Geometric RSK correspondence, Whittaker functions and random polymers

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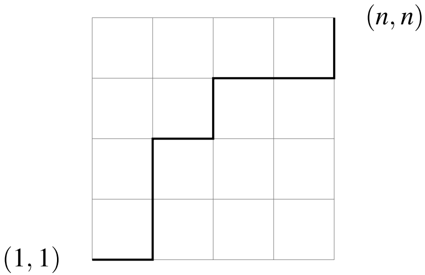
Based on joint work with T. Seppäläinen and N. Zygouras

Background

A random polymer model

$$Z_n = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



Let $a, b \in \mathbb{R}^n$ with $a_i + b_j > 0$ and define

$$\mathbb{P}(dX) = \prod_{ij} \Gamma(a_i + b_j)^{-1} x_{ij}^{-a_i - b_j - 1} e^{-1/x_{ij}} dx_{ij}.$$

This model was introduced by Seppalainen (2010), who computed the free energy explicitly and obtained sharp estimates on fluctuations.

Background

Theorem (Corwin-O'C-Seppäläinen-Zygouras 11)

Under $\mathbb{P}$, the partition function Z_n has the same distribution as the first marginal of the Whittaker measure *on $\mathbb{R}_+^n$ defined by*

$$\mu_{a,b}(dx) = \prod_{ij} \Gamma(a_i + b_j)^{-1} e^{-1/x_n} \Psi_a(x) \Psi_b(x) \prod_{i=1}^{n} \frac{dx_i}{x_i}.$$

c.f. last passage percolation and random matrices.

Analogous result for semi-discrete polymer was obtained in [O'C 2009].

Proofs based on A.N. Kirillov's (2000) geometric RSK correspondence, and multi-dimensional variants of Pitman's $2M - X$ theorem.

This approach does not extend to polymer models with symmetry constraints.

Outline

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- Geometric RSK (gRSK)
- Main result: on the link between gRSK and Whittaker functions
- Key ingredient of proof: new ‘local moves’ description of gRSK
- Applications to random polymers with symmetry

Whittaker functions

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- In the context of $GL(n,\mathbb{R})$, they can be considered as functions $\Psi_\lambda(x)$ on $(\mathbb{R}_{>0})^n$, indexed by a (spectral) parameter $\lambda \in \mathbb{C}^n$
- The following 'Gauss-Givental' representation for Ψ_λ is due to Givental (97), Joe-Kim (03), Gerasimov-Kharchev-Lebedev-Oblezin (06)

Whittaker functions

A *triangle* P with shape $x \in (\mathbb{R}_{>0})^n$ is an array of positive real numbers:

$$P = \begin{array}{ccccccc} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots\cdots & & & z_{n1} \end{array}$$

with bottom row $z_{n\cdot} = x$.

Denote by $\Delta(x)$ the set of triangles with shape x.

Whittaker functions

Let

$$P = \begin{matrix} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots & & & z_{n1} \end{matrix}$$

Define

$$P^\lambda = R_1^{\lambda_1} \left(\frac{R_2}{R_1}\right)^{\lambda_2} \cdots \left(\frac{R_n}{R_{n-1}}\right)^{\lambda_n}, \qquad \lambda \in \mathbb{C}^n, \qquad R_k = \prod_{i=1}^k z_{ki}$$

Whittaker functions

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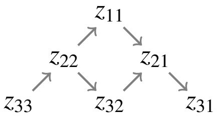
Define

$$P^{\lambda} = R_1^{\lambda_1} \left(\frac{R_2}{R_1} \right)^{\lambda_2} \cdots \left(\frac{R_n}{R_{n-1}} \right)^{\lambda_n}, \qquad \lambda \in \mathbb{C}^n, \qquad R_k = \prod_{i=1}^{k} z_{ki}$$

$$\mathcal{F}(P) = \sum_{a \to b} \frac{z_a}{z_b}$$



Whittaker functions

The Whittaker functions are defined, for $\lambda \in \mathbb{C}^n$ and $x \in (\mathbb{R}_{>0})^n$, by

$$\Psi_\lambda(x) = \int_{\Delta(x)} P^{-\lambda} e^{-\mathcal{F}(P)} dP,$$

where $dP = \prod_{1 \le i \le k < n} dz_{ki}/z_{ki}$.

For $n = 2$,

$$\Psi_{(\nu/2,-\nu/2)}(x) = 2K_\nu \left(2\pi \sqrt{x_2/x_1} \right).$$

Geometric RSK correspondence

A.N. Kirillov (00), Noumi-Yamada (04): geometric lifting of Robinson-Schensted-Knuth (RSK) correspondence.

Bi-rational map

$$T : (\mathbb{R}_{>0})^{n\times n} \to (\mathbb{R}_{>0})^{n\times n}$$

$$X = (x_{ij}) \mapsto (t_{ij}) = T = T(X).$$

For $n = 2$,

$$\begin{matrix} & t_{21} & \\ t_{11} & & t_{22} \\ & t_{12} & \end{matrix} \quad = \quad \begin{matrix} & x_{11}x_{21} & \\ x_{12}x_{21}/(x_{12}+x_{21}) & & x_{11}x_{22}(x_{12}+x_{21}) \\ & x_{11}x_{12} & \end{matrix}$$

Geometric RSK correspondence

We identify T with a pair of triangles (P, Q) of the same shape $(t_{nn}, \dots, t_{11})$:

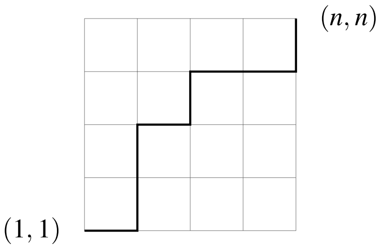
$$\begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{array} \quad = \quad (P, Q)$$

$$P = \begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \end{array} \qquad Q = \begin{array}{ccccc} & & t_{13} & & \\ & t_{12} & & t_{23} & \\ t_{11} & & t_{22} & & t_{33} \end{array}$$

Connection to polymers

From Kirillov's definition of the map T in terms of lattice paths:

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



Main result

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

Theorem (O'C-Seppäläinen-Zygouras 12)

- *The map* $(\log x_{ij}) \to (\log t_{ij})$ *has Jacobian* ± 1
- *For* $\nu, \lambda \in \mathbb{C}^n$,
$$\prod_{ij} x_{ij}^{\nu_i + \lambda_j} = P^\lambda Q^\nu$$
- *The following identity holds:*
$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

A Whittaker integral identity

It follows that

$$\prod_{ij} x_{ij}^{-\nu_i-\lambda_j} e^{-1/x_{ij}} \frac{dx_{ij}}{x_{ij}} = P^{-\lambda} Q^{-\nu} e^{-1/t_{11}-\mathcal{F}(P)-\mathcal{F}(Q)} \prod_{ij} \frac{dt_{ij}}{t_{ij}}.$$

Integrating both sides gives, for $\Re(\nu_i + \lambda_j) > 0$:

Corollary (Stade 02)

$$\prod_{ij} \Gamma(\nu_i + \lambda_j) = \int_{\mathbb{R}_+^n} e^{-1/x_n} \Psi_\nu(x) \Psi_\lambda(x) \prod_{i=1}^{n} \frac{dx_i}{x_i}.$$

As observed by Gerasimov-Kharchev-Lebedev-Oblezin (06), this is equivalent to a Whittaker integral identity which was conjectured by Bump (89) and proved by Stade (02). In our setting, it is the analogue of Cauchy-Littlewood.

Local moves

Proof of main result uses a new description of the gRSK map T as a composition of a sequence of 'local moves' applied to the input matrix

$$\begin{array}{ccccc} & & x_{31} & & \\ & x_{21} & & x_{32} & \\ x_{11} & & x_{22} & & x_{33} \\ & x_{12} & & x_{23} & \\ & & x_{13} & & \end{array}$$

This description is a re-formulation of Noumi and Yamada's (2004) geometric row insertion algorithm.

Local moves

The basic move is:

b

a $\qquad$ d

c

Local moves

The basic move is:

$$
\begin{array}{ccc}
 & b & \\
\dfrac{bc}{ab+ac} & & bd+cd \\
 & c &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

$$\begin{array}{ccccc} & & c & & \\ & b & & f & \\ a & & e & & i \\ & d & & h & \\ & & g & & \end{array}$$

Local moves

This can be applied at any position in the matrix:

		c		
	b		f	
a		e		i
	d		h	
		g		

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a & & e & & i \\
 & d & & h & \\
 & & g & &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

$$\begin{array}{ccccc} & & c & & \\ & b & & f & \\ a & & e & & i \\ & d & & {\color{red}h} & \\ & & g & & \end{array}$$

Local moves

This can be applied at any position in the matrix:

$$
\begin{array}{ccccc}
 & & c & & \\
 & b & & f & \\
a & & e & & i \\
 & \dfrac{eg}{de+dg} & & eh+gh & \\
 & & g & &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

$$
\begin{array}{ccccc}
 & & c & & \\
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a & & e & & i \\
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 & & \color{red}{g} & &
\end{array}
$$

Local moves

This can be applied at any position in the matrix:

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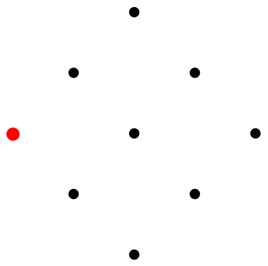
Local moves

Start with:

$$\begin{array}{ccccc} & & x_{31} & & \\ & x_{21} & & x_{32} & \\ x_{11} & & x_{22} & & x_{33} \\ & x_{12} & & x_{23} & \\ & & x_{13} & & \end{array}$$

Local moves

Apply the local moves in the following order:



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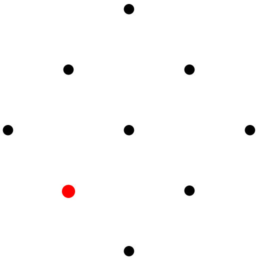

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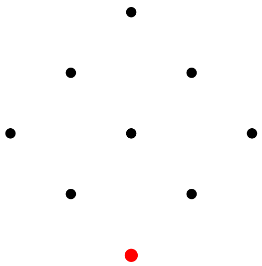

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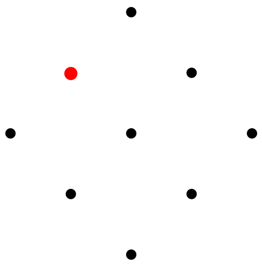

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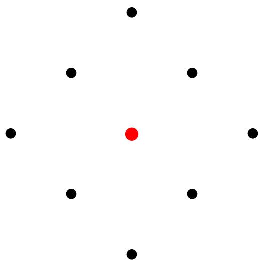

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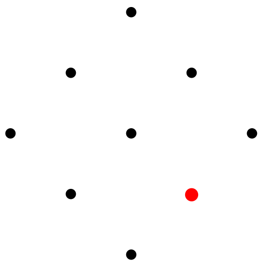

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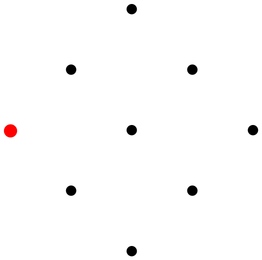

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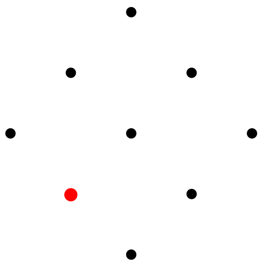

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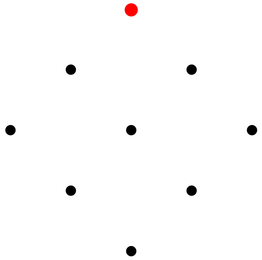

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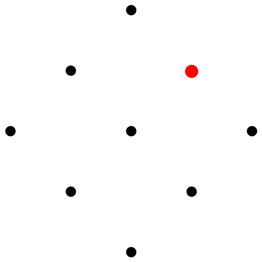

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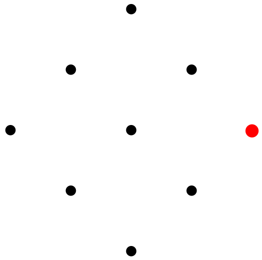

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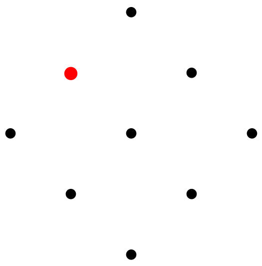

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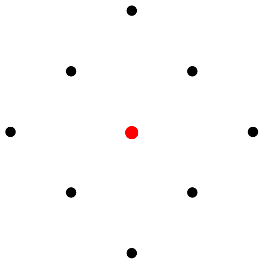

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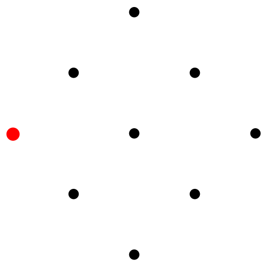

Local moves

To arrive at:

$$\begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{array}$$

Local moves

Here goes:

		c		
	b		f	
a		e		i
	d		h	
		g		

Local moves

Here goes:

c

b f

a e i

d h

g

Local moves

Here goes:

c

$b \quad f$

$a \quad e \quad i$

$d \quad h$

g

Local moves

Here goes:

$$\begin{array}{ccccc} & & c & & \\ & b & & f & \\ a & & e & & i \\ & \color{red}{d} & & h & \\ & & g & & \end{array}$$

Local moves

Here goes:

		c		
	b		f	
a		e		i
	ad		h	
		g		

Local moves

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 & & adg & &
\end{array}
$$

Local moves

Here goes:

c

$ab \qquad f$

$a \qquad \color{red}{e} \qquad i$

$ad \qquad h$

adg

Local moves

Here goes:

$$\begin{array}{ccccc} & & c & & \\ & ab & & f & \\ \frac{bd}{b+d} & & ae(b+d) & & i \\ & ad & & h & \\ & & adg & & \end{array}$$

Local moves

Here goes:

$$c$$

$$ab \qquad\qquad f$$

$$\frac{bd}{b+d} \qquad ae(b+d) \qquad i$$

$$ad \qquad\qquad \color{red}{h}$$

$$adg$$

Local moves

Here goes:

$$c$$

$$ab \qquad\qquad f$$

$$\frac{bd}{b+d} \qquad\qquad ae(b+d) \qquad\qquad i$$

$$\frac{(b+d)eg}{be+de+dg} \qquad ah(be+de+dg)$$

$$adg$$

Local moves

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$$\frac{bd}{b+d} \qquad ae(b+d) \qquad i$$

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Local moves

Here goes:

$$abc$$

$$ab \qquad \qquad {\color{red} f}$$

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Local moves

Here goes:

$$abc$$

$$\frac{bce(b+d)}{b^2c+be(b+d)} \qquad abcf+ae(b+d)f$$

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$$\frac{bd}{b+d} \qquad ae(b+d) \qquad i$$

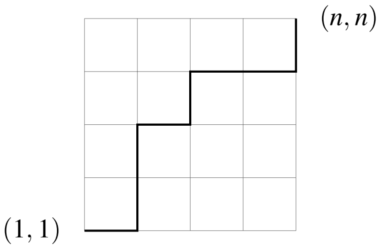
$$\frac{bdeg}{be+de+dg} \qquad ah(be+de+dg)$$

$$adg$$

Geometric RSK correspondence

Kirillov's definition of the map T in terms of lattice paths:

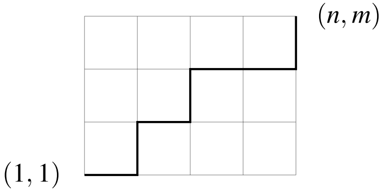
$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



Geometric RSK correspondence

Kirillov's definition of the map T in terms of lattice paths:

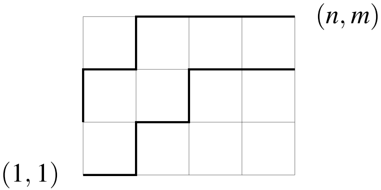
$$t_{nm} = \sum_{\phi \in \Pi_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$



Geometric RSK correspondence

Kirillov's definition of the map T in terms of lattice paths:

$$t_{n-k+1,m-k+1}\dots t_{nm} = \sum_{\phi\in\Pi^{(k)}_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$

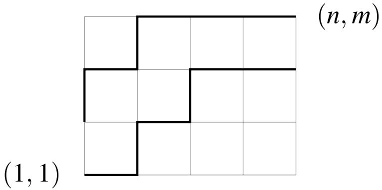


Geometric RSK correspondence

Kirillov's definition of the map T in terms of lattice paths:

$$t_{n-k+1,m-k+1} \dots t_{nm} = \sum_{\phi \in \Pi^{(k)}_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$

$$T(X)' = T(X')$$



Symmetric input matrix

Symmetry properties of gRSK:

$$T(X') = T(X)'$$

$$X \mapsto (P, Q) \qquad \Longleftrightarrow \qquad X' \mapsto (Q, P).$$

$$X = X' \qquad \Longleftrightarrow \qquad P = Q$$

Theorem (O'C-Seppäläinen-Zygouras 2012)

The restriction of T to symmetric matrices is volume-preserving.

Combined with the first main result, this yields formulas for the distribution of partition functions for polymer models with symmetry constraints.

Symmetric random polymer

Corollary

Let $\alpha_i > 0$ for each i and define

$$\mathbb{P}_\alpha(dX) = Z_\alpha^{-1} \prod_i x_{ii}^{-\alpha_i} \prod_{i<j} x_{ij}^{-\alpha_i-\alpha_j} e^{-\frac{1}{2}\sum_i \frac{1}{x_{ii}} - \sum_{i<j} \frac{1}{x_{ij}}} \prod_{i\le j} \frac{dx_{ij}}{x_{ij}}.$$

Then

$$\mathbb{P}_\alpha(sh\, P \in dx) = c_\alpha^{-1} e^{-1/2x_n} \Psi_\alpha^n(x) \prod_i \frac{dx_i}{x_i},$$

where

$$c_\alpha = \prod_i \Gamma(\alpha_i) \prod_{i<j} \Gamma(\alpha_i + \alpha_j).$$

Interpolating ensembles (cf. Baik-Rains 01)

Corollary

Let $\zeta > 0$ *and* $\alpha_i > 0$ *for each i, and define*

$$\mathbb{P}_{\alpha,\zeta}(dX) = Z_{\alpha,\zeta}^{-1} \prod_i x_{ii}^{-\alpha_i-\zeta} \prod_{i<j} x_{ij}^{-\alpha_i-\alpha_j} e^{-\frac{1}{2}\sum_i \frac{1}{x_{ii}} - \sum_{i<j} \frac{1}{x_{ij}}} \prod_{i\le j} \frac{dx_{ij}}{x_{ij}}.$$

Then

$$\mathbb{P}_\alpha(sh\, P \in dx) = c_{\alpha,\zeta}^{-1} f(x)^\zeta e^{-1/2x_n} \Psi_\alpha^n(x) \prod_i \frac{dx_i}{x_i},$$

where

$$f(x) = \prod_i x_i^{(-1)^i},$$

$$c_{\alpha,\zeta} = 2^{\sum_{i=1}^n (\alpha_i+\zeta)} \prod_i \Gamma(\alpha_i+\zeta) \prod_{i<j} \Gamma(\alpha_i+\alpha_j).$$

Random polymer above a wall (cf. Gueudre-La Doussal 12) (absorbing boundary conditions)

Let $\alpha_i > 0$ for each i and define

$$\mathbb{Q}_\alpha(dX) = \tilde{Z}_\alpha^{-1} \prod_{i<j\leq n} x_{ij}^{-\alpha_i-\alpha_j} e^{-\frac{1}{2}\sum_i \frac{1}{x_{ii}} - \sum_{i<j} \frac{1}{x_{ij}}} \prod_{i<j\leq n} \frac{dx_{ij}}{x_{ij}}.$$

Let

$$z_n = \sum_\phi \prod_{(i,j)\in\phi} x_{ij}$$

where the sum is over above-diagonal paths from $(1,2)$ to $(n-1,n)$.

Theorem (O'C-Seppäläinen-Zygouras 12 (v2, to appear))

Law of z_n under $\mathbb{Q}_\alpha$ is same as law of $2t_{n-1,n-1}$ under $\mathbb{P}^{(n-1)}_{(\alpha_1,\dots,\alpha_{n-1}),\alpha_n}$.