TCC Linear Algebraic Groups

Assignment 2 **February 2022**

If you are taking this course for credit you should submit solutions to some of the problems by **10am (UK time)** on **THURSDAY 17 FEBRUARY** (The start of the fifth lecture).

What does *some* mean? Well that depends on your background, and how well you wish to follow the course. I would say the minimum is a correct solution to 2 or 3 problems (if you only did 2 on the first you should probably attempt at least 3 this time). But in all cases if you unsure just ask me via email (and if you have a suggestion for an alternative way to show your engagement).

Unless otherwise stated G is a linear algebraic group and X is a G-space.

(Hints for some of the exercises can be found in Malle-Testerman)

1. (a) Show that $X^g := \{x \in X | g \cdot x = x\}$ is closed for all $g \in G$ and thus conclude that $X^G = \{x \in X | g \cdot x = x \text{ for all } g \in G\}$ is closed.
   (b) Let Y, Z be closed subsets of X and define the transporter of Y into Z as follows $\mathrm{Tran}_G(Y, Z) := \{x \in G | x \cdot Y \subseteq Z\}$. Show that the transporter of Y into Z is a closed subset of G.
   (c) Prove that $N_G(H)$ is closed for any closed subgroup $H \leq G$.
2. Suppose that G is connected with nilpotent Borel subgroups. Prove that G is soluble. Use this to conclude that any 2-dimensional connected linear algebraic group is soluble.
3. Suppose that G is connected. Prove that $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive.
4. Prove that $R(G) = (\cap_B B)^\circ$ where B runs over all Borel subgroups of G. Use this to prove that Sp_{2n} is semisimple.
5. Show that the inversion morphism $i : G \to G$ has differential $di(X) = -X$ for $X \in \mathrm{Lie}(G)$.
6. (a) Show that $Z(G)$ is a subgroup of $\ker(\mathrm{Ad})$.
   (b) Determine $\ker(\mathrm{Ad})$ for GL_n, U_n, T_n.
7. Suppose that k has characteristic $p > 0$ and that $G = \mathrm{SL}_3$. Define $\phi : \mathbb{G}_a \to G$ via
$$\phi(t) = \begin{pmatrix} 1 & t & t^p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
   (a) Let $H = \mathrm{im}(\phi)$ and show that H is isomorphic to $\mathbb{G}_a$. What is $C_G(H)$?
   (b) Determine $\mathrm{Lie}(H)$ as a subalgebra of $\mathrm{Lie}(G) \cong \mathfrak{sl}_3$. (Recall Theorem 7.4(b) and the fact $\mathrm{Lie}(H)$ is a vector subspace of $\mathrm{Lie}(G)$).
   (c) Calculate $C_G(\mathrm{Lie}(H)) := \{g \in G | \mathrm{Ad}(g)X = X \text{ for all } X \in \mathrm{Lie}(H)\}$ and check that it is not the same as $C_G(H)$. (Recall that the action of $\mathrm{Ad}(g)$ on X is just conjugation by g so $\mathrm{Ad}(g)X = gXg^{-1}$).
   (d) Calculate $C_{\mathrm{Lie}(G)}(H) := \{X \in \mathrm{Lie}(G) | \mathrm{Ad}(h)X = X \text{ for all } h \in H\}$ and check that it is not the same as $\mathrm{Lie}(C_G(H))$ when $p = 3$.
8. Any other exercise from Malle-Testerman in the range 10.15–10.28.