LARGE DEVIATIONS FOR THE EMPIRICAL FLOW OF CONTINUOUS TIME MARKOV CHAINS

Davide Gabrielli

University of L'Aquila

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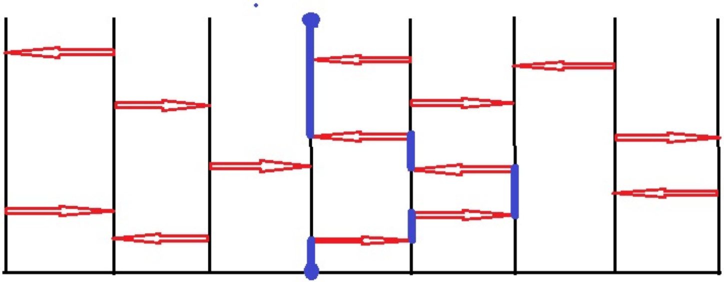
Continuous time Markov chains

- $\{\xi_t^x\}_{t\in\mathbb{R}^+}$ Continuous time Markov chain on V, countable state space, such that $\xi_0^x = x$
- $r(y,z)$ rate of jump from $y \in V$ to $z \in V$
- (V,E) oriented graph (not necessarily locally finite) where

$$E = \{(y,z) \,:\, r(y,z) > 0\}$$

- $r(y) = \sum_z r(y,z)$ holding time at y
- Graphical construction

Harris graphical construction



Continuous time Markov chains

BASIC ASSUMPTIONS

- ξ_t^x does not explode a.e.
- irreducibility
- There exist an unique invariant probability measure π

$$\sum_z \pi(y) r(y,z) = \sum_z \pi(z) r(z,y)$$

- Generator: if $\sum_z r(y,z)|f(z)| < +\infty$ for any y then

$$Lf(y) = \sum_z r(y,z)\left[f(z) - f(y)\right]$$

Empirical measure and flow

$$(\xi_t^x)_{t\in[0,T]} \in D\left([0,T],V\right) \text{ a sample path}$$

Empirical measure $\rho_T \in \mathcal{M}^{+,1}(V)$

$$\rho_T(y) = \frac{1}{T}\int_0^T \delta_{\xi_s^x}(y)\,ds$$

Empirical flow $Q_T : E \to \mathbb{R}^+$

$$Q_T(y,z) = \frac{\sharp \text{ jumps from } y \curvearrowright z \text{ in } [0,T]}{T}$$

Empirical process

$$(\xi_t^x)_{t\in[0,T]} \in D\left([0,T],V\right) \text{ a sample path}$$

Empirical process $\mathcal{R}_T \in \mathcal{M}^{+,1}_{\text{st}}\Big(D\left((-\infty,+\infty),\,V\right)\Big)$

$$(\xi_t)_{t\in[0,T]} \Rightarrow \left(\widetilde{\xi}_t\right)_{t\in(-\infty,+\infty)} = \text{double infinite periodic extemsion}$$

$$t\in[0,T) \Rightarrow \widetilde{\xi}_t = \xi_t \qquad \widetilde{\xi}_{t+T} = \widetilde{\xi}_t$$

$$\mathcal{R}_T = \frac{1}{T}\int_0^T \delta_{\tau_s \widetilde{\xi}}\, ds\,, \qquad \tau = \text{shift}$$

Properties

1) $$\rho_T(y) = \mathbb{E}_{\mathcal{R}_T}\left(\delta_y\left(\widetilde{\xi}_t\right)\right), \qquad \forall t$$

2) $$\operatorname{div} Q_T(y) = \sum_z \left(Q_T(y,z) - Q_T(z,y)\right) = 0$$

3) $$Q_T(y,z) = \mathbb{E}_{\mathcal{R}_T}\left(\sharp \text{ jumps } y \curvearrowright z \text{ in } [0,1]\right)$$

Donsker-Varadhan $c(\sigma)$ conditions

There exists a sequence $u_n : V \to \mathbb{R}^+$ such that

- For any y and n it holds $\sum_z r(y,z)u_n(z) < +\infty$
- The sequence u_n is uniformly bounded from below
- The sequence u_n is uniformly in n bounded from above on compacts of V
- The sequence $v_n = -\frac{Lu_n}{u_n}$ converges point-wise to $v : V \to \mathbb{R}$
- The function v has compact level sets
- There exists positive constants σ and C such that

$$v \geq \sigma r - C$$

LDP empirical measure

Under condition $c(0)$, ρ_T satisfies a LDP on $\mathcal{M}^{+,1}(V)$ with weak convergence

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathbb{P}_x\left(\rho_T \in \mathcal{C}\right) \leq -\inf_{\mu\in\mathcal{C}} I_1(\mu)\,, \qquad \forall\, \mathcal{C}\,, \text{closed}$$

$$\liminf_{T\to+\infty} \frac{1}{T} \log \mathbb{P}_x\left(\rho_T \in \mathcal{O}\right) \geq -\inf_{\mu\in\mathcal{O}} I_1(\mu)\,, \qquad \forall\, \mathcal{O}\,, \text{open}$$

Rate function I_1 has variational representation

$$I_1(\mu) = \sup_{f>0} -\mathbb{E}_\mu\left(\frac{Lf}{f}\right)$$

The reversible case

If the detailed balance condition holds

$$\pi(y)r(y,z) = \pi(z)r(z,y)$$

then the variational problem can be solved and we get an explicit expression

$$I_1(\mu) = \frac{1}{2}\sum_{(y,z)\in E}\left(\sqrt{\mu(y)r(y,z)} - \sqrt{\mu(z)r(z,y)}\right)^2$$

LDP empirical process

Under condition $c(0)$ the empirical process $\mathcal{R}_T$ satisfies a LDP on $\mathcal{M}^{+,1}\left(D((-\infty,+\infty),V)\right)$ with rate functional

$$\mathcal{I}(\mathcal{R}) = \text{density of relative entropy}$$

The rate functional $\mathcal{I}$ is affine

$$\mathcal{I}\Big(c\mathcal{R}^1 + (1-c)\mathcal{R}^2\Big) = c\mathcal{I}\Big(\mathcal{R}^1\Big) + (1-c)\mathcal{I}\Big(\mathcal{R}^2\Big)$$

LDP for empirical flow

Under condition $c(\sigma)$ with $\sigma > 0$ we have that (μ_T, Q_T) satisfies a joint LDP on $\mathcal{M}^{1,+}(V) \times L^1_+(E)$

- On $\mathcal{M}^{1,+}(V)$ the weak topology
- On $L^1_+(E)$ the bounded weak* topology

weak* topology is the smallest topology such that

$$Q \to \sum_{(y,z)\in E} Q(y,z)\phi(y,z)$$

is continuous for any $\phi \in C^0(E)$

Additional conditions to have strong topology

The rate functional

The joint rate function of (μ_T, Q_T) is

$$I(\mu, Q) = \begin{cases} \sum_{(y,z)\in E} \Phi\Big(Q(y,z), \mu(y)r(y,z)\Big) & \text{if } \operatorname{div} Q = 0 \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\Phi(Q, \lambda) = Q \log \frac{Q}{\lambda} + \lambda - Q$$

is the rate function associated to a Poisson process of rate λ

♡ Is it possible to get this LDP from the Graphical construction?

The rate functional

Under condition $c(\sigma)$

$$\sum_{(y,z)\in E} Q^\pi(y,z) = \sum_{(y,z)\in E} \pi(y)r(y,z) = \mathbb{E}_\pi(r) < +\infty$$

and (π, Q^π) is the unique zero.

The rate functional I is convex

The rate functional I is affine. Let K_i be the connected components of the graph $(V(\mu), E(Q))$ then

$$\begin{cases} I(\mu,Q) = \sum_j \mu(K_j) I(\mu_j, Q_j) \\ (\mu,Q) = \sum_j \mu(K_j)\,(\mu_j, Q_j) \end{cases}$$

Contraction

Fenchel–Rockafellar Theorem

Let ϕ, ψ two convex extended functions on a topological vector space X, under some additional conditions it holds

$$\inf_{x \in X} \{\phi(x) + \psi(x)\} = \sup_{f \in X^*} \{-\phi^*(-f) - \psi^*(f)\}$$

$$\sup_{f>0} -\mathbb{E}_\mu \left(\frac{Lf}{f} \right) = \inf_Q I(\mu, Q)$$

The divergence free constraint transforms in a gradient constraint

Current fluctuations

Current across a bond

$$J(y,z) = Q(y,z) - Q(z,y)$$

J is a discrete vector field $J(y,z) = -J(z,y)$ and $\operatorname{div} J = \operatorname{div} Q$
Joint LDP for the empirical measure and current by contraction

$$\tilde{I}(\mu, J) = \inf_{Q} I(\mu, Q) = I(\mu, Q^{\mu,J})$$

where

$$Q^{\mu,J}(y,z) = \frac{J(y,z) + \sqrt{J^2(y,z) + 4\mu(y)\mu(z)r(y,z)r(z,y)}}{2}$$

Ideas around the proof

Tilting

1) perturbed rates; gradient perturbations (Donsker–Varadhan) are not enough; $F : E \to \mathbb{R}$

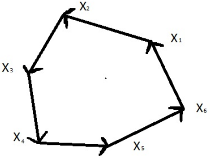
$$r^F(y,z) = r(y,z)e^{F(y,z)}$$

2) Cyclic decomposition of divergence free flows

Cycle $C = (x_1, \dots, x_n)$ if $(x_i, x_{i+1}) \in E$

$\mathbb{I}_C$ divergence free flow associated to C

$\mathbb{I}_C(y,z) = 1$ if $(y,z) \in C$ and zero otherwise



Ideas around the proof

if $\operatorname{div} Q = 0$ and exists V_n invading sequence such that (zero flow towards infinity)

$$\lim_{n \to +\infty} \sum_{y \in V_n \,, z \notin V_n} Q(y,z) = 0$$

then

$$Q = \sum_{C \in \mathcal{C}} \widehat{Q}(C) \mathbb{I}_C$$

Contraction from level 3

1) A generalized contraction principle
2) Cyclic decomposition

An example

Birth and death chain; $V = \mathbb{N} \cup \{0\}$, $E = (y, z)$ such that $|y - z| = 1$

$r(y, y+1) = b_y\,, y \geq 0$ births rates and $r(y, y-1) = d_y\,, y \geq 1$ deaths rates

If

$$\begin{cases} \lim_{y \to +\infty} d_y = +\infty \\ \limsup_{y \to +\infty} \frac{b_y}{d_y} < 1 \end{cases}$$

then condition $c(\sigma)$ is satisfied and our LDP holds.

If moreover

$$\limsup_{y \to +\infty} \frac{b_y}{d_y} = 0$$

then the LDP for empirical flow holds also in the strong L^1 topology

Applications

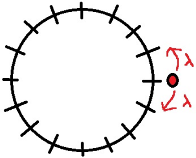
Current fluctuations of a simple random walk on a ring with N sites

Minimize $I_N(\mu, Q)$ with the constraint
$Q(y, y+1) - Q(y+1, y) = j$ you get

$$W_N(j) = Nj \log\left(\frac{Nj}{\lambda} + \sqrt{\left(\frac{Nj}{\lambda}\right)^2 + 1}\right) - \lambda\sqrt{\left(\frac{Nj}{\lambda}\right)^2 + 1} + \lambda$$

In the diffusive rescaling $\lambda = \lambda_N = \alpha N^2$ we have

$$\lim_{N\to+\infty} W_N(j) = \frac{j^2}{2\alpha}$$



Applications

Lattice diffusions

Random walk on $(\epsilon\mathbb{Z})^d$ in a confining potential V

LDP empirical measure and current $I_\epsilon(\mu, J)$

In the diffusive rescaling

$$I_\epsilon(\mu, J) \to \int \frac{[J(x) - J(\rho(x))]^2}{\rho(x)}\, dx$$

Applications

Current fluctuations for interacting particle systems

Projection from configuration space to physical space

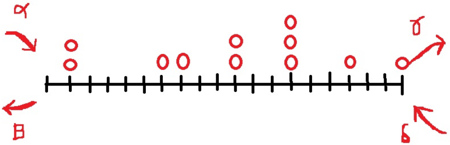
$$\widetilde{Q}(y, y+1) = \sum_{\eta} Q(\eta, \eta^{y,y+1})$$

Current

$$\widetilde{J}(y, y+1) = \widetilde{Q}(y, y+1) - \widetilde{Q}(y+1, y)$$

divergence free discrete vector field (no condensation)

Exclusion models, Zero Range (almost exactly solved), others



Applications

Gallavotti-Cohen functional

$$W_T = \frac{1}{T} \log \frac{d\mathbb{P}_\pi|_{[0,T]}}{d\mathbb{P}^*_\pi|_{[0,T]}}$$

is a function of the empirical current

Total activity

$$N_T = \sharp \text{ jumps in } [0, T]$$

Bodineau-Toninelli: Phase transition in the East Model