicates a $\mathcal{B}'$ assembly of $\mathcal{C}^{\bullet}$-structures

The Dissymmetry Theorem

The dissymmetry theorem for connected graphs gives us the relation:

$$\mathcal{C} + \mathcal{B}^{\bullet}(\mathcal{C}^{\bullet}) = \mathcal{C}^{\bullet} + \mathcal{B}(\mathcal{C}^{\bullet}) \tag{D1}$$

There is a combinatorial interpretation of what each term means:

- $\mathcal{C}$ is a connected graph

The Dissymmetry Theorem

The dissymmetry theorem for connected graphs gives us the relation:

$$\mathcal{C} + \mathcal{B}^{\bullet}(\mathcal{C}^{\bullet}) = \mathcal{C}^{\bullet} + \mathcal{B}(\mathcal{C}^{\bullet}) \tag{D1}$$

There is a combinatorial interpretation of what each term means:

- $\mathcal{C}$ is a connected graph
- $\mathcal{C}^{\bullet}$ is a rooted connected graph

The Dissymmetry Theorem

The dissymmetry theorem for connected graphs gives us the relation:

$$\mathcal{C} + \mathcal{B}^{\bullet}(\mathcal{C}^{\bullet}) = \mathcal{C}^{\bullet} + \mathcal{B}(\mathcal{C}^{\bullet}) \tag{D1}$$

There is a combinatorial interpretation of what each term means:

- $\mathcal{C}$ is a connected graph
- $\mathcal{C}^{\bullet}$ is a rooted connected graph
- $\mathcal{B}^{\bullet}(\mathcal{C}^{\bullet})$ is a connected graph rooted at any block, vertex or articulation point except for one

The Dissymmetry Theorem

The dissymmetry theorem for connected graphs gives us the relation:

$$\mathcal{C} + \mathcal{B}^{\bullet}(\mathcal{C}^{\bullet}) = \mathcal{C}^{\bullet} + \mathcal{B}(\mathcal{C}^{\bullet}) \tag{D1}$$

There is a combinatorial interpretation of what each term means:

- $\mathcal{C}$ is a connected graph
- $\mathcal{C}^{\bullet}$ is a rooted connected graph
- $\mathcal{B}^{\bullet}(\mathcal{C}^{\bullet})$ is a connected graph rooted at any block, vertex or articulation point except for one
- $\mathcal{B}(\mathcal{C}^{\bullet})$ is a connected graph rooted at a particular block

The Dissymmetry Theorem

- The Dissymmetry theorem relies on the connection with $\mathcal{B}'$-enriched trees

The Dissymmetry Theorem

- The Dissymmetry theorem relies on the connection with $\mathcal{B}'$-enriched trees
- The idea is to view connected graphs as being made up of 'trees of 2-connected graphs'

The Dissymmetry Theorem

- The Dissymmetry theorem relies on the connection with $\mathcal{B}'$-enriched trees
- The idea is to view connected graphs as being made up of 'trees of 2-connected graphs'
- This extra structure adds more to the interpretation of what we get from Lagrange Inversion

Conclusions and Open Questions

- We have obtained convergence conditions for infinitely many species in the virial expansion

Conclusions and Open Questions

- We have obtained convergence conditions for infinitely many species in the virial expansion
- Lagrange-Good inversion generalises precisely what one needs to do to get a virial expansion from the cluster expansion in the multispecies case

Conclusions and Open Questions

- We have obtained convergence conditions for infinitely many species in the virial expansion
- Lagrange-Good inversion generalises precisely what one needs to do to get a virial expansion from the cluster expansion in the multispecies case
- The explanation of the virial coefficients representing 2-connected coloured graphs is possible in some circumstances

Conclusions and Open Questions

- We have obtained convergence conditions for infinitely many species in the virial expansion
- Lagrange-Good inversion generalises precisely what one needs to do to get a virial expansion from the cluster expansion in the multispecies case
- The explanation of the virial coefficients representing 2-connected coloured graphs is possible in some circumstances
- There are some difficulties in applying some of the conditions and the dissymmetry theorem to particular examples. The issue relies on understanding how to gain appropriate lower bounds on the derivative.

Further Work/Open Questions

- Langrange Inversion and the Dissymmetry Theorem run in parallel to provide in the former case a method of computing coefficients exactly and in the latter case an interpretation of the coefficients in terms of combinatorial structures

Further Work/Open Questions

- Langrange Inversion and the Dissymmetry Theorem run in parallel to provide in the former case a method of computing coefficients exactly and in the latter case an interpretation of the coefficients in terms of combinatorial structures
- Further work could be done in understanding what models can precisely fit the requirements of our paper

Further Work/Open Questions

- Langrange Inversion and the Dissymmetry Theorem run in parallel to provide in the former case a method of computing coefficients exactly and in the latter case an interpretation of the coefficients in terms of combinatorial structures
- Further work could be done in understanding what models can precisely fit the requirements of our paper
- There are notable difficulties in considering such a problem from an implicit-function theorem point of view