TCC Linear Algebraic Groups

Assignment 3 **February 2022**

If you are taking this course for credit you should submit solutions to **BOTH** of the problems by **10am (UK time)** on **THURSDAY 3 MARCH** (The start of the seventh lecture).

The last two assignments are slightly different. There will be fewer questions and they should be possible for anyone following the course (and if you don't think so please just email me). There are plenty of exercises in Malle-Testerman but as we start to skip around the book and concentrate and different things I want to give some exercises in doing relevant calculations. This time it is with the representation theory of some simple linear algebraic groups.

1. In Lecture 5 we saw lots of concrete calculations for SL_3. In this question you will do the same things for $G = G_2$. Appendix A will help you with some of the parts, which is fine to look at if you need it. The Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ - this tells us that $\langle \alpha_1, \alpha_1^\vee \rangle = 2 = \langle \alpha_2, \alpha_2^\vee \rangle$, $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -3$.

   (a) Write down the 12 roots in the root system G_2 with base α_1 (short root) and α_2 (long root) (if you are struggling you may look this up in many places). Draw them in $\mathbb{R}^2$.

   (b) What are the fundamental dominant weights, written in the root lattice (so write $\lambda_i = j_{i,1}\alpha_1 + j_{i,2}\alpha_2$).

   (c) The Weyl group of G is Dih_{12} (the Dihedral group of order 12). What is its orbit on λ_1? (The discussion at the bottom of page 126/top of 127 in Malle-Testerman may be useful or use your diagram)

   (d) Using the previous question, what are the weights of $L(\lambda_1)$? (You may use the fact that the Weyl orbit of any non-zero dominant weight has size at least 2)

   (e) What is the highest (dominant) weight of the adjoint module for G? (Hint: it should be straightforward to work out the root using Chevalley's Commutator Formula and then turn it into a weight)

2. Recall Weyl's dimension formula for the dimension of an irreducible module $L(\lambda)$ with dominant highest weight λ in characteristic 0 (which is assumed to be the case for the whole question):

$$\dim(L(\lambda)) = \frac{\prod_{\alpha\in\Phi^+} \langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

   where $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+} \beta$.

   We saw that any dominant weight for G of rank 2 can be written as $m_1\lambda_1 + m_2\lambda_2$ for integers $m_i \geq 0$ and fundamental dominant weights λ_1 and λ_2 (given explicitly for SL_3 as characters in Lecture 5).

(a) When $G = SL_3$, of type A_2, what is ρ when written as a linear combination of λ_1 and λ_2? Use this to verify that

$$\dim(L(m_1\lambda_1 + m_2\lambda_2)) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2).$$

(b) The weights of $L(\lambda_1)$ are $\lambda_1, -\lambda_1+\lambda_2$ and $-\lambda_2$. Using this and the previous formula prove that $L(\lambda_1) \otimes L(\lambda_1) = L(2\lambda_1) + L(\lambda_2)$. (Hint: the weights of a tensor product are just the sum of a weight from each factor).

(c) When $G = G_2$ what is ρ when written as a linear combination of λ_1 and λ_2? Use this to verify that

$$\dim(L(b_1\lambda_1+b_2\lambda_2)) = \frac{1}{5!}(b_1+1)(b_2+1)(b_1+b_2+2)(b_1+2b_2+3)(b_1+3b_2+4)(2b_1+3b_2+5).$$

(d) What is $L(\lambda_1) \otimes L(\lambda_1)$, written as a sum of irreducible modules? You may assume that all modules are completely reducible in characteristic 0. (Part (d) of Question 1 will be helpful).