Poisson-Dirichlet statistics for the extremes of log-correlated Gaussian fields

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joint work with
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Random combinatorial structures & Stat Mech, Venice, May 2013

Outline

1. Log-Correlated Fields
   - Hierarchical cases (Polymers on tree, BBM)
   - Non-Hierarchical cases (2DGFF, MRM)

2. Main Result: Poisson-Dirichlet Statistics of the Gibbs weights
   (Gibbs measure is supported on extremes)

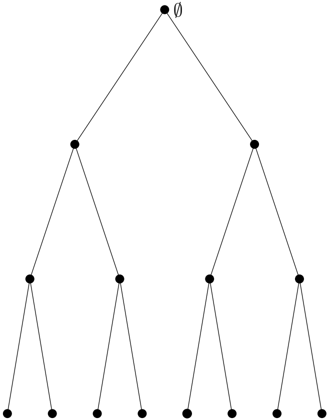
3. Ideas of the Proof
   - Multiscale Decomposition
   - Tree approximation: Bolthausen, Deuschel, Giacomin
   - Spin Glass tools: Ghirlanda-Guerra Identities, Bovier-Kurkova Lemma

4. Beyond the Gibbs measure: the Extremal Process

1. Examples of log-correlated fields

An example: Gaussian Field on a binary tree

- Consider a binary tree with n generations.
  Let $\mathcal{T}_n$ be the leaves $|\mathcal{T}_n| = 2^n$.



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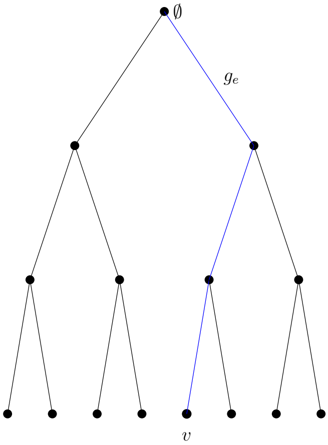

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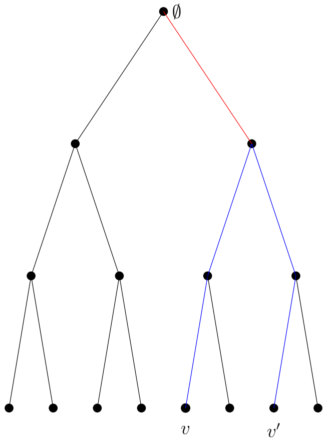

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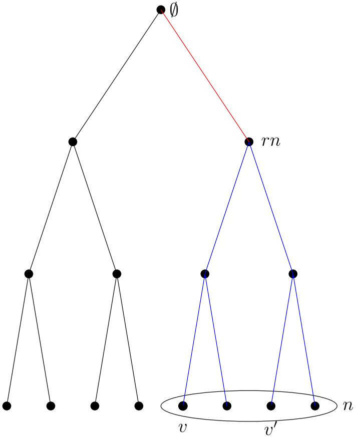
$$\#\{v' : \mathbb{E}[X_v X_{v'}] \geq rn\} = \frac{2^n}{2^{rn}}$$



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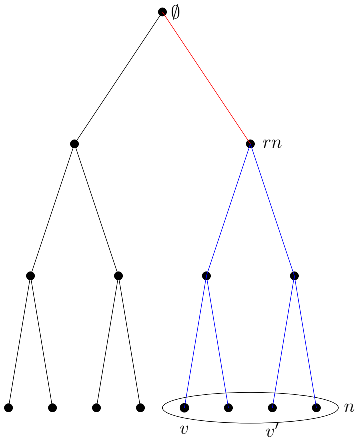

- Correlations are hierarchical:
  two correlations of a triplet v, v', v'' must be equal.

Log-correlated Gaussian fields

- Consider a Gaussian field

$$(X_v, v \in \mathcal{X}_n)$$

indexed by 2^n points in Euclidean space, say $[0,1]$.

- $\mathbb{E}[X_v^2] = n$ and for $c(v,v') := \mathbb{E}[X_v X_v']$

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Non-hierarchical example: 2D discrete GFF

- Consider a box $\mathcal{V}_n \subset \mathbb{Z}^2$ with 2^n points.
- $(X_v, v \in \mathcal{V}_n)$ Gaussian field with

$$\mathbb{E}[X_v X_{v'}] = E^v \left[\sum_{k=0}^{\tau_{\partial V_n}} 1_{\{S_k = v'\}} \right] .$$

$(S_k)_{k \geq 0}$ SRW starting at v.

- The field is log-correlated

$$\mathbb{E}[X_v^2] = \frac{1}{\pi} \log 2^n + O(1)$$

$$\mathbb{E}[X_v X_{v'}] = \frac{1}{\pi} \log \frac{2^n}{\|v - v'\|} + O(1)$$

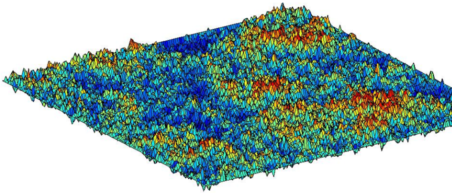

Figure by Samuel April

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Multifractal Random Measure.
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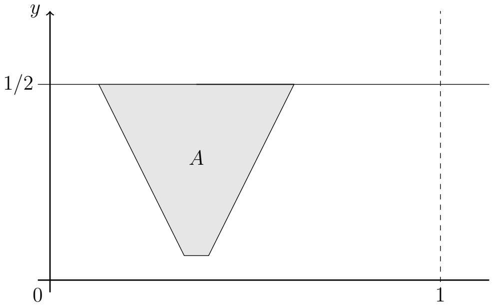

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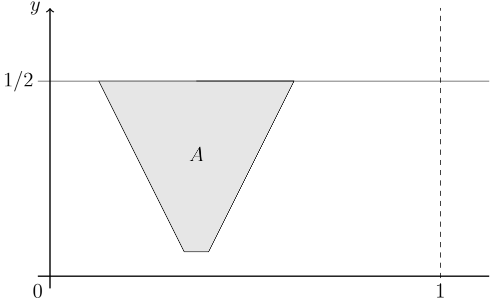

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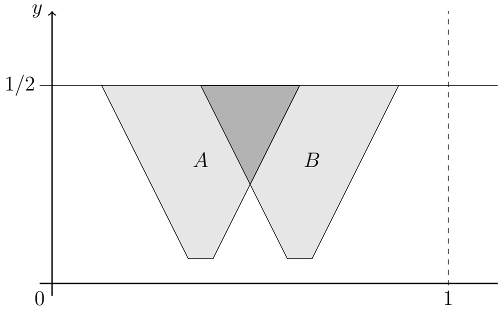
3. $A \cap B = \emptyset \Leftrightarrow$
   $\mu(A) \perp \mu(B)$.



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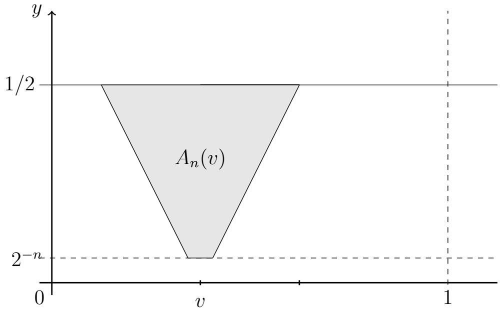

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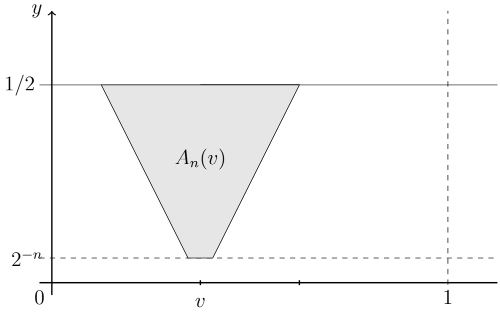

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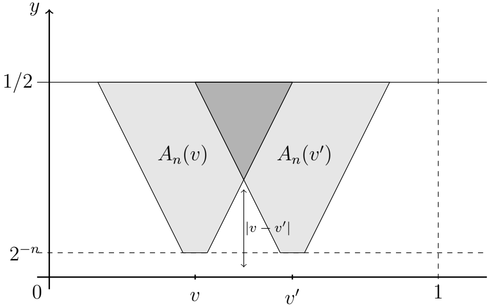

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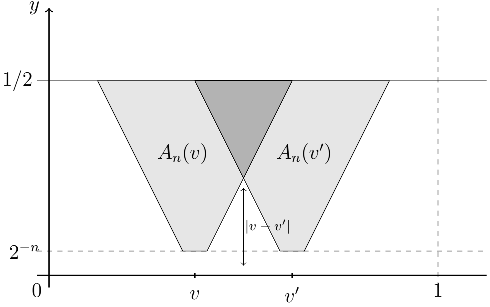

$$\mathbb{E}[X_v^2] = \int_{A_n(v)} y^{-2} dx dy = n \log 2 + O(1)$$

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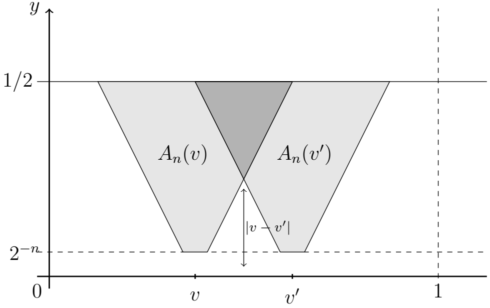

Correlations are not hierarchical :
For a triplet v, v', v'', equality of two correlations is not ensured.

2. Main results: PD statistics

Correlations and Extremal Statistics

Goal: Understand the statistics of the high values (extremes) of the log-correlated field $(X_v, v \in \mathcal{X}_n)$

- Gaussian fields considered have strong correlations, of the order of the variance ($\sim$ spin glasses).
- The problem of determining fine statistics (law of the maximum, point process of extremes) is typically very hard.
- A more robust approach, but coarser, is through the Gibbs weights.

Correlations and Extremal Statistics

Goal: Understand the statistics of the high values (extremes) of the log-correlated field $(X_v, v \in \mathcal{X}_n)$ as $n \to \infty$.

- Free energy: $\beta > 0$, $Z_n(\beta) = \sum_{v \in \mathcal{X}_n} e^{\beta X_v}$

$$f(\beta) := \lim_{n \to \infty} \frac{1}{\log 2^n} \log Z_n(\beta)$$

- Gibbs measure: Measure on $\mathcal{X}_n$ concentrating on extremes as $\beta \to \infty$

$$\langle \ \cdot \ \rangle_{\beta,n} = \frac{\sum_{v \in \mathcal{X}_n} (\cdot) e^{\beta X_v}}{Z_n(\beta)}$$

- Overlap distribution: Normalized covariance $q(v, v') = \frac{\mathbb{E}[X_v X_v']}{\mathbb{E}[X_v^2]}$

$$x_{\beta,n}(r) := \mathbb{E}\left[\langle \ 1_{\{q(v,v') \le r\}} \ \rangle_{\beta,n}^{\times 2}\right]$$

Extremal Statistics for IID variables (REM model)

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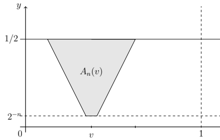
- Gibbs measure: For $\beta > \beta_c$, the Gibbs weights

$$\left(\frac{e^{\beta X_v}}{Z_n(\beta)}, v \in \mathcal{X}_n\right)_{\downarrow} \longrightarrow \mathrm{PD}(\beta_c/\beta)$$

The REM is said to exhibit 1-RSB.

Main results: Free energy

The Gaussian field $(X_v, v \in \mathcal{X}_N)$ (cones) is 1-RSB:



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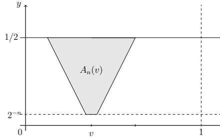

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The free energy is the same as the REM:

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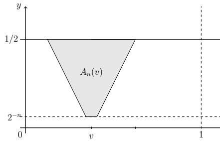

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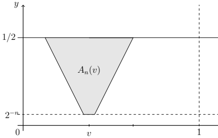

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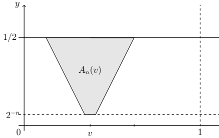

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- The result follows from the works on 2DGFF of Bolthausen, Deuschel & Giacomin '01, and Daviaud '06.

Main results: 1-RSB and PD weights

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2. *Let* F *be a smooth function of the overlaps of* s *points.*

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- Open questions: What about other test-functions ? Conjectured in Duplantier, Rhodes, Sheffield & Vargas '12

3. Some ideas of the proof

Ideas of the Proofs

The method of proof is robust and is applicable to other log-correlated fields

- Bacry & Muzy construction on $[0,1]^d$ (Multifractal Random Measure)
- 2D discrete Gaussian free field

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1. Spin glass: GG Identities and AC Stochastic Stability
2. Multi-scale decomposition
3. Spin glass: Bovier & Kurkova technique '04
4. Tree approximation (Bolthausen-Deuschel-Giacomin '01, Daviaud '06)

1. Gibbs Measures of Gaussian Fields

Theorem (Panchenko '10)

If the free energy is differentiable at $\beta > 0$, then the field concentrates:

$$\frac{1}{\log 2^n}\mathbb{E}\langle \Big| X_v - \mathbb{E}[\langle X_v\rangle_{\beta,n}] \Big| \rangle_{\beta,n} \to 0 \ .$$

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In particular, by integration by parts, for any smooth F,

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&\qquad\qquad + \frac{1}{s}\sum_{k=2}^{s}\mathbb{E}\left[\left\langle q(v_1,v_k)F(\{q(v_i,v_j)\})\right\rangle_{\beta,n}^{\times s}\right] + o(1)
\end{aligned}$$

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Ghirlanda-Guerra Identities

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$$+ \frac{1}{s}\sum_{k=2}^{s} \mathbb{E}\left[\Big\langle q(v_1,v_k) F(\{q(v_i,v_j)\}) \Big\rangle_{\beta,n}^{\times s} \right] + o(1)$$

- In the case where $q(v,v') \to \delta_{vv'}$, the GG identities characterizes PD distributions (Talagrand '03).
- GG identities are at the core of Ultrametricity (Aizenman-A '08, Panchenko '09 '12).

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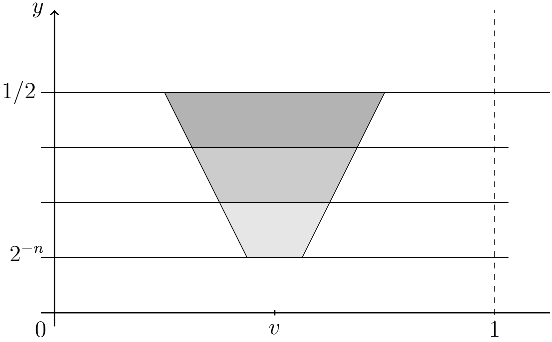
Reduces the problem to computing $x_\beta(r)$.

2. Multi-scale Decomposition

- Independence of disjoint sets → multiscale decomposition in strips



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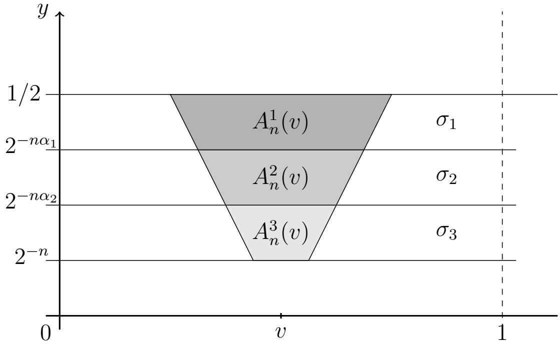
- Independence of disjoint sets → multiscale decomposition in strips
- Pick $\alpha = (\alpha_1, \alpha_2)$, $0 < \alpha_1 < \alpha_2 < 1$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.



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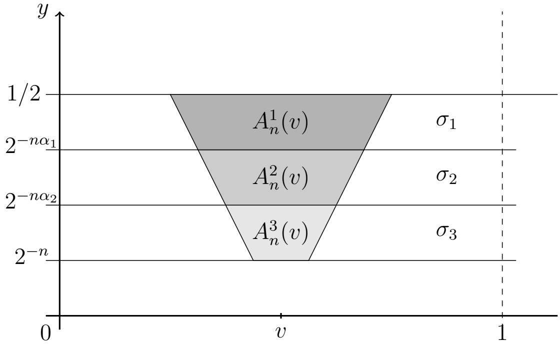
- Write $Y^{(\sigma,\alpha)} = (Y_v^{(\sigma,\alpha)}, n \in \mathcal{X}_n)$ for the Gaussian field

$$Y_v^{(\sigma,\alpha)} = \sigma_1\ \mu(A_n^1(v)) + \sigma_2\ \mu(A_n^2(v)) + \sigma_3\ \mu(A_n^3(v))$$



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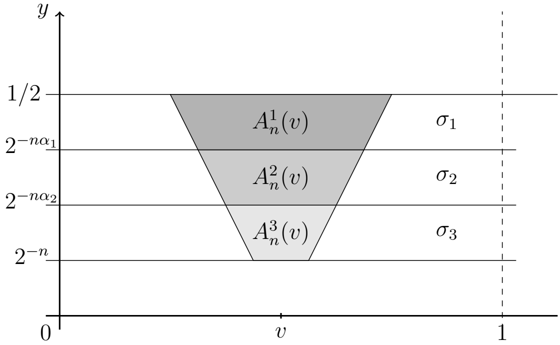
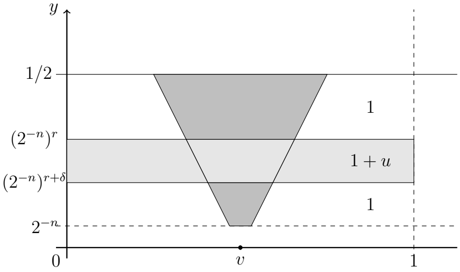
- This is similar to a GREM (Derrida '85).



3. The Bovier-Kurkova technique

- Bovier & Kurkova '04 obtained the overlap distribution of a continuous version of the GREM by considering perturbation of the model.
- For $0 < r < 1$, δ and u small.



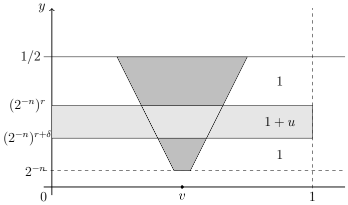
3. The Bovier-Kurkova technique

- For $0 < r < 1$, δ small and u close to 1

$$x_\beta(r) = \lim_{n\to\infty} \mathbb{E}\Big\langle 1_{\{q(v,v')\le r\}} \Big\rangle_{\beta,n}^{\times 2} \qquad Z_n^{(u,r,\delta)}(\beta) = \sum_{v\in\mathcal{X}_n} e^{\beta Y^{(u,r,\delta)(v)}}$$

Lemma (Bovier & Kurkova '04)

$$\beta^2 \int_r^{r+\delta} x_\beta(s)ds = \frac{d}{du}\Big(\lim_{n\to\infty} \frac{1}{n\log 2}\mathbb{E}\log Z_n^{(u,r,\delta)}(\beta)\Big)\Big|_{u=0}$$



4. BDG tree approximation

To compute the free energy, it suffices to compute the log-number of high points

$$\mathcal{E}^{(\sigma,\alpha)}(\gamma) = \lim_{n\to\infty} \frac{\log \#\{v \in \mathcal{X}_n : Y_v^{(\sigma,\alpha)} \geq \gamma\sqrt{2}\log 2n\}}{\log 2^n} \quad \text{in prob.}$$

Theorem

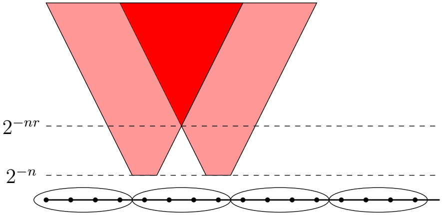
- *Daviaud '06: Case* $\sigma_1 = \sigma_2 = \sigma_3 = 1$

$$\mathcal{E}(\gamma) = 1 - \gamma^2 \quad \textit{(like IID)}$$

- *A-Zindy '12: The number of high points* $\mathcal{E}^{(\sigma,\alpha)}(\gamma)$ *is the same as for the GREM*(σ,α)*.*

4. BDG tree approximation

- Divide the 2^n points into 2^{nr} boxes with $2^{n(1-r)}$ points/box (offspring)
- Contribution at scale 2^{-nr} is not the same for the points in the box.
- Log-Miracle #1:
  Non-common part is smaller than the common part: $1 \ll rn \log 2$.



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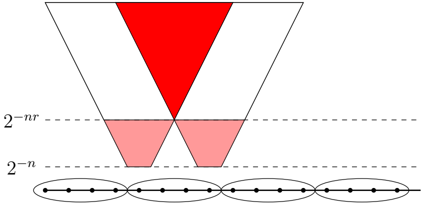

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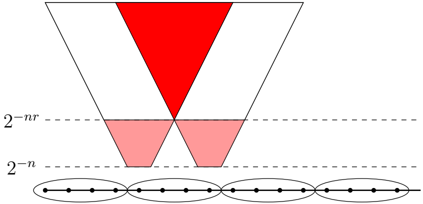
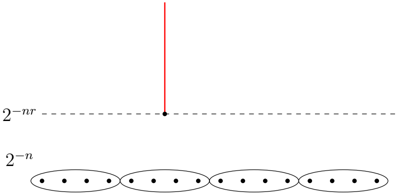
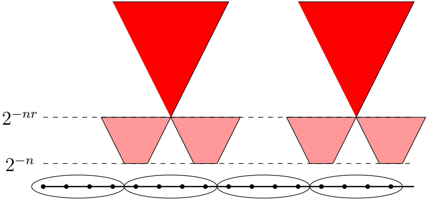




4. BDG tree approximation

- The offspring within a box are not independent.
- Log-Miracle #2:
  The offspring of two boxes are independent at scale below 2^{-nr}.
  Enough independent boxes for the offspring to reach a high value.



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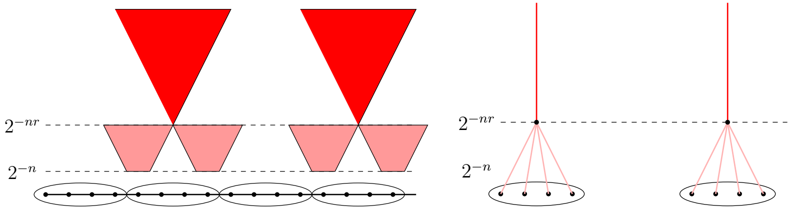

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4. Beyond the Gibbs measure: the Extremal process

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The analysis of the extremal process

$$(X_v, v \in \mathcal{X}_n) \text{ close to } \max_v X_v$$

is much more delicate than the one of the Gibbs measure.

Not in the same universality class as the REM.

Hierarchical case

- Bramson '78: for BBM, $\max_v X_v - m(n)$ converges as $n \to \infty$ for an appropriate $m(n)$.
- The limit law is not Gumbel as in the REM.
- $(X_v - m(n), v \in \mathcal{T}_n)$ converges to a Poisson cluster process (A, Bovier & Kistler '11, Aïdekon. Berestycki, Brunet & Shi '11).

Beyond the Gibbs measure: the Extremal Process

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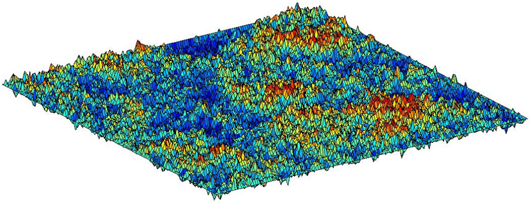
Universality class of log-correlated fields

Non-Hierarchical case (cones, 2DGFF, etc)

- The extremal process should be like the one of BBM: Carpentier & Ledoussal '00, Fyodorov & Bouchaud '08.
- Recent results: BDG '01, Bramson & Zeitouni '10, Ding & Zeitouni '12, Duplantier, Rhodes, Sheffield & Vargas '12
- Convergence of max: Bramson, Ding, Zeitouni '13
- Result on Extremal process: Biskup & Louidor '13



Thank you !