Extension of Caputo evolution equations with time-nonlocal initial condition

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T. (2018).
Stochastic classical solutions for space-time fractional evolution equations on bounded domain.
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And time permitting

Du, T., Zhou (2018).
Stochastic solutions for time-nonlocal evolution equations.
Submission: Sept. 2018.

Hernández-Hernández, Kolokoltsov, T. (2017).
Generalised fractional evolution equations of Caputo type.
Chaos, Solitons & Fractals, 102: 184-196.

Main idea

Let $\partial_{t,\infty}^{\beta}$ be the Marchaud derivative (extension of Caputo derivative, $\beta \in (0,1)$).
Consider the *extension* of Caputo evolution equations with time-nonlocal initial condition

$$\begin{cases} \partial_{t,\infty}^{\beta} u(t,x) = \Delta u(t,x), & \text{in } (0,T] \times \mathbb{R}^d, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty,0] \times \mathbb{R}^d. \end{cases} \tag{1}$$

The **stochastic representation** is

$$u(t,x) = \mathbb{E}\left[\phi\left(-W(t), B_{E(t)}^{x}\right)\right].$$

Here $W(t)$ is the waiting time of $B_{E(t)}^{x}$ (the fractional kinetic process).
Question: Are time-nonlocal initial conditions meaningful for applications?

Overview

Caputo evolution equation (EE)

Consider the Caputo evolution equation

$$\begin{cases} \partial_{t,0}^{\beta} u(t,x) = \Delta u(t,x), & \text{in } (0,T] \times \mathbb{R}^d, \\ u(0,x) = \phi(0,x), & \text{in } \{0\} \times \mathbb{R}^d. \end{cases} \tag{2}$$

where the Caputo derivative $\partial_{t,0}^{\beta}$, $\beta \in (0,1)$, is defined as

$$\partial_{t,0}^{\beta} u(t) := \int_0^t u'(r) \, \frac{(t-r)^{-\beta} \, dr}{\Gamma(1-\beta)}. \tag{3}$$

It is well known that the stochastic solution reads $u(t,x) = \mathbb{E}[\phi(0, B^x_{E(t)})]$, where B is a Brownian motion and $E(t)$ is an independent inverse β-stable subordinator **[**Saichev, Zaslavsky '97**]**, **[**Beaumer, Meerschaert '01**]**, **[**Meerschaert, Scheffler '04**]**.

Notable properties of the fractional kinetic $Y_t = B^x_{E(t)}$:

1. **Universality**: Y_t is the quenched scaling limit of random conductance models **[**Barlow, Černý '11**]**. Note that Y_t is non-Markovian process (with memory), but it is the limit of Markovian processes (without memory).
2. **Subdiffusion**: Mean squared displacement $\mathbb{E}[Y_t^2] = t^\beta < t = \mathbb{E}[B_t^2]$.
3. **A model for trappings**: the continuous non-Markovian time change $E(t)$ is i.o. constant on time intervals.
4. **Universality for** $\beta = 1/2$: Y_t is the intermediate time behaviour of perturbed cellular flows **[**Hairer et al. '18**]**.
5. **Connection to 4th order PDEs**: Y_t, $\beta = 1/2$ is the fundamental solution to $\partial_t u = \Delta^2 u + \Delta\phi(0)/\sqrt{\pi t}$ **[**Meerschaert, Nane '09**]** (hence the solution is positivity preserving).

Marchaud to Caputo derivative

Consider the Marchaud derivative

$$\partial^{\beta}_{t,\infty} u(t) := \int_{-\infty}^{t} u'(r) \, \frac{(t-r)^{-\beta} \, dr}{\Gamma(1-\beta)}. \tag{4}$$

If $u(t) = u(0)$ for all $t < 0$ the Marchaud derivative equals the Caputo derivative, as

$$\partial^{\beta}_{t,\infty} u(t) = \int_{0}^{t} u'(r) \, \frac{(t-r)^{-\beta} \, dr}{\Gamma(1-\beta)} = \partial^{\beta}_{t,0} u(t).$$

Probabilistically $-\partial^{\beta}_{t,\infty}$ is the generator of the inverted β-stable-subordinator $-X^{\beta}_{s}$, easily observed from the representation

$$-\partial^{\beta}_{t,\infty} u(t) = \int_{0}^{\infty} (u(t-r) - u(t)) \, \frac{r^{-1-\beta} \, dr}{-\Gamma(-\beta)}. \tag{5}$$

Marchaud to Caputo evolution equation

Consider the Marchaud evolution equation

$$\begin{cases} \partial^{\beta}_{t,\infty} u(t,x) = \Delta u(t,x), & \text{in } (0,T] \times \mathbb{R}^d, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty, 0] \times \mathbb{R}^d. \end{cases} \tag{6}$$

Then, if $\phi(t) = \phi(0)$ for all $t < 0$, then $\partial^{\beta}_{t,\infty} = \partial^{\beta}_{t,0}$ the EE (6) becomes the standard Caputo EE.

- The Marchaud EE (6) is the natural fractional counterpart the time-nonlocal evolution equations proposed in **[**Chen, Du, Li, Zhou '17**]** and **[**Du, Yang, Zhou '17**]**. In **[**Allen '17**]** uniqueness of weak solutions is considered.
- With a little extra work, existence/regularity results follow from results about inhomogeneous Caputo EEs, such as **[**Allen, Caffarelli, Vasseur '16**]**, **[**Baeumer, Kurita, Meerschaert '05**]**.

The Theorem

Here is a rough statement of the main result.

Theorem

Assuming certain regularity on ϕ, there exists a unique classical solution to the Marchaud EE

$$\begin{cases} \partial_{t,\infty}^{\beta} u(t,x) = \Delta u(t,x), & \text{in } (0,T] \times \mathbb{R}^d, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty,0] \times \mathbb{R}^d. \end{cases} \tag{7}$$

Moreover, the solution allows the stochastic representation

$$u(t,x) = \mathbb{E}\left[\phi\left(-W(t), B^x_{E(t)}\right)\right], \tag{8}$$

where $W(t)$ is the waiting/trapping time of the fractional kinetic process $B^x_{E(t)}$.

1. Marchaud evolution equation
2. Stochastic representation
   - Intuition
   - Motivation
3. Proof of Theorem
4. Generalised Marchaud evolution equations

Stochastic representation: Intuition

Denote by $Y_t^x = B_{E(t)}^x$ the fractional kinetic process.
The solution

$$\mathbb{E}\left[\phi\Big(-W(t), Y_t^x\Big)\right]$$

weights the initial condition with respect to the duration of the holding time $W(t)$ of the process Y_t.

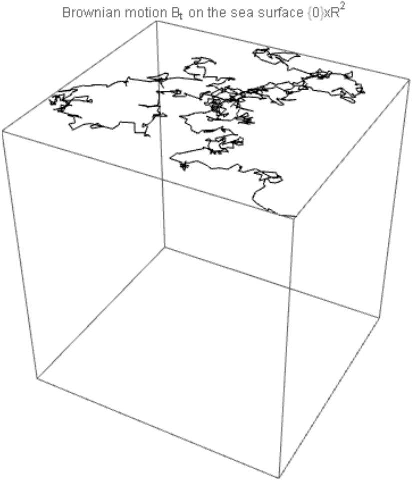
Example

The initial condition $\phi(t,x) = \mathbf{1}_{(-\infty,-1]}(t)\tilde{\phi}(x)$, results in

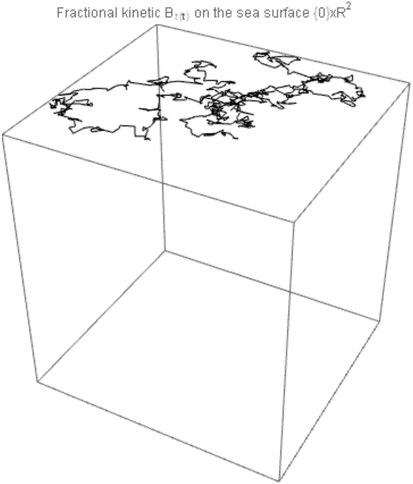
$$\mathbb{E}\left[\tilde{\phi}\Big(Y_t^x\Big) \big| Y_t^x \text{ is trapped for more than 1 time-unit}\right],$$

We will now plot $t \mapsto (-W(t), Y_t) \in (-\infty, 0] \times \mathbb{R}^2$, where the values of $-W(t)$ are thought of as the depth underneath a surface $\{0\} \times \mathbb{R}^2$ (and not the past).

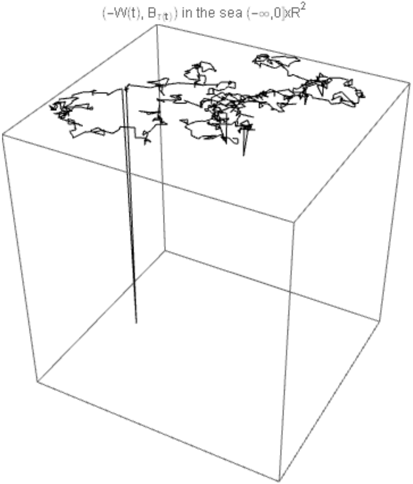
Brownian motion on the sea surface: $(0, B_t) \in \{0\} \times \mathbb{R}^2$



Fractional kinetic on the sea surface: $(0, Y_t) \in \{0\} \times \mathbb{R}^2$



Fractional kinetic in the sea: $(-W(t), Y_t) \in (-\infty, 0] \times \mathbb{R}^2$



Key remark

Probabilistically $-\partial^{\beta}_{t,\infty}$ is the generator of the inverted β-stable-subordinator $-X^{\beta}_{s}$, easily observed from the representation

$$-\partial^{\beta}_{t,\infty} u(t) = \int_0^\infty (u(t-r) - u(t)) \frac{r^{-1-\beta}\, dr}{-\Gamma(-\beta)}. \tag{9}$$

Stochastic representation: Motivation

$$\begin{cases} \mathcal{G}u = 0, \text{ in } \Omega, & \mathcal{G} \text{ Markovian generator of } G_s \\ u = \phi, \text{ in } \partial\Omega, & \end{cases}$$

should be solved by $u(\omega) = \mathbb{E}\left[\phi\left(G^{\omega}_{\tau_{\partial\Omega}(\omega)}\right)\right]$, where $\tau_{\partial\Omega}(\omega) := \inf\{s : G^{\omega}_s \notin \Omega\}$.
Now set $\mathcal{G} \equiv (-\partial^{\beta}_{t,\infty} + \Delta)$, $\Omega \equiv (0, T] \times \mathbb{R}^d$, and $\partial\Omega \equiv (-\infty, 0] \times \mathbb{R}^d$. Then

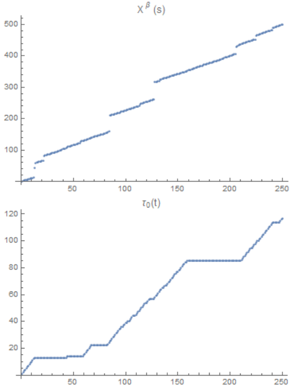
$$G^{\omega}_s = (t - X^{\beta}_s, B^x_s), \qquad t - X^{\beta} \perp B^x, \, \omega = (t, x)$$

$$\tau_{\partial\Omega}(\omega) = \tau_0(t) := \inf\{s : t - X^{\beta}_s \leq 0\} = \inf\{s : t < X^{\beta}_s\} =: E(t)$$

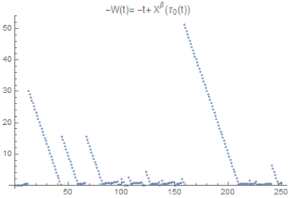
$$u(t,x) = \mathbb{E}\left[\phi\left(t - X^{\beta}_{\tau_0(t)}, B^x_{\tau_0(t)}\right)\right] = \mathbb{E}\left[\phi\left(-W(t), B^x_{E(t)}\right)\right],$$

where $W(t)$ is the waiting time of $B^x_{\tau_0(t)}$.

Stochastic representation: Motivation $X^{\beta}_{\tau_0(t)} - t = W(t)$



(a) X^{β} and $t \mapsto \tau_0(t) = E(t)$



(b) $t \mapsto W(t)$

Definition of classical solution for Marchaud EE

Definition

1. $u \in C_{b,\partial\Omega}((-\infty, T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega)$,
2. $\partial_t u \in L^1((0,T] \times \Omega)$
3. $u(t,x) \to \phi(0,x)$, as $t \downarrow 0$, for each $x \in \Omega$, and
4. u satisfies

$$\begin{cases} \partial^{\beta}_{t,\infty} u(t,x) = -(-\Delta^{\frac{\alpha}{2}})u(t,x), & \text{in } (0,T] \times \Omega, \\ u(t,x) = \phi(t,x), & \text{in } (-\infty, 0) \times \Omega, \\ u(t,x) = 0, & \text{in } (0,T] \times \Omega^c, \end{cases} \quad (10)$$

for a given time-nonlocal initial condition ϕ, where $-(-\Delta^{\frac{\alpha}{2}})$, $\alpha \in (0,2)$ is the fractional Laplacian.

Theorem's statement

Let B^x be the rotationally symmetric α-stable Lévy process killed on exiting Ω, $\alpha \in (0, 2)$.

Theorem (T. '18)

Let $\Omega \subset \mathbb{R}^d$ be a regular set. Assume that $\phi \in C^1_{b,\partial\Omega}((-\infty, 0]; Dom((-\Delta^{\frac{\alpha}{2}})^k))$, for some $k > -1 + (3d+4)/(2\alpha)$, and $\partial_t \phi$ is Lipschitz at 0. Then

$$u(t,x) = \mathbb{E}\left[\phi\left(-W(t), B^x_{\tau_0(t)}\right)\right]$$

is the unique classical solution to the Marchaud EE (10).

The heat kernel is

$$H^{t,x}_{\beta,\alpha}(r,y) = \int_0^t \frac{-\Gamma(-\beta)^{-1}}{(z-r)^{1+\beta}} \left(\int_0^\infty p^\Omega_s(x,y) p^\beta_s(t-z)\, ds\right) dz,$$

where $p^\Omega_s(x)$ is the law of B^x_s and p^β_s is the law of X^β_s.

Proof: Rewrite Marchaud EE as an inhomogeneous Caputo EE

Observe that if u equals ϕ for $t \leq 0$, then for $t > 0$

$$\partial^{\beta}_{t,\infty} u(t) = \int_0^t u'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)} - \int_{-\infty}^0 \phi'(r) \frac{(t-r)^{-\beta} dr}{-\Gamma(1-\beta)}$$
$$= \partial^{\beta}_{t,0} u(t) - f_\phi(t),$$

and so we solve the inhomogeneous Caputo EE

$$\begin{cases} \partial^{\beta}_{t,0} u(t,x) = \Delta u(t,x) + f_\phi(t,x), & \text{in } (0,T] \times \mathbb{R}^d, \\ u(t,x) = \phi(0,x), & \text{in } \{0\} \times \mathbb{R}^d. \end{cases}$$

(In short: (Caputo, IC $= \phi(0)$, FT$= f_\phi$).)

Proof: Obtain the stochastic representation (1)

The stochastic representation for the inhomogeneous EE (Caputo, IC $= \phi(0)$, FT$= f_\phi$) is expected to be

$$u(t,x) = \mathbb{E}\left[\phi\left(0, B^x_{\tau_0(t)}\right)\right] + \mathbb{E}\left[\int_0^{\tau_0(t)} f_\phi\left(t - X^\beta_s, B^x_s\right) ds\right].$$

Now note that for ϕ extended to $\phi(0)$ on $(0,T]$

$$-f_\phi(t) = \int_{-\infty}^{t} \phi'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)} = -\partial^\beta_{t,\infty}\phi(t),$$

and by Dynkin formula

$$\mathbb{E}\left[\phi\left(0, B^x_{\tau_0(t)}\right)\right] = \phi(0,x) + \mathbb{E}\left[\int_0^{\tau_0(t)} \Delta\phi\left(t - X^\beta_s, B^x_s\right) ds\right].$$

Proof: Obtain the stochastic representation (2)

Recombining and by Dynkin formula the solution to (Caputo, IC $= \phi(0)$, FT$= f_\phi$)

$$
\begin{aligned}
u(t,x) &= \mathbb{E}\left[\phi\left(0, B^x_{\tau_0(t)}\right)\right] + \mathbb{E}\left[\int_0^{\tau_0(t)} f_\phi\left(t - X^\beta_s, B^x_s\right) ds\right] \\
&= \phi(0,x) + \mathbb{E}\left[\int_0^{\tau_0(t)} (-\partial^\beta_{t,\infty} + \Delta)\phi\left(t - X^\beta_s, B^x_s\right) ds\right] \\
&= \mathbb{E}\left[\phi\left(t - X^\beta_{\tau_0(t)}, B^x_{\tau_0(t)}\right)\right] = u(t,x),
\end{aligned}
$$

the solution to (Marchaud, IC $= \phi$).

Proof: Small summary

1. Solutions to (Marchaud, IC= ϕ) = solutions to (Caputo, IC=$\phi(0)$, FT=f_ϕ).
2. Feynman-Kac for (Marchaud, IC= ϕ) = Feynman-Kac for (Caputo, IC=$\phi(0)$, FT=f_ϕ).

Theorem (T. '18)

And so, as the unique classical solution to (Caputo, IC=$\phi(0)$, FT=f) is

$$u(t,x) = \mathbb{E}\left[\phi\left(0, B^x_{\tau_0(t)}\right)\right] + \mathbb{E}\left[\int_0^{\tau_0(t)} f\left(t - X^\beta_s, B^x_s\right) ds\right],$$

if $\phi(0) \in Dom((-\Delta^{\frac{\alpha}{2}})^k)$, $f \in C^1([0,T]; Dom((-\Delta^{\frac{\alpha}{2}})^k))$, for some $k > -1 + (3d+4)/(2\alpha)$,

simply select ϕ such that $f_\phi \in C^1([0,T]; \mathrm{Dom}((-\Delta^{\frac{\alpha}{2}})^k))$.

Proof: Plan for (Caputo EE, IC$=\phi(0)$, FT$=f$)

1. **Prove that the candidate stochastic representation is a weak solution:**
   using BVP point of view in the motivation slide, not discussed.
2. **Prove smoothness of the candidate stochastic representation:**
   extends [Chen, Meerschaert, Nane '12] using separation of variables.
3. **Uniqueness of classical solution**:
   easy by separation of variables, not discussed.

(2) Homogeneous term as in [Chen, Meerschaert, Nane '12]

Denote by $\{\lambda_n, \varphi_n : n \geq 1\}$ the eigenvalue/eigenfunctions of the restricted fractional Lapacian $(-\Delta^{\alpha/2})_\Omega$. Then

$$\begin{aligned}\mathbb{E}\left[\phi\left(0, B^x_{\tau_0(t)}\right)\right] &= \int_0^\infty \mathbb{E}[\phi(B^x_s)] d_s \mathbb{P}[\tau_0(t) \leq s] \\ &= \sum_{n\geq 1} \langle \phi, \varphi_n \rangle \varphi_n(x) \int_0^\infty e^{-\lambda_n s} d_s \mathbb{P}[\tau_0(t) \leq s] \\ &= \sum_{n\geq 1} \langle \phi, \varphi_n \rangle \varphi_n(x) \mathbb{E}[e^{-\lambda_n \tau_0(t)}],\end{aligned}$$

where $\mathbb{E}[e^{-\lambda_n \tau_0(t)}] = E_\beta(\lambda_n t^\beta) := \sum_{m\geq 0} \frac{(-\lambda_n t^\beta)^m}{\Gamma(m\beta+1)}$, the Mittag-Leffler function that solves the homogeneous Caputo IVP

$$\partial^\beta_{t,0} g(t) = -\lambda_n g(t), \quad g(0) = 1.$$

(2) Inhomogeneous term

For the inhomogeneous term we compute

$$\mathbb{E}\left[\int_0^{\tau_0(t)} f\left(t - X_s^\beta, B_s^x\right) ds\right]$$

$$= \sum_{n\geq 0} \varphi_n(x)\mathbb{E}\left[\int_0^{\tau_0(t)} e^{-\lambda_n s}\langle f\left(t - X_s^\beta\right), \varphi_n\rangle\, ds\right]$$

$$= \sum_{n\geq 0} \varphi_n(x) E_{\beta,\lambda_n} \star \langle f(\cdot), \varphi_n\rangle(t),$$

where the Mittag-Leffler convolution

$$E_{\beta,\lambda} \star \langle f, \varphi_n\rangle(t) \equiv -\lambda_n^{-1} \int_0^t \langle f(r), \varphi_n\rangle \partial_t E_\beta(-\lambda_n(t-r)^\beta) dr$$

is the solution to the inhomogeneous Caputo IVP
$\partial_{t,0}^\beta g(t) = -\lambda_n g(t) + \langle f, \varphi_n\rangle$, $g(0) = 0$.

(2) Inhomogeneous term

Convergence of the first derivative in time of the series depends on bounds on the function

$$\partial_t E_{\beta,\lambda} \star f(t) = \partial_t \int_0^t f(r)(t-r)^{\beta-1} \beta E'_\beta(-\lambda(t-r)^\beta) dr.$$

If f is $C^1([0,T])$ we can hit f with ∂_t to access the bound

$$|\partial_t E_{\beta,\lambda} \star f(t)| \leq \frac{c}{\lambda} \left(\|f'\|_\infty + f(0) \frac{\lambda t^{\beta-1}}{1+\lambda t^\beta} \right).$$

Generalised Marchaud evolution equations

Perform the natural probabilistic generalisation $\partial^{\beta}_{t,\infty} u(t) \mapsto \partial^{(\nu)}_{t,\infty} u(t) := \int_0^\infty (u(t) - u(t-r))\, \nu(t, dr)$, and consider

$$\begin{cases} \partial^{(\nu)}_{t,\infty} u(t,x) = \Delta u(t,x), & \text{in } (0,T] \times \mathbb{R}^d, \\ u(t,x) = \phi(t,x), & \text{in } (-\delta, 0] \times \mathbb{R}^d, \end{cases} \tag{11}$$

where δ is the length of the support of the Lévy-type kernel ν.

A (simplified) theorem reads

Theorem (Du, T., Zhou '18)

Suppose that $\nu(t,dr) \equiv \nu(r)dr$, with $\int_0^\infty \nu(r)dr = \infty$ and let $\phi \in L^\infty(-\infty, 0; H^1(\mathbb{R}^d))$. Then $u(t,x) = \mathbb{E}\left[\phi\left(-X^{t,(\nu)}_{\tau_0(t)}, B^x_{\tau_0(t)}\right)\right]$ is a weak solution to (11).

Summary

1. **Marchaud-type fractional derivatives allow to meaningfully define time-nonlocal initial conditions for EEs** (extending Caputo-type EEs).
2. **The stochastic representation for the solution provides intuition for the time-nonlocal initial condition**, as the trapping time of the anomalous diffusion weights the initial condition.
3. **Marchaud-type EEs can be solved in terms of inhomogeneous Caputo-type EEs**.

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Thank you!