The East model

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Motivations

Glasses are intriguing materials: mechanical rigidity similar to crystalline media, despite absence of long–range order.

Some key features of their evolution:

- **Extremely slow dynamics**
- **Dynamic heterogeneity: a very broad range of relaxation times with transient spatial fluctuations in local dynamics (simulations,measurements of relaxation spectra)**

Dynamic heterogeneity

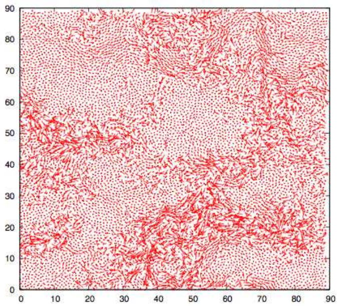


Figure: **Particle displacements in a 2D supercooled liquid. Physics 4, 42 (2011) L. Berthier**

East model $d = 1$

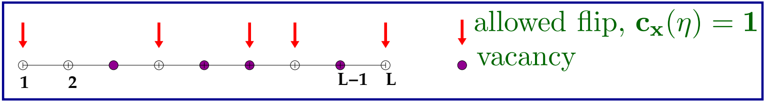
- $q \in (0,1)$ parameter
- box $[1, L] \subset \mathbb{N}$
- configurations $\eta \in \{0,1\}^L$, $\eta_x = \begin{cases} 1 & \text{particle at } x \\ 0 & \text{vacancy at } x \end{cases}$
- Spin–flip dynamics:
  - → for each site x, wait at x an exponential time of mean 1
  - → afterwards, if $c_x(\eta) = 1$, refresh η_x as $\begin{cases} 1 & \text{prob. } 1-q \\ 0 & \text{prob. } q \end{cases}$

Dynamical constraint

Dynamical constraint at $\mathbf{x}$:

$$\mathbf{c_x}(\eta) = \begin{cases} \mathbb{1}(\eta_{\mathbf{x-1}} = \mathbf{0}) & \text{if } \mathbf{2 \le x \le L} \\ \mathbf{1} & \text{if } \mathbf{x = 1} \end{cases}$$

Site 1 behaves as with a frozen vacancy at 0



Waves of vacancies

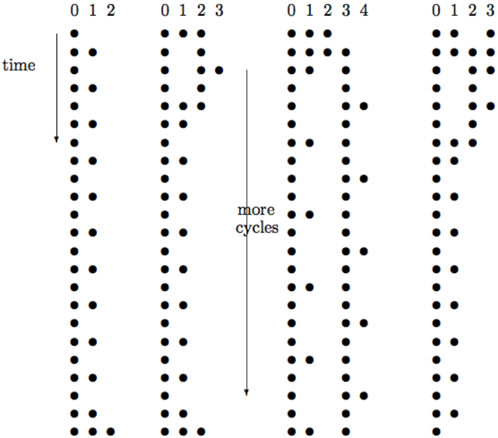


Figure: Source: Aldous, Diaconis JSP **107**

Energetic considerations

- $\mathbf{q} = \frac{\mathbf{e}^{-\beta}}{\mathbf{1}+\mathbf{e}^{-\beta}}$, β **inverse temperature**
- **interesting regime: low temperature,** $\mathbf{q} \downarrow \mathbf{0}$
- **Hamiltonian** $\mathbf{H}(\eta) = -\sum_{\mathbf{x}=\mathbf{1}}^{\mathbf{L}} \eta_{\mathbf{x}}$
  trivial energy landscape
- **Ground state: filled configuration** $\mathbb{1}$
- $\pi := (\mathbf{1}-\mathbf{q})$**–Bernoulli probability measure on** $\{\mathbf{0},\mathbf{1}\}^{\mathbf{L}}$
- π **reversible (detailed balance OK)**
- $\mathbf{1}/\mathbf{q} =$ **equilibrium mean distance consec. vacancies**

Random walk on subgraph of hypercube

State space: $\{0,1\}^L$
Edges: allowed jumps

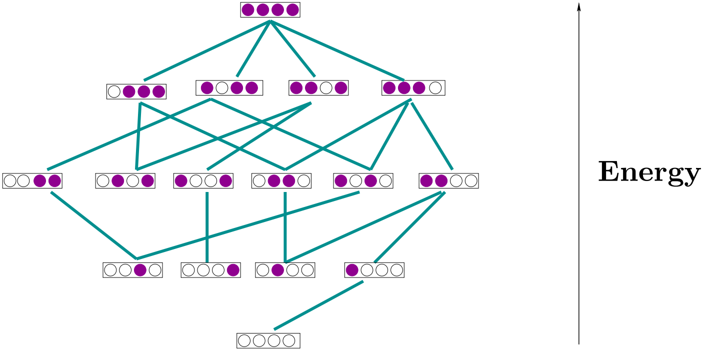


Figure: Underlying graph for East dynamics, $L = 4$

Slow dynamics

$T_{rel}(L)$: relaxation time of East model on $[1, L]$
$T_{rel}(L)$ increasing in L

Theorem

Let $n := \lceil \log_2 L \rceil$, i.e. $2^{n-1} < L \le 2^n$

- $\forall L \; c(n)\frac{1}{q^n} \le T_{rel}(L) \le c'(n)\frac{1}{q^n}$
- $L \le d/q \Rightarrow \frac{n!}{q^n 2^{\binom{n}{2}}} q^{\alpha} \le T_{rel}(L) \le \frac{n!}{q^n 2^{\binom{n}{2}}} q^{-\alpha'}$
  where α, α' depend on d
- $L = O(1/q) \Rightarrow T_{rel}(L) = \left(\frac{1}{q}\right)^{\frac{n}{2}(1+o(1))}$
- *(CMRT)* $T_{rel}(\infty) = \left(\frac{1}{q}\right)^{\frac{n}{2}(1+o(1))}$

Hierarchical structure

Combinatorics for the East Model

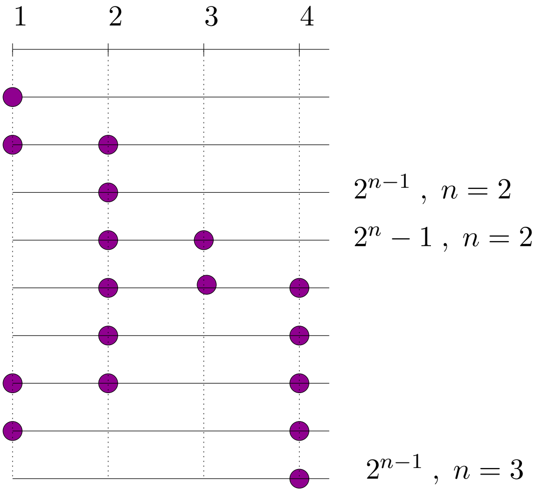
F.Chung, P.Diaconis, R.Graham

$\mathbf{V(n)}$: configurations reachable from $\mathbb{1}$ by at most n vacancy, on $\{1, 2, 3, \dots\}$

Theorem

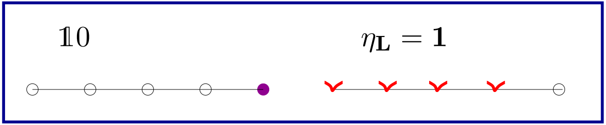
- $\mathbf{2^{\binom{n}{2}} n! c_1^n \leq |V(n)| \leq 2^{\binom{n}{2}} n! c_2^n}, \qquad \mathbf{c_1, c_2 \in (0, 1)}$
- $\max\{\mathbf{x \geq 1} : \exists \eta \in \mathbf{V(n)} \textit{ with } \eta_\mathbf{x} = \mathbf{0}, \ \eta_\mathbf{y} = \mathbf{1} \forall \mathbf{y} \neq \mathbf{x}\} = \mathbf{2^{n-1}}$
- $\max\{\mathbf{x \geq 1} : \exists \eta \in \mathbf{V(n)} \textit{ with } \eta_\mathbf{x} = \mathbf{0}\} = \mathbf{2^n - 1}$

- $\max\{\mathbf{x} : \exists\eta \in \mathbf{V}(\mathbf{n}) \text{ with } \eta_{\mathbf{x}} = \mathbf{0}\,,\ \eta_{\mathbf{y}} = \mathbf{1}\forall\mathbf{y} \neq \mathbf{x}\} = \mathbf{2^{n-1}}$
- $\max\{\mathbf{x} : \exists\eta \in \mathbf{V}(\mathbf{n}) \text{ with } \eta_{\mathbf{x}} = \mathbf{0}\} = \mathbf{2^{n} - 1}$



Characteristic times

- **Relaxation time**: $\mathbf{T_{rel}(L) = 1/gap(\mathcal{L}^{(L)})}$
- **Mixing time**:
  $\mathbf{T_{mix}(L)} = \inf_{\mathbf{t \geq 0}} \left\{ \|\mathbf{P_t}(\eta, \cdot) - \pi\|_{\mathbf{TV}} \leq \mathbf{1/4}\,, \forall \eta \right\}$
- **Hitting time**: $\mathbb{E}_{\mathbb{1}0}\left[\tau_{\eta_{\mathbf{L}}=\mathbf{1}}\right]$



All characteristic times are increasing in L

Equivalence of characteristic times

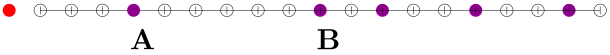
Theorem

- **If** $\mathbf{L} = \mathbf{L}(\mathbf{q}) \leq \mathbf{d}/\mathbf{q}$, **then** $\mathbf{T}_{\mathbf{rel}}(\mathbf{L})$, $\mathbf{T}_{\mathbf{mix}}(\mathbf{L})$ **and** $\mathbb{E}_{\mathbb{1}0}\left[\tau_{\eta_{\mathbf{L}}=\mathbf{1}}\right]$ **are equivalent**:

$$\mathbf{T}_{\mathbf{mix}}(\mathbf{L})/\mathbf{T}_{\mathbf{rel}}(\mathbf{L}) \quad , \quad \mathbb{E}_{\mathbb{1}0}\left[\tau_{\eta_{\mathbf{L}}=\mathbf{1}}\right]/\mathbf{T}_{\mathbf{rel}}(\mathbf{L}) \ \in \ [\mathbf{c}(\mathbf{d}), \mathbf{c}'(\mathbf{d})] \, .$$

Hitting time $\mathbb{E}_{\mathbb{1}0}\left[\tau_{\eta_{\mathbf{L}}=1}\right]$

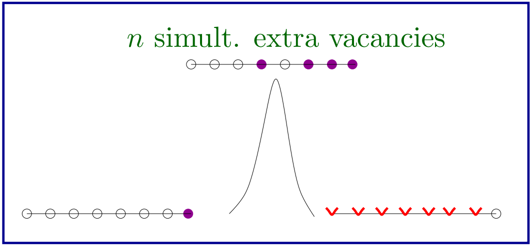
Take a generic initial configuration



- **Until an initial vacancy is not removed, it behaves as the frozen vacancy at 0**
- **It creates waves of vacancies**
- **Interested in the time a wave created by A kills the vacancy at B**

Energy and entropy

[CDG] if $\mathbf{2^{n-1} < L \leq 2^n}$, i.e. $\mathbf{n} = \lceil \log_2 \mathbf{L} \rceil$, then



- $\mathbf{L}$ =const $\Rightarrow \mathbf{T_{rel}(L) \sim (1/q)^n}$
- Case $\mathbf{L = d/q^{\gamma}}$, $\gamma \in (\mathbf{0}, \mathbf{1}]$
  1. $\mathbf{T_{rel}(L) \ll (1/q)^n}$
  2. Up to $\tau_{\eta_{\mathbf{L}}=\mathbf{1}}$, max. $n+k$ simultaneous extra vacancies
  3. Entropic corrections

Effective dynamics: some basic observations

Regime $\mathbf{qL(q) \downarrow 0}$

(i) $\mathbf{P}\big(\text{vacancies } \eta(\mathbf{s}) \supset \text{vacancies } \eta(\mathbf{t})\big) = \mathbf{1} - \mathbf{o}(\mathbf{1}), \ \mathbf{s} < \mathbf{t}$

(ii) $\pi(\mathbb{1}) = \mathbf{1} - \mathbf{o}(\mathbf{1})$, $\mathbb{1}$ **filled configuration**

$\Rightarrow$ **Annihilation effective dynamics**

Regime $\mathbf{qL(q) \to d}$

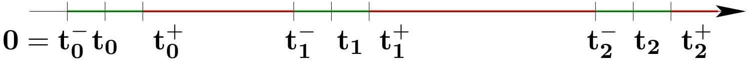
$\Rightarrow$ **No annihilation effective dynamics**

Effective dynamics for finite volume observations

Complete analysis:
[ES, F.-Martinelli-Roberto-Toninelli CMP 309 (2012)]

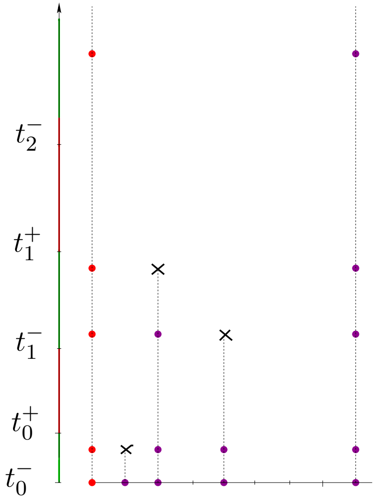
Recall: $\mathbf{T}_{\mathrm{rel}}(\mathbf{L}) \sim \mathbf{q}^{-\mathbf{n}}$ **if** $\mathbf{n} = \lceil \log_{\mathbf{2}} \mathbf{L} \rceil$

Set $\mathbf{t_n} = \mathbf{q}^{-\mathbf{n}}, \ \mathbf{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$



1. **In stalling periods no evolution**
2. **In active period** $[\mathbf{t_n^-}, \mathbf{t_n^+}]$ **coalescence dynamics, domains of length L s.t.** $\mathbf{n} = \lceil \log_{\mathbf{2}} \mathbf{L} \rceil$ **coalesce with their right domains**

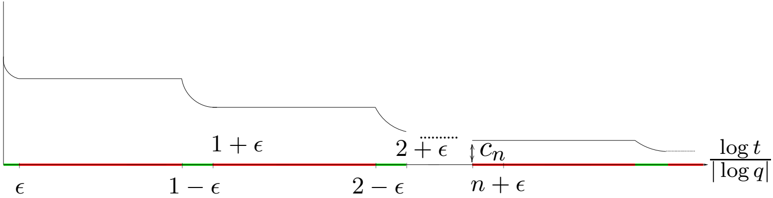
Effective coalescence process



Plateau behavior of persistence and correlation functions

- **Hierarchical coalesce process**
  HCP starting with IID domain lengths: scaling limit with universality classes
- **East model on $\mathbb{Z}$ starting with IID domain lengths with finite mean and a vacancy at 0, then**

$$\mathbb{P}(\eta_t(0) = 0) \sim \mathbb{P}(\eta_s(0) = 0 \forall s \in [0,t]) \sim$$



$$\mathbf{c_n := (1/(2^n + 1))^{(1+o(1))}}$$

More precisely:

- $\lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{P}(\sigma_t(0)=0) - \left(\frac{1}{2^n+1}\right)^{(1+o(1))} \right| = 0$
- asymptotic monotonicity in active period

Similar results (plateau behavior) of time–time correlation functions

Comparison with simulations

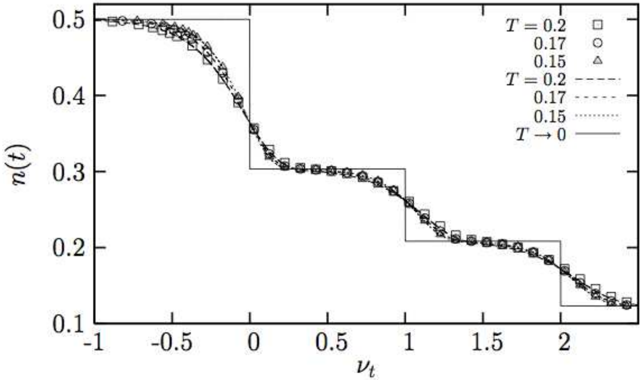
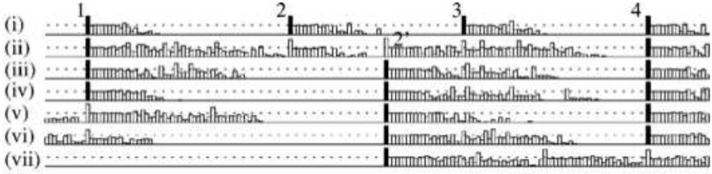


Figure: (Léonard et al) "Vacancy density n(t) for three different temperatures. Symbols represent the simulation results. The theoretical prediction is shown at finite T with dotted lines, while its $T \to 0$ limit is shown as a full line".

Equilibrium effective dynamics

- $\mathbf{L \sim \Delta/q}$, $\mathbf{\Delta \gg 1}$
- $\mathbf{T_{\mathrm{rel}}(1/q) \leq t \leq T_{\mathrm{rel}}(\Delta/q)}$

Evans, Sollich: superdomains dynamics



Based on the continuous timescale separation hypothesis

Timescale separation

Given $\mathbf{L}'(\mathbf{q}) \geq \mathbf{L}(\mathbf{q})$ we write

$$\mathbf{T}_{\mathrm{rel}}\big(\mathbf{L}'(\mathbf{q})\big) \succ \mathbf{T}_{\mathrm{rel}}\big(\mathbf{L}(\mathbf{q})\big)$$

and say there is timescale separation between $\mathbf{L}'(\mathbf{q}), \mathbf{L}(\mathbf{q})$ if $\exists \beta > \mathbf{0}$ such that

$$\mathbf{T}_{\mathrm{rel}}(\mathbf{L}'(\mathbf{q})) \geq \mathbf{q}^{-\beta}\mathbf{T}_{\mathrm{rel}}(\mathbf{L}(\mathbf{q}))$$

- **Example: finite volume**

Take $\mathbf{L}', \mathbf{L}$ constant. Then

$$\mathbf{T}_{\mathrm{rel}}(\mathbf{L}') \succ \mathbf{T}_{\mathrm{rel}}(\mathbf{L})$$

iff $\lceil \log_{\mathbf{2}} \mathbf{L}' \rceil > \lceil \log_{\mathbf{2}} \mathbf{L} \rceil$

Continuous timescale separation hypothesis [ES]

Evans, Sollich
[Phys. Rev. Lett. 83 (1999). Phys. Rev. E **68** (2003)]

Conjecture (ES)

$\mathbf{T_{rel}}\left(\frac{\mathbf{d'}}{\mathbf{q}}\right) \succ \mathbf{T_{rel}}\left(\frac{\mathbf{d}}{\mathbf{q}}\right)$ **for all** $\mathbf{d' > d}$

More precisely

$$\mathbf{T_{rel}}\left(\frac{\mathbf{d}}{\mathbf{q}}\right) = \left(\frac{\mathbf{1}}{\mathbf{q}}\right)^{\mathbf{f(d)+o(1)}} \mathbf{T_{rel}}\left(\frac{\mathbf{1}}{\mathbf{q}}\right)$$

with $\mathbf{f}$ **strictly increasing**

The ES hypothesis is:

1. **supported by simulations**
2. **at the base of the "superdomain dynamics", effective dynamics near to equilibration**

Failure of the ES hypothesis

Theorem

- **The ES hypothesis fails:**

$$\mathbf{T_{rel}}\left(\frac{\mathbf{d}}{\mathbf{q}}\right) = \left(\frac{\mathbf{1}}{\mathbf{q}}\right)^{\mathbf{f(d)+o(1)}} \mathbf{T_{rel}}\left(\frac{\mathbf{1}}{\mathbf{q}}\right); \quad \mathbf{f} \equiv \mathbf{0}$$

- **No timescale separation between $\mathbf{d/q}$ and $\mathbf{d'/q}$**

What do simulations refer to?

- **Timescale separation at level $\mathbf{O(1/q^{\gamma})}$, $\gamma \in (\mathbf{0}, \mathbf{1})$**
- **Not clear nature:**

$$\begin{cases} \text{continuous} \\ \text{discrete} \end{cases} \qquad \mathbf{T_{rel}}\left(\frac{\mathbf{d'}}{\mathbf{q}^{\gamma}}\right) \succ \mathbf{T_{rel}}\left(\frac{\mathbf{d}}{\mathbf{q}^{\gamma}}\right); \qquad \mathbf{d'} > \mathbf{d}\,,$$

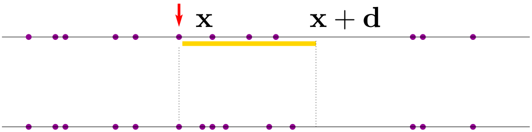
Aldous–Diaconis equilibrium effective dynamics

- **East process on** $\mathbb{N} = \{\mathbf{1}, \mathbf{2}, \dots\}$
- **Map** $\eta_{\mathbf{t}} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbb{N}}$ **to** $\xi_t = \{\mathbf{x} \in \mathbb{N} : \eta_{\mathbf{x}} = \mathbf{0}\}$

Conjecture:

- $\left\{\mathbf{q}\xi_{\mathbf{t}\mathbf{T}_{\mathbf{rel}}(\mathbf{1}/\mathbf{q})}\right\}_{\mathbf{t}\geq \mathbf{0}} \to \{\boldsymbol{\Theta}_{\mathbf{t}}\}_{\mathbf{t}\geq\mathbf{0}}$
- $\boldsymbol{\Theta}_{\mathbf{t}}$ **Poisson point process rate 1**

- **Each point creates "wave" of length $> d$ with rate $G(d)$**
- **The wave $(x, x+d]$ deletes points in $(x, x+d]$ and replace them by a Poisson point process rate 1**



References

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  2. *Time scale separation in the low temperature East model: Rigorous results.* **J. Stat. Mech. (2013) L04001**