Branching Brownian motion: extremal process and ergodic theorems

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with Louis-Pierre Arguin and Nicola Kistler

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hausdorff center for mathematics
Plan

1. BBM
2. Maximum of BBM
3. The Lalley-Sellke conjecture
4. The extremal process of BBM
5. Ergodic theorems
6. Universality
Branching Brownian Motion

Branching Brownian motion is one of the fundamental models in probability. It combines two classical objects:
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Galton-Watson process

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Pure random genealogy
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BBM is the canonical model of a spatial branching process.
Galton-Watson tree and corresponding BBM
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BBM as Gaussian process
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- Fix GW-tree. Label individuals at time $t$ as $i_1(t), \ldots, i_n(t)$.
BBM as Gaussian process

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BBM special case of models where

\[ \mathbb{E}x_k(t)x_\ell(t) = tA(t^{-1}d(i_k(t), i_\ell(t))) \quad \text{for } A : [0, 1] \to [0, 1]. \]
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$\Rightarrow$ GREM models of spin-glasses.

A. Bovier
First question: how big is the biggest?
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To compare:

Single Brownian motion:

$$P [X(t) \leq x \sqrt{t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) \, dz$$

Two independent Brownian motions:

$$P [\max_{k=1,\ldots,n} \epsilon t X_k(t) \leq \epsilon t x] \rightarrow e^{-\sqrt{\frac{4}{\pi}} x}$$
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  \]

- \( e^t = \mathbb{E} n(t) \) independent Brownian motions:
  \[
  \mathbb{P} \left[ \max_{k=1,\ldots,e^t} X_k(t) \leq t \sqrt{2} - \frac{1}{2\sqrt{2}} \ln t + x \right] \to e^{-\sqrt{4\pi e^{-\sqrt{2x}}}}
  \]
The KPP-F equation
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- **birth**: $v$,
- **death**: $-v^2$,
- **diffusive migration**: $\partial_x^2 v$. 
KPP-F equation and the maximum of of BBM

Maximum of BBM

McKean, 1975:

\( u \) solves the F-KPP equation, i.e.

\[
\partial_t u = \frac{1}{2} \partial^2_x u + u^2 - u,
\]

\( u(0, x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \)

Bramson, 1978:

\[
\begin{align*}
\begin{align*} u(t, x + m(t)) & \to \omega(x), \\
m(t) & = \sqrt{2t - \frac{3}{2}} \sqrt{2 \ln t} 
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\]

where \( \omega(x) \) solves

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\frac{1}{2} \partial^2_x \omega + \sqrt{2} \partial_x \omega + \omega^2 - \omega = 0.
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\[ u(t, x) \equiv \mathbb{P} \left[ \max_{k=1 \ldots n(t)} x_k(t) \leq x \right] \]
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\partial_t u = \frac{1}{2} \partial_x^2 u + u^2 - u, \quad u(0, x) = \begin{cases} 
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\[ u(t, x+m(t)) \to \omega(x), \quad m(t) = \sqrt{2} t - \frac{3}{2\sqrt{2}} \ln t \]
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The derivative martingale

Lalley-Sellke, 1987: \( \omega(x) \) is random shift of Gumbel-distribution

\[ \omega(x) = E \left[ e^{-CZ} e^{-\sqrt{2}x} \right], \]

\[ Z(t) = \lim_{t \to \infty} Z(t), \]

where \( Z(t) \) is the derivative martingale,

\[ Z(t) = \sum_{k \leq n(t)} \left\{ \sqrt{2}t - x_k(t) \right\} e^{-\sqrt{2} \left\{ \sqrt{2}t - x_k(t) \right\}}, \]

Lalley-Sellke conjecture:

\[ P\text{-a.s., for any } x \in \mathbb{R}, \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T 1_{\{ \max_{n \in [1]} x_n(t) - m(t) \leq x \}} = \exp \left( -CZ e^{-\sqrt{2}x} \right) \]
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**Lalley-Sellke conjecture:** $\mathbb{P}$-a.s., for any $x \in \mathbb{R}$,

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_{0}^{T} \mathbb{I}\left\{ \max_{k=1}^{n(t)} x_k(t) - m(t) \leq x \right\} = \exp \left( -CZ e^{-\sqrt{2}x} \right)$$
Looking at BBM from the top

Closer look at the extremes: Zooming into the top

Can we describe the asymptotic structure of the largest points, and their genealogical structure?
Classical Poisson convergence for many BMs

From classical extreme values statistics one knows:

Let $X_i(t)$, $i \in \mathbb{N}$, iid Brownian motions. Then, the point process

$$\mathcal{P}_t \equiv \sum_{i=1}^{e^t} \delta_{X_i(t)-\sqrt{2}t+\frac{1}{2\sqrt{2}} \ln t} \to \text{PPP} \left( \sqrt{4\pi e^{-x}} \, dx \right),$$

where $\text{PPP}(\mu)$ is Poisson point process with intensity measure $\mu$. 
Extensions to correlated processes

GREM [Derrida '82]: Recall

\[ x_i(\tau) = tA(\tau - 1) d(i_\ell(\tau), i_k(\tau)) \]

A increasing step function.

Extreme behaviour relatively insensitive to correlations: If \( A(x) < x \), \( \forall x \in (0, 1) \), then no change in the extremal process.

Poisson cascades: If \( A \) takes only finitely many values, and \( A(x) > x \); for some \( x \in (0, 1) \), the extremal process is known (Derrida, B-Kurkova) and given by Poisson cascade process.

Borderline: If \( A \) takes only finitely many values, and \( A(x) \leq x \), for all \( x \in [0, 1] \), but \( A(x) = x \), for some \( x \in (0, 1) \), the extremal process is again Poisson, but with reduced intensity (B-Kurkova).

What happens at the natural border \( A(x) = x \)??
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A. Bovier () Branching Brownian motion: extremal process and ergodic theorems
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What happens at the natural border $A(x) = x$??
The extremal process of BBM

The life of BBM

General principle: follow history of the leading particles!

There are three phases with distinct properties and effects:
- the early years
- midlife
- before the end

Let us look at them...
The life of BBM

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Let us look at them.......
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Both
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Both \( r \) and \( r' \) occur with positive probability, independent of \( t! \)

In the second case, all particles at time \( r \) have the same chance to have offspring that is close to the maximum.
The early years...

Randomness **persists** for all times from what happened in the early history:

Both and occur with positive probability, independent of $t$!

In the second case, all particles at time $r$ have the same chance to have offspring that is close to the maximum.

**Two consequences:**

- the random variable $Z$, the "derivative martingale"
- particles near the maximum at time $t$ can have common ancestors at finite, $t$-independent times (when $t \uparrow \infty$).
...Midlife...
...Midlife...

Key fact: The function $m(t)$ is convex:

![Graph showing the convex function $m(t)$](image)

The function $m(t)$
The extremal process of BBM

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Key fact: The function \( m(t) \) is convex:

- Descendants of a particle maximal at time \( 0 \ll s \ll t \) cannot be maximal at time \( t \)!

The function \( m(t) \)
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- Particles realising the maximum at time $t$ have ancestors at times $s$ that are selected from the very many particles that are a lot below the maximum at time $s$. 

![The function $m(t)$](image)
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- Particles realising the maximum at time $t$ have ancestors at times $s$ that are selected from the very many particles that are a lot below the maximum at time $s$.
- Offspring of the selected particles is atypical!
- Only one descendant of a selected particle at times $0 \ll s \ll t$ can be at finite distance from the maximum at time $t$. 
...just before the end

Any particle that arrives close to the maximum at time $t$ can have produced offspring shortly before. These will be only a finite amount smaller than their brothers.

Hence, particles near the maximum come in small families.
The extremal process

Let \( E_t \equiv \sum_{i=1}^{n(t)} \delta x_i(t) - m(t) \),

Let \( Z \) be the limit of the derivative martingale, and set

\[ P_Z = \sum_{i \in \mathbb{N}} \delta p_i \equiv \text{PPP}(CZe - \sqrt{2}x dx) \]

Let \( L(t) \equiv \{ \max_{j \leq n(t)}(x_j(t)) > \sqrt{2}t \} \) and

\[ \Delta(t) \equiv \sum_k \delta x_k(t) - \max_{j \leq n(t)}(x_j(t)) \text{ conditioned on } L(t) \].

Law of \( \Delta(t) \) under \( P(\cdot | L(t)) \) converges to law of point process, \( \Delta \). Let \( \Delta(i) \) be iid copies of \( \Delta \), with atoms \( \Delta(j,i) \).
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\[ \mathcal{P}_Z = \sum_{i \in \mathbb{N}} \delta_{p_i} \equiv \text{PPP} \left( CZ e^{-\sqrt{2}x} \, dx \right) \]
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Law of \( \Delta(t) \) under \( \mathbb{P}(\cdot | \mathcal{L}(t)) \) converges to law of point process, \( \Delta \). Let \( \Delta^{(i)} \) be iid copies of \( \Delta \), with atoms \( \Delta^{(i)}_j \).
The extremal process

Theorem (Arguin-B-Kistler, 2011 (PTRF 2013))

With the notation above, the point process \(E_t\) converges in law to a point process \(E\), given by

\[
E \equiv \sum_{i, j \in \mathbb{N}} \delta_{p_i + \Delta(i) j}
\]

Similar result obtained independently by A. Idékon, Brunet, Berestycki, and Shi.
The extremal process

**Theorem (Arguin-B-Kistler, 2011 (PTRF 2013))**

With the notation above, the point process $\mathcal{E}_t$ converges in law to a point process $\mathcal{E}$, given by

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Ergodic theorem for the max

Alternative look: what happens if we consider time averages? Naively one might expect a law of large numbers:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\left\{ \max_{k=1}^n x_k(t) - m(t) \leq x \right\}} = \mathbb{E} \exp \left( -CZ e^{-\sqrt{2}x} \right) \text{ a.s.}$$

But this cannot be true!
Ergodic theorem for the max

Alternative look: what happens if we consider time averages? Naively one might expect a law of large numbers:

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T \mathbb{I} \left\{ \max_{k=1}^n x_k(t) - m(t) \leq x \right\} = \mathbb{E} \exp \left(-CZ e^{-\sqrt{2}x} \right) \quad \text{a.s.}$$

But this cannot be true! Lalley and Sellke conjectured a random version:

**Theorem (Arguin, B, Kistler, 2012 (EJP 18))**

\[ \mathbb{P}-\text{a.s., for any } x \in \mathbb{R}, \]

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T \mathbb{I} \left\{ \max_{k=1}^n x_k(t) - m(t) \leq x \right\} = \exp \left(-CZ e^{-\sqrt{2}x} \right)$$
Ergodic theorem for the extremal process

We prove this conjecture and extend it to the entire extremal process:
Ergodic theorem for the extremal process

We prove this conjecture and extend it to the entire extremal process:

**Theorem (Arguin, B, Kistler (2012))**

\[ E_t \text{ converges } \mathbb{P}\text{-almost surely weakly under time-average to the Poisson cluster process } E_Z. \text{ That is, } \mathbb{P}\text{-a.s., } \forall f \in C_c^+(\mathbb{R}), \]

\[ \frac{1}{T} \int \exp \left( - \int f(y)E_{t,\omega}(dy) \right) \, dt \to E \left[ \exp \left( - \int f(y)E_Z(dy) \right) \right] \]

Here \( E_Z \) is the process \( E \) for given value \( Z \) of the derivative martingale, \( E \) is w.r.t. the law of that process, given \( Z \).
Elements of the proof

For $\varepsilon > 0$ and $R_T \ll T$, decompose

$$\frac{1}{T} \int_0^T \exp \left( - \int f(y) \mathcal{E}_{y,\omega}(dy) \right) dt$$
Elements of the proof

For $\varepsilon > 0$ and $R_T \ll T$, decompose

$$\frac{1}{T} \int_0^T \exp \left( - \int f(y) \mathcal{E}_{t,\omega}(dy) \right) dt = \frac{1}{T} \int_0^\varepsilon T \exp \left( - \int f(y) \mathcal{E}_{t,\omega}(dy) \right) dt$$

$(I)$: vanishes as $\varepsilon \downarrow 0$
Elements of the proof

For $\varepsilon > 0$ and $R_T \ll T$, decompose

$$
\frac{1}{T} \int_0^T \exp \left( - \int f(y) E_{t,\omega}(dy) \right) dt = \frac{1}{T} \int_0^T \exp \left( - \int f(y) E_{t,\omega}(dy) \right) dt
$$

(\text{i): vanishes as } \varepsilon \downarrow 0

$$
+ \frac{1}{T} \int_{\varepsilon T}^T \mathbb{E} \left[ \exp \left( - \int f(y) E_{t,\omega}(dy) \right) \bigg| \mathcal{F}_{R_T} \right] dt
$$

(\text{ii): what we want}
Elements of the proof

For $\varepsilon > 0$ and $R_T \ll T$, decompose

$$\frac{1}{T} \int_0^T \exp \left( - \int f(y) \mathcal{E}_{t,\omega}(dy) \right) dt = \frac{1}{T} \int_0^T \exp \left( - \int f(y) \mathcal{E}_{t,\omega}(dy) \right) dt$$

\[\text{(I): vanishes as } \varepsilon \downarrow 0\]

$$+ \frac{1}{T} \int_{\varepsilon T}^T \mathbb{E} \left[ \exp \left( - \int f(y) \mathcal{E}_{t,\omega}(dy) \right) \bigg| \mathcal{F}_{R_T} \right] dt$$

\[\text{(II): what we want}\]

$$+ \frac{1}{T} \int_{\varepsilon T}^T Y_t(\omega) dt$$

\[\text{(III): needs LLN}\]
Elements of the proof: the LLN
Elements of the proof: the LLN

\[(III) \equiv \exp \left( - \int_{\epsilon T}^{T} f(y) \mathcal{E}_{t,\omega}(dy) \right) - \mathbb{E} \left[ \exp \left( - \int_{\epsilon T}^{T} f(y) \mathcal{E}_{t,\omega}(dy) \right) \middle| \mathcal{F}_{R_{T}} \right] \]

should vanish by a law of large numbers.
Elements of the proof: the LLN

\[(III) \equiv \exp \left( - \int_{\epsilon T}^{T} f(y) \mathcal{E}_{t,\omega}(dy) \right) - \mathbb{E} \left[ \exp \left( - \int_{\epsilon T}^{T} f(y) \mathcal{E}_{t,\omega}(dy) \right) \bigg| \mathcal{F}_{RT} \right] \]

should vanish by a law of large numbers.
We use a criterion which is an adaptation of the theorem due to Lyons:

**Lemma**

Let \( \{Y_s\}_{s \in \mathbb{R}_+} \) be a.s. uniformly bounded and \( \mathbb{E}[Y_s] = 0 \) for all \( s \). If

\[
\sum_{T=1}^{\infty} \frac{1}{T} \mathbb{E} \left[ \left| \frac{1}{T} \int_{0}^{T} Y_s \, ds \right|^2 \right] < \infty,
\]

then

\[
\frac{1}{T} \int_{0}^{T} Y_s \, ds \rightarrow 0, \text{ a.s.}
\]
Requires covariance estimate:

**Lemma**

Let $Y_s$ from (III). For $R_T = o(\sqrt{T})$ with $\lim_{T \to \infty} R_T = +\infty$, there exists $\kappa > 0$, s.t.

$$
E[Y_s Y_{s'}] \leq Ce^{-R_T^\kappa} \quad \text{for any } s, s' \in [\varepsilon T, T] \text{ with } |s - s'| \geq R_T.
$$
Universality

The new extremal process of BBM should not be limited to BBM:

- Branching random walk (Aïdeckon, Madaule)
- Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin, Bramson, Zeitouni, Biskup and Louisdor (!) ....]
- Cover times of random walks [Ding, Zeitouni, Sznitman,....]
- Spin glasses with log-correlated potentials [Fyodorov, Bouchaud,..]

and building block for further models:

- Extensions to stronger correlations: beyond the borderline [Fang-Zeitouni ’12]...
- Extension back to spin glasses: some of the observations made give hope.... see Louis-Pierre’s talk
References


Thank you for your attention!