Extending the Khintchine-Meinardus probabilistic method to general funcions

Dudley Stark

Queen Mary, University of London

Random Combinatorial Structures and Statistical Mechanics
Venice, Friday 10 May
Joint work with Boris Granovsky.

Mathematical set-up and motivation

We are interested in the asymptotics of some sequence c_n.
For example, let c_n be the number of interger partitions of n.
We can write 3 as

$$3 = 1 + 1 + 1 = 2 + 1 \quad \text{so } c_3 = 3.$$

$$\begin{aligned} \sum_{n=1}^{\infty} c_n z^n &= (1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)\cdots \\ &= \prod_{k=1}^{\infty} (1 - z^k)^{-1} \end{aligned}$$

Mathematical set-up and motivation

We are interested in the asymptotics of some sequence c_n.
For example, let c_n be the number of interger partitions of n.
We can write 3 as

$$3 = 1 + 1 + 1 = 2 + 1 \quad \text{so } c_3 = 3.$$

$$\begin{aligned} \sum_{n=1}^{\infty} c_n z^n &= (1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)\cdots \\ &= \prod_{k=1}^{\infty} (1 - z^k)^{-1} \end{aligned}$$

Theorem (Hardy-Ramanujan, (1918))

$$c_n \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

- In combinatorics, c_n is the number of decomposable structures of size n.
    - integer partitions of n.

- In combinatorics, c_n is the number of decomposable structures of size n.
  - integer partitions of n.
  - More generally, weighted partitions, for which there are b_k types of parts of size k.

b_k could be the number of types of parts of size k or the number of primes in an arithmetic semigroup.

- In combinatorics, c_n is the number of decomposable structures of size n.
  - integer partitions of n.
  - More generally, weighted partitions, for which there are b_k types of parts of size k.

  b_k could be the number of types of parts of size k or the number of primes in an arithmetic semigroup.

$$\sum_{n=1}^{\infty} c_n z^n = \prod_{k=1}^{\infty} (1 - z^k)^{-b_k}$$

Generally, we will have

$$b_k \asymp k^{r-1}, \quad r > 0.$$

- In physics, a partition is an assembly of particles such that its full energy is n.
- Define the generating function $f(z) = \sum_{n=1}^{\infty} c_n z^n$.

- In physics, a partition is an assembly of particles such that its full energy is n.
- Define the generating function $f(z) = \sum_{n=1}^{\infty} c_n z^n$.
- We assume $f(z)$ is of the form (Vershik, 1996)

$$f(z) = \prod_{k=1}^{\infty} S_k(z).$$

Examples:

- Unrestricted partitions, i.e. Bose-Einstein statistics:

$$f(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-b_k},$$

Examples:

- Unrestricted partitions, i.e. Bose-Einstein statistics:

$$f(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-b_k},$$

- Partitions having no repeated parts, i.e. Fermi-Dirac statistics:

$$f(z) = \prod_{k=1}^{\infty} (1 + z^k)^{b_k},$$

Examples:

- Unrestricted partitions, i.e. Bose-Einstein statistics:

$$f(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-b_k},$$

- Partitions having no repeated parts, i.e. Fermi-Dirac statistics:

$$f(z) = \prod_{k=1}^{\infty} (1 + z^k)^{b_k},$$

- Assemblies (labelled combinatorial objects),
  i.e. "Maxwell-Boltzman" model, let $c_n \mapsto c_n/n!$, $b_k \mapsto b_k/k!$,

$$f(z) = \exp\left(\sum_{k=1}^{\infty} b_k z^k\right) = \prod_{k=1}^{\infty} e^{b_k z^k},$$

Our goal is to find the asymptotics of c_n.
We use a Khintchine type probabilistic representation of c_n.

Our goal is to find the asymptotics of c_n.
We use a Khintchine type probabilistic representation of c_n.

- Assume that

$$S_k(z) = \sum_{j \geq 0} d_k(j) z^{kj}, \quad k \geq 1.$$

Our goal is to find the asymptotics of c_n.
We use a Khintchine type probabilistic representation of c_n.

- Assume that

$$S_k(z) = \sum_{j \geq 0} d_k(j) z^{kj}, \quad k \geq 1.$$

For $\delta > 0$ fixed, define Y_k, $k \geq 1$ to be integer valued random variables

$$\mathbb{P}(Y_k = jk) = \frac{d_k(j) e^{-\delta kj}}{S_k(e^{-\delta})}, \quad j \geq 0, \quad k \geq 1$$

Our goal is to find the asymptotics of c_n.
We use a Khintchine type probabilistic representation of c_n.

- Assume that

$$S_k(z) = \sum_{j \geq 0} d_k(j) z^{kj}, \quad k \geq 1.$$

For $\delta > 0$ fixed, define Y_k, $k \geq 1$ to be integer valued random variables

$$\mathbb{P}(Y_k = jk) = \frac{d_k(j) e^{-\delta kj}}{S_k(e^{-\delta})}, \quad j \geq 0, \quad k \geq 1$$

  - For unrestricted partitions, $\frac{1}{k} Y_k \sim \mathrm{NegBinomial}(b_k, e^{-\delta k})$.
  - For restricted partitions, $\frac{1}{k} Y_k \sim \mathrm{Binomial}\left(b_k, \frac{e^{-\delta k}}{1+e^{-\delta k}}\right)$.
  - For assemblies, $\frac{1}{k} Y_k \sim \mathrm{Poisson}(b_k e^{-\delta k})$.

- Define

$$f_n(z) = \prod_{k=1}^{n} S_k(z).$$

- Define

$$Z_n = \sum_{k=1}^{n} Y_k$$

Then, we have the following representation, true for any $\delta > 0$,

Fundamental Representation

$$c_n = e^{n\delta} f_n(e^{-\delta}) \, \mathbb{P}(Z_n = n)$$

Proof

Let $z^{-\delta+2\pi i\alpha}$, $\alpha \in [0,1]$ and let δ be a free parameter. Then

$$
\begin{aligned}
c_n &= e^{n\delta} \int_0^1 f(e^{-\delta+2\pi i\alpha}) e^{-2\pi i\alpha n} d\alpha \\
&= e^{n\delta} \int_0^1 \prod_{k=1}^{n} \left(S_k \left(e^{-\delta+2\pi i\alpha} \right) \right) e^{-2\pi i\alpha n} d\alpha \\
&= e^{n\delta} f_n(e^{-\delta}) \int_0^1 \prod_{k=1}^{n} \left(\frac{S_k \left(e^{-\delta+2\pi i\alpha} \right)}{S_k(e^{-\delta})} \right) e^{-2\pi i\alpha n} d\alpha
\end{aligned}
$$

We choose $\delta = \delta_n$ such that $\mathbb{E}(Z_n) = n$, so that if Z_n is asymptotically normal, we can expect that

Local Limit Lemma

$$\mathbb{P}(Z_n = n) \sim \frac{1}{\sqrt{2\pi \mathrm{Var}(Z_n)}}.$$

Estimating $e^{n\delta_n}$ and $f_n(e^{-\delta_n})$ involves approximating δ_n up to order $o(n^{-1})$.

Meinardus (1954) invented a method for the asymptotic enumeration of weighted partitions. We have adapted his method to estimate δ_n and to prove the Local Limit Lemma in the three examples above (unrestricted partitions, partitions containing at most one part of a given size, and assemblies).

Meinardus (1954) invented a method for the asymptotic enumeration of weighted partitions. We have adapted his method to estimate δ_n and to prove the Local Limit Lemma in the three examples above (unrestricted partitions, partitions containing at most one part of a given size, and assemblies).

There are three Meinardus conditions:

Meinardus (1954) invented a method for the asymptotic enumeration of weighted partitions. We have adapted his method to estimate δ_n and to prove the Local Limit Lemma in the three examples above (unrestricted partitions, partitions containing at most one part of a given size, and assemblies).

There are three Meinardus conditions:

(*I*) The Dirichlet generating function $D(s) = \sum_{k=1}^{\infty} b_k k^{-s}$ is meromorphic in a domain containing the half plane $\{\Re s \geq 0\}$, with a pole ρ_r on the real axis.

Meinardus (1954) invented a method for the asymptotic enumeration of weighted partitions. We have adapted his method to estimate δ_n and to prove the Local Limit Lemma in the three examples above (unrestricted partitions, partitions containing at most one part of a given size, and assemblies).

There are three Meinardus conditions:

(*I*) The Dirichlet generating function $D(s) = \sum_{k=1}^{\infty} b_k k^{-s}$ is meromorphic in a domain containing the half plane $\{\Re s \geq 0\}$, with a pole ρ_r on the real axis.

(*II*) $D(s)$ grows at most polynomially as $|\Im s| \to \infty$.

Meinardus (1954) invented a method for the asymptotic enumeration of weighted partitions. We have adapted his method to estimate δ_n and to prove the Local Limit Lemma in the three examples above (unrestricted partitions, partitions containing at most one part of a given size, and assemblies).

There are three Meinardus conditions:

(*I*) The Dirichlet generating function $D(s) = \sum_{k=1}^{\infty} b_k k^{-s}$ is meromorphic in a domain containing the half plane $\{\Re s \geq 0\}$, with a pole ρ_r on the real axis.

(*II*) $D(s)$ grows at most polynomially as $|\Im s| \to \infty$.

(*III*) Let $\tau = \delta + 2\pi i\alpha$. Let $g(\tau) = \sum_{k=1}^{\infty} b_k e^{-k\tau}$. There are constants $C_2 > 0$, $\varepsilon > 0$, such that

$$\Re(g(\tau)) - g(\delta) \leq -C_2\delta^{-\varepsilon}, \quad |\arg \tau| > \frac{\pi}{4}, 0 \neq \alpha \leq 1/2.$$

Improvements:

- We weaken Condition (III) to Condition $(III)'$: For $\delta > 0$ small enough and any $\epsilon > 0$,

$$2\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi k\alpha) \geq \left(1 + \frac{\rho_r}{2} + \epsilon\right) \mathcal{M} |\log \delta|,$$

$$\sqrt{\delta} \leq |\alpha| \leq 1/2, \quad i = 1, 2, 3,$$

for a constant $\mathcal{M}$.

Improvements:

- We weaken Condition (III) to Condition $(III)'$: For $\delta > 0$ small enough and any $\epsilon > 0$,

$$2\sum_{k=1}^{\infty} b_k e^{-k\delta} \sin^2(\pi k\alpha) \geq \left(1 + \frac{\rho_r}{2} + \epsilon\right) \mathcal{M} |\log \delta|,$$

$$\sqrt{\delta} \leq |\alpha| \leq 1/2, \quad i = 1, 2, 3,$$

  for a constant $\mathcal{M}$.
- We also allow $D(s)$ to have multiple poles on the real axis. This allows us to consider weights $b_k = \binom{k+l}{l}$ for $l \geq 2$. The corresponding function $D(s)$ has poles at integers $1, 2, \ldots, l+1$. (Benvenuti, Feng Hanany, He (2007) *J. High Energy Physics*)

To get an asymptotic equation for δ_n, we use the Mellin transform of the Gamma function

$$e^{-u} = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} u^{-s}\Gamma(s)ds, \quad \Re(s) = v > 0$$

to get

$$\sum_{k=1}^{\infty} b_k e^{-k\delta} = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} \delta^{-s}\Gamma(s)D(s)ds,$$

for v in the half-plane of convergence of $D(s)$. Suppose $D(s)$ has exactly one pole at ρ_r. Shifting v to $\Re(v) < 0$ and taking the derivative of the preceding expression and setting it equal to $-n$ gives

$$\sum_{k=1}^{\infty} b_k e^{-k\delta} = A_r \delta^{-\rho_r}\Gamma(\rho_r) + O(\delta^{C_0}),$$

where A_r is the residue of $D(s)$ at $s = \rho_r$.

In the case of assemblies, (for which $\frac{1}{k}Y_k \sim \mathrm{Poisson}(b_k e^{-\delta k})$). taking derivatives with respect to δ and replacing δ by δ_n gives

$$n = A_r \Gamma(\rho_r + 1)\delta_n^{-\rho_r - 1} + O(\delta_n^{C_0 - 1}).$$

Therefore,

$$\delta_n = \left(\frac{n}{A_r \Gamma(\rho_r + 1)} \right)^{\frac{1}{\rho_r + 1}}.$$

Extending the method to general functions

We restrict to

$$f(z) = \prod_{k=1}^{\infty} S(a_k z^k)^{b_k}$$

for a function $S(z)$ and sequences a_k, b_k.

- We assume a_k and b_k grow at most polynomially in k.

Extending the method to general functions

We restrict to

$$f(z) = \prod_{k=1}^{\infty} S(a_k z^k)^{b_k}$$

for a function $S(z)$ and sequences a_k, b_k.

- We assume a_k and b_k grow at most polynomially in k.
- We assume $\frac{d^2}{d\delta^2} \log S(e^{-\delta}) > 0$.

Extending the method to general functions

We restrict to

$$f(z) = \prod_{k=1}^{\infty} S(a_k z^k)^{b_k}$$

for a function $S(z)$ and sequences a_k, b_k.

- We assume a_k and b_k grow at most polynomially in k.
- We assume $\frac{d^2}{d\delta^2} \log S(e^{-\delta}) > 0$.
- We assume $\log S(z) = \sum_{j=1}^{\infty} \xi_j z^j$ with radius of convergence 1.

Extending the method to general functions

We restrict to

$$f(z) = \prod_{k=1}^{\infty} S(a_k z^k)^{b_k}$$

for a function $S(z)$ and sequences a_k, b_k.

- We assume a_k and b_k grow at most polynomially in k.
- We assume $\frac{d^2}{d\delta^2} \log S(e^{-\delta}) > 0$.
- We assume $\log S(z) = \sum_{j=1}^{\infty} \xi_j z^j$ with radius of convergence 1.

Define

$$\Lambda_k = \sum_{j|k} b_j a_j^{k/j} \xi_{k/j},$$

so that

$$\log f(z) = \sum_{k=1}^{\infty} \Lambda_k z^k.$$

Define

$$D(s) = \sum_{k=1}^{\infty} \Lambda_k k^{-s}.$$

Theorem

Under three Meinardus conditions on $D(s)$ and Λ_k,

$$c_n \sim H n^{-\frac{2+\rho_r-2A_0}{2(\rho_r+1)}} \exp\left(\sum_{l=0}^{r} P_l \, n^{\frac{\rho_l}{\rho_r+1}} + \sum_{l=0}^{r} h_l \sum_{s:\lambda_s \le \rho_l} K_{s,l} \, n^{\frac{\rho_l-\lambda_s}{\rho_r+1}} \right)$$

for constants A_0, H, h_l, P_l, $K_{s,l}$ where the poles of $D(s)$ occur at the ρ_l and the λ_s are distances between the ρ_l.

Example: Gentile statistics

Gentile statistics interpolate between Bose-Einstein and Fermi-Dirac statistics. They allow no more than $\eta - 1$, $\eta \geq 2$, repetitions of a part of any size. For Gentile statistics,

$$S(z) = 1 + z + z^2 + \cdots + z^{\eta-1} = \frac{1 - z^\eta}{1 - z},$$

$a_k = b_k = 1$.

Example: Gentile statistics

Gentile statistics interpolate between Bose-Einstein and Fermi-Dirac statistics. They allow no more than $\eta - 1$, $\eta \geq 2$, repetitions of a part of any size. For Gentile statistics,

$$S(z) = 1 + z + z^2 + \cdots + z^{\eta-1} = \frac{1 - z^\eta}{1 - z},$$

$a_k = b_k = 1$.

$$c_n \sim \sqrt{\frac{\kappa}{4\pi\eta}} n^{-3/4} e^{2\kappa\sqrt{n}},$$

where $\kappa = \sqrt{\zeta(2)(1 - \eta^{-1})}$ as predicted by Tran, Murthy Bhaduri (2004) *Ann. Physics*.

Example: weighted partitions having no repeated part sizes

$\sum_{n=1}^{\infty} c_n z^n = \prod_{k=1}^{\infty}(1 + b_k z^k)$ enumerates the number of partitions with at most 1 part of any given size. Suppose $b_k \sim y^k k^{-q}$. We take $S(z) = 1 + z$, $a_k = k^{-q}$ for $q > 0$, $b_k = 1$.

Example: weighted partitions having no repeated part sizes

$\sum_{n=1}^{\infty} c_n z^n = \prod_{k=1}^{\infty}(1 + b_k z^k)$ enumerates the number of partitions with at most 1 part of any given size. Suppose $b_k \sim y^k k^{-q}$. We take $S(z) = 1 + z$, $a_k = k^{-q}$ for $q > 0$, $b_k = 1$.

- For $0 < q < 1$, we can apply our theorem. $D(s)$ has poles at $1 - q, 1 - 2q, 1 - 3q, \ldots$.

Example: weighted partitions having no repeated part sizes

$\sum_{n=1}^{\infty} c_n z^n = \prod_{k=1}^{\infty}(1 + b_k z^k)$ enumerates the number of partitions with at most 1 part of any given size. Suppose $b_k \sim y^k k^{-q}$. We take $S(z) = 1 + z$, $a_k = k^{-q}$ for $q > 0$, $b_k = 1$.

- For $0 < q < 1$, we can apply our theorem. $D(s)$ has poles at $1 - q, 1 - 2q, 1 - 3q, \ldots$.
- For $q = 1$ (polynomials over finite fields) $c_n \sim e^{-\gamma}$, where γ is Euler's constant (Knuth, Greene).

Example: weighted partitions having no repeated part sizes

$\sum_{n=1}^{\infty} c_n z^n = \prod_{k=1}^{\infty}(1 + b_k z^k)$ enumerates the number of partitions with at most 1 part of any given size. Suppose $b_k \sim y^k k^{-q}$. We take $S(z) = 1 + z$, $a_k = k^{-q}$ for $q > 0$, $b_k = 1$.

- For $0 < q < 1$, we can apply our theorem. $D(s)$ has poles at $1 - q, 1 - 2q, 1 - 3q, \ldots$.
- For $q = 1$ (polynomials over finite fields) $c_n \sim e^{-\gamma}$, where γ is Euler's constant (Knuth, Greene).
- For $q > 1$, our method does not apply, but by elementary methods we get

$$c_n \sim V(q) n^{-q}$$

for a constant $V(q)$.

Based on

- J. London Math. Soc. (2006) B. Granovsky and D. Stark
- Adv. Appl. Math (2008) B. Granovsky, D. Stark and M. Erlihson
- Comm. Math. Physics (2012) B. Granovsky and D. Stark
- Work in progress with B. Granovsky

Thank You!