TCC Linear Algebraic Groups

Assignment 1 **January 2022**

If you are taking this course for credit you should submit solutions to some of the problems by **10am (UK time)** on **THURSDAY 3 FEBRUARY** (The start of the third lecture). What does *some* mean? Well that depends on your background, and how well you wish to follow the course. I would say the minimum is a correct solution to 2 problems. But in all cases if you unsure just ask me via email (and if you have a suggestion for an alternative way to show your engagement).

Most exercises are taken from the book Linear Algebraic Groups by Gunter Malle and Donna Testerman. If you are looking for more exercises you could try any of the other Linear Algebraic Groups books (by Borel, Springer etc), and just following some of the proofs/theory we have skipped from the algebraic geometry side is a good exercise in itself.

Unless otherwise stated G is a linear algebraic group.

1. Read the construction/definition of the orthogonal groups on page 8 and 9. If you are not familiar with them then reading the start of a book/lecture notes on classical groups might be helpful - if you Google classical groups you'll find lots (we're not too worried about the intricacies over finite fields for now).

2. Given an example of an affine variety which is not irreducible but is connected (in the Zariski topology). [Of course, we saw for an algebraic group this is not possible!]

3. (a) Prove that $C_G(x) := \{g \in G \mid xg = gx\}$ is a closed subgroup of G. Hence prove that the centre of G is a closed subgroup.

   (b) Prove that if H is a subgroup of G then so is its closure $\bar{H}$. (for a hint, see Exercise 10.3 in the book)

   (c) Prove that if H is subgroup of G containing a non-empty open subset of $\bar{H}$ then H is closed.

4. By considering their coordinate algebras, show that T_n, U_n and D_n are connected and find their dimension.

5. Show that the subset G_u of unipotent elements of G is closed. (Hint: consider the element's characteristic polynomial).

6. (a) Show that the algebraic group automorphisms of $\mathbb{G}_a$ are the multiplications by non-zero elements of k.

   (b) Show that $\text{End}(\mathbb{G}_m) := \{\phi : \mathbb{G}_m \to \mathbb{G}_m \mid \phi \text{ a morphism of algebraic groups}\}$ is isomorphic to $\mathbb{Z}$. Hence show that the group of algebraic group automorphisms of $\mathbb{G}_m$ is $\mathbb{Z}_2$ (the cyclic group of order 2).