Maps between isolated points on modular curves

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Introduction

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Theorem (Frey, 1994)

C has infinitely many points of degree at most d if and only if there is an infinite family of such points parametrized by $\mathbb{P}^1$ or a positive rank abelian variety.

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Question

When does C have infinitely many points of degree d?

Theorem (Frey, 1994)

C has infinitely many points of degree at most d if and only if

1. *there exists a map $C \to \mathbb{P}^1$ of degree at most d, or*
2. *the image of the map $C^{(d)} \to \mathrm{Jac}(C)$ contains a translate of a positive rank abelian variety.*

Introduction

Definition

Let $x \in C$ be a degree d closed point, and $\varphi : C^{(d)} \to \mathrm{Jac}(C)$ denote the map $y \mapsto [y - x]$. We say that

1. x is $\mathbb{P}^1$-isolated if there does not exist a point $y \neq x \in C^{(d)}$ such that $\varphi(y) = \varphi(x)$.
2. x is AV-isolated if there does not exist a positive rank abelian subvariety A of $\mathrm{Jac}(C)$ such that $\varphi(x) + A \subseteq \varphi(C^{(d)})$.
3. x is isolated if it is both $\mathbb{P}^1$-isolated and AV-isolated.

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Theorem (Bourdon, Ejder, Liu, Odumodu, Viray, 2019)

C has infinitely many points of degree d if and only if there is a non-isolated degree d point on C.

Introduction

Theorem (Bourdon, Gill, Rouse, Watson, 2020)

Let x be a non-CM, non-cuspidal isolated point of odd degree with rational j-invariant on the modular curve $X_1(N)$, for some $N \geq 1$. Then $j(x) \in \{-\frac{140625}{8}, \frac{351}{4}\}$.

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Theorem (Ejder, 2022)

Let $p > 7$ be prime and n be a positive integer. Let x be a non-CM, non-cuspidal isolated point with rational j-invariant on the modular curve $X_1(p^n)$. Then $p = 37$ and $j(x) \in \{9317, -7 \cdot 137^3 \cdot 2083^3\}$.

Introduction

Theorem (Bourdon, Ejder, Liu, Odumodu, Viray, 2019)

Let $f : C \to D$ *be a non-constant morphism of curves over* K. *Let* $x \in C$ *be a closed point and let* $y = f(x) \in D$. *Suppose that* x *is isolated and* $\deg(x) = \deg(f) \cdot \deg(y)$. *Then* y *is also isolated.*

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1. Are there other situations in which this theorem holds?
2. Can we extend this result to non-geometrically integral curves?

Isolated divisors

Let C be a smooth, projective, one-dimensional scheme over a number field K. Let $\mathbf{Pic}_{C/K}$ denote the Picard scheme of C, and $\mathbf{Pic}^0_{C/K}$ be the connected component of the identity. Let $\varphi : C^{(d)} \to \mathbf{Pic}_{C/K}$ be the canonical map given by $\alpha \mapsto [\alpha]$.

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Definition

Let $\alpha \in C^{(d)}$ be an effective degree d divisor on C. We say that

1. α is $\mathbb{P}^1$-isolated if there does not exist a point $\beta \neq \alpha \in C^{(d)}$ such that $\varphi(\beta) = \varphi(\alpha)$.
2. α is AV-isolated if there does not exist a positive rank abelian subvariety A of $\mathbf{Pic}^0_{C/K}$ such that $\varphi(\alpha) + A \subseteq \varphi(C^{(d)})$.
3. α is isolated if it is both $\mathbb{P}^1$-isolated and AV-isolated.

Maps between isolated divisors

Theorem

Let $f : C \to D$ be a dominant morphism of smooth, projective, one-dimensional schemes over a number field K. Let $\alpha \in C^{(d)}$ be an effective degree d divisor on C, and $\beta \in D^{(e)}$ be an effective degree e divisor on D.

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1. *If $f_*(\alpha)$ is isolated, then α is isolated.*
2. *If $f^*(\beta)$ is isolated, then β is isolated.*

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1. *If $f_*(\alpha)$ is isolated, then α is isolated.*
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Theorem

Let $f : C \to D$ be as above, $x \in C$ be a degree d closed point, and $y \in D$ be a degree e closed point, with $f(x) = y$.

1. *$f_*(x)$ is a closed point (and equal to y) if and only if $\deg(x) = \deg(y)$.*
2. *$f^*(y)$ is a closed point (and equal to x) if and only if $\deg(x) = \deg(f) \cdot \deg(y)$.*

Isolated points on modular curves

Let $H \leq H' \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, with $-I \in H$, and let $f : X_H \to X_{H'}$ be the natural map of modular curves.

Isolated points on modular curves

Let $H \leq H' \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, with $-I \in H$, and let $f : X_H \to X_{H'}$ be the natural map of modular curves.

Let $x = [(E, \alpha)]_H \in X_H$ be a closed point, where $E/\mathbb{Q}(j(E))$ and $j(E) \notin \{0, 1728\}$. Let $y = f(x) = [(E, \alpha)]_{H'} \in X_{H'}$.

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Write $G = \pm\bar{\rho}_{E,N}(G_{\mathbb{Q}(j(E))})$.

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Write $G = \pm\bar{\rho}_{E,N}(G_{\mathbb{Q}(j(E))})$.

Theorem

1. *Suppose that $H' = (G \cap H')H$. If x is isolated, then so is y.*
2. *Suppose that $G \cap H' = G \cap H$. If y is isolated, then so is x.*

Applications

Theorem

Let $H \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$*, with* $-I \in H$*, and let* K *be a number field.*

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Theorem

Let $H \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, *with* $-I \in H$, *and let* K *be a number field. Let* $x = [(E, \alpha)]_H \in X_H$ *be a non-CM, non-cuspidal isolated point with* E/K *and* $\mathbb{Q}(j(E)) = K$. *Let* $G = \pm\bar{\rho}_{E,N}(G_K)$.

Applications

Theorem

Let $H \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, with $-I \in H$, and let K be a number field. Let $x = [(E, \alpha)]_H \in X_H$ be a non-CM, non-cuspidal isolated point with E/K and $\mathbb{Q}(j(E)) = K$. Let $G = \pm\bar{\rho}_{E,N}(G_K)$.

Then the point $y = [(E, \alpha)]_G \in X_G$ is an isolated K-rational point.

Applications

Theorem

Let x be a non-CM, non-cuspidal isolated point with rational j-invariant on a modular curve X_H of level 7. Then $j(x) = \frac{2268945}{128}$, and one of the rows of the following table holds.

Generators	$\overline{X_H}$	g	$\deg x$
$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	6	3	18
$\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	3	3	9
$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$	2	3	6
$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$	2	3	6
$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$	6	1	6
$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	1	3	3
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$	3	1	3
$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$	2	1	2
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$	1	1	1

Applications

Theorem

Let $x \in X_1(2p)$ be a non-CM, non-cuspidal isolated point with rational j-invariant. Then $p = 37$ and $j(x) \in \{-9317, -7 \cdot 137^3 \cdot 2083^3\}$.