ON THE LOG-LOCAL PRINCIPLE FOR THE TORIC BOUNDARY

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ABSTRACT. Let $X$ be a smooth projective complex variety and let $D = D_1 + \cdots + D_l$ be a reduced normal crossing divisor on $X$ with each component $D_j$ smooth, irreducible, and nef. The log-local principle put forward in [12] conjectures that the genus 0 log Gromov–Witten theory of maximal tangency of $(X, D)$ is equivalent to the genus 0 local Gromov–Witten theory of $X$ twisted by $\bigoplus_{j=1}^l \mathcal{O}(-D_j)$. We prove that an extension of the log-local principle holds for $X$ a (not necessarily smooth) $\mathbb{Q}$-factorial projective toric variety, $D$ the toric boundary, and descendent point insertions.

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1. INTRODUCTION

Let $X$ be a smooth projective complex variety of dimension $n$ and let $D = D_1 + \cdots + D_l$ be an effective reduced normal crossing divisor with each component $D_j$ smooth, irreducible and nef. We can then consider two, a priori very different, geometries associated to the pair $(X, D)$:

- the $n$-dimensional log geometry of the pair $(X, D)$,
- the $(n + l)$-dimensional local geometry of the total space $\text{Tot} \left( \bigoplus_{j=1}^l \mathcal{O}_X(-D_j) \right)$.

The genus zero log Gromov–Witten invariants of $(X, D)$ virtually count rational curves $f : \mathbb{P}^1 \to X$ of a fixed degree $f_*[\mathbb{P}^1] \in \text{H}_2(X, \mathbb{Z})$, with insertions, such as passing through a number of general points, and with prescribed intersections with $D$. Such an $f$ is said to be of maximal tangency if $f(\mathbb{P}^1)$ meets each $D_j$ in only one point of full tangency. On the other hand, the local Gromov-Witten theory of $\text{Tot} \left( \bigoplus_{j=1}^l \mathcal{O}_X(-D_j) \right)$ is a way to study the local contribution of $X$ to the

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enumerative geometry of a compact \((n+l)\)-dimensional variety \(Y\) containing \(X\) with normal bundle \(\bigoplus_{j=1}^l O_X(-D_j)\).

The existence of a relation between the log and the local theory of \((X, D)\) was introduced by the log-local principle of \([12, \text{Conjecture 1.4}]\):

**Conjecture 1.1.** Let \(d\) be an effective curve class such that \(d \cdot D_j > 0\) for all \(1 \leq j \leq l\). After dividing by \(\prod_{j=1}^l (-1)^{d-D_j} d \cdot D_j\), the genus 0 log Gromov-Witten invariants of maximal tangency and class \(d\) of \((X, D)\) equal the genus 0 local Gromov-Witten invariants of class \(d\) of \(\text{Tot} \left( \bigoplus_{j=1}^l O_X(-D_j) \right)\) (with the same insertions).

**Theorem 1.2** (\([12]\)). The log-local principle holds if \(X\) is a smooth projective variety and \(D\) is smooth nef.

There are two natural directions to generalise the log-local principle further. The first is to investigate the correspondence at the level of BPS invariants \([7–9, 14, 23]\); this is proven for the pair of \(\mathbb{P}^2\) and smooth cubic in \([3]\) and will be investigated for toric Calabi-Yau 4-folds in \([4]\). The second natural question is to what extent the log-local principle generalises to the case when \(X\) and \(D\) are not smooth: log Gromov-Witten theory is indeed well-defined for any pair \((X, D)\) which is log smooth, but it is unclear how to define a local geometry in such generality. In the present paper, we consider a situation that goes beyond the smoothness assumptions of Conjecture 1.1 and where both log and local sides can be defined: we take for \(X\) a \(\mathbb{Q}\)-factorial projective toric variety and for \(D\) the toric boundary divisor of \(X\). As \(X\) is \(\mathbb{Q}\)-factorial, it makes sense to require that the components \(D_j\) of \(D\) are nef. We show in Proposition 2.1 that requiring each \(D_j\) to be nef forces \(X\) to be a product of fake weighted projective spaces. While such an \(X\) is not necessarily smooth, and \(D\) is typically not normal crossing, \((X, D)\) can naturally be viewed as a log smooth variety, and so log Gromov-Witten invariants of \((X, D)\) are well-defined. On the other hand, \(X\) can be naturally viewed as a smooth Deligne–Mumford stack, and the local geometry \(\text{Tot} \left( \bigoplus_{j=1}^l O_X(-D_j) \right)\) makes sense in the category of orbifolds. The local Gromov-Witten invariants can be defined using orbifold Gromov–Witten theory \([2]\), and it thus makes sense to ask if the genus 0 log invariants of maximal tangency of such a pair \((X, D)\) are related in the sense of Conjecture 1.1 to the corresponding local invariants. Our main result is the following Theorem 1.3; we refer to Theorems 3.1–3.3 for precise statements.

**Theorem 1.3.** Let \(X\) be a \(\mathbb{Q}\)-factorial projective toric variety and let \(D\) be the toric boundary divisor of \(X\). Assume that all the components \(D_j\) of \(D\) are nef. Then the genus 0 log Gromov-Witten invariants of maximal tangency of \((X, D)\), and the genus 0 local Gromov-Witten invariants of \((X, D)\), both with descendent point insertions, can be computed in closed form for all degrees. As a corollary, the log-local principle holds for the resulting invariants.

By dimension reasons, the log and local Gromov–Witten invariants of \((X, D)\) are generally non-zero only for one or two point insertions, and our proof proceeds by calculating both sides to obtain explicit closed formulas for these invariants for all \((X, D)\) (Theorems 3.1 and 3.2). To compute
the log invariants we use the tropical correspondence result [18] and an algorithm of [19] for the tropical multiplicity. The log-local principle of Conjecture 1.1 then predicts an explicit formula in all degrees for the local invariants, which we verify using local mirror symmetry techniques and a reconstruction result from small to big quantum cohomology.

Relation to [21] and [4]. After this paper was finished, we received the manuscript [21] where the log-local principle is considered for simple normal crossings divisors. The respective strategies have different flavours in the proof and complementary virtues in the outcome: [21] prove the log/local correspondence for $X$ smooth and $D_j$ a hyperplane section, with a beautiful geometric argument reducing the simple normal crossings case to the case of smooth pairs, and with no restrictions on $X$. The combinatorial pathway we pursued in the toric setting allows on the other hand to relax the hypotheses on the smoothness of $X$, the normal crossings nature of $D$, and the very ampleness of $D_j$, and it lends itself to a wider application to the case when $D$ is not the toric boundary and to a possible refinement to include all-genus invariants. We consider this specifically in the follow-up paper [4], where we prove the log-local principle for log Calabi–Yau surfaces with the components of the anticanonical divisor smooth and nef and suitably reformulate it to, and verify it for, the higher genus theory in these cases.

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2. Setup

2.1. Notation. Let $X$ be a $\mathbb{Q}$-factorial projective toric variety of dimension $n_X$ and let $D = D_1 + \cdots + D_l_D$ be the toric boundary divisor of $X$. In the foregoing discussion, we write $r_X := \text{rank } \text{Pic}(X)$ for the rank of the Picard group of $X$, so that $l_D = n_X + r_X$, and $\chi_X = \chi(X) := \dim_{\mathbb{C}} H(X, \mathbb{C})$ for the dimension of the cohomology of $X$. The variety $X$ has a natural presentation as a GIT quotient $C_{n_X+r_X}^+ / / ((\mathbb{C}^*)^{r_X} \times G_X)$ for $G_X$ a finite abelian group; for every $1 \leq j \leq l_D$, we write $D_j$ for the divisor corresponding to the $(\mathbb{C}^*)^{r_X} \times G_X$ reduction to $X$ of the $j$th coordinate hyperplane in $C_{n_X+r_X}^+$. Note in particular that $\sum_{j=1}^{l_D} D_j = -K_X$.

We also fix a further piece of notation, which will turn out to be convenient when dealing with the book-keeping of indices for products of fake weighted projective spaces. Let $m \in \mathbb{N}_0$. If $v = (v_1, \ldots, v_m) \in \mathbb{N}_0^m$ is a lattice point in the non-negative $m$-orthant, we write $|v| = \sum_{i=1}^m v_i$ for its 1-norm; in the following we will consistently use serif fonts for orthant points and italic fonts for their Cartesian coordinates. For $R$ a finitely generated commutative monoid with generators $\alpha_1, \ldots, \alpha_m$, $x = \alpha_1^{n_1} \cdots \alpha_m^{n_m} \in R$ a reduced word in $\alpha_i$, and $v \in \mathbb{N}_0^m$, we write $x^v$ for the product $\prod_i \alpha_i^{v_i n_i} \in R$. We introduce partial orders on the $m$-orthant by saying that $v \prec w$ (resp. $v \preceq w$)
if \( v_i < w_i \) (resp. \( v_i \leq w_i \)) for all \( i = 1, \ldots, m \). Also, we will write \( Q_{ij}^X \in \mathbb{Z}, i = 1, \ldots, r_X, j = 1, \ldots, n_X + r_X \), for the weight of the \( i \)-th factor of the \( (\mathbb{C}^*)^{r_X} \) torus action on the \( j \)-th affine factor of \( \mathbb{C}^{n_X + r_X} \).

**Definition 2.2.** A nef toric pair \((X, D)\) is a pair given by \( X \) a \( \mathbb{Q} \)-factorial complex projective toric variety with toric boundary divisor \( D = D_1 + \cdots + D_{l_D} \), such that all the components \( D_j \) are nef.

Nefness of all the components \( D_j \) of the toric boundary divisor imposes strong conditions on \( X \), as the Proposition 2.1 below shows.

**Definition 2.1.** Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric variety, and let \( \mathbb{C}^{n_X + r_X} \rightarrow \mathbb{C}^{n_X + r_X} \times G_X \) be its natural GIT description. We say that \( X \) is a fake weighted projective space if \( \mathbb{C}^{n_X + r_X} \rightarrow \mathbb{C}^{n_X + r_X} \times G_X \) is a weighted projective space.

**Proposition 2.1.** Let \( X \) be a \( \mathbb{Q} \)-factorial projective toric variety such that every effective divisor on \( X \) is nef. Then \( X \) is a product of fake weighted projective spaces.

**Proof.** By [11, Proposition 5.3], \( X \) admits a finite surjective toric morphism \( \prod \mathbb{P}^{n_i} \rightarrow X \). Let \( \Sigma \subset N \otimes \mathbb{R} \) be the fan of \( \prod \mathbb{P}^{n_i} \) and \( \Sigma' \subset N' \otimes \mathbb{R} \) the fan of \( X \). Then we have an injective morphism of lattices \( N \rightarrow N' \) of finite index. Identifying \( N \) with its image in \( N' \), \( \Sigma = \Sigma' \). It follows that \( X \) is the quotient of \( \prod \mathbb{P}^{n_i} \) by \( N'/N \). Hence \( X \) is a product of fake weighted projective spaces. \( \square \)

By Proposition 2.1, there is \( n_X \in \mathbb{N}^{r_X} \) such that \( n_X = |n_X| \) and \( X \) is a product of \( r_X \), \( n_i := (n_X)_i \)-dimensional fake weighted projective spaces,

\[
X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i} \left( w_X^{(i)} \right),
\]

with \( w_X^{(i)} = ((w_X)_1^{(i)}, \ldots, (w_X)_n^{(i)}) \in \mathbb{N}^{n_i+1} \), which we may assume not to have any common factors, and

\[
\mathbb{P}^{G_i} \left( w_X^{(i)} \right) := \mathbb{P} \left( w_X^{(i)} \right) \, G_i,
\]

for \( G_i \) a finite abelian group. Notice that, for fixed \( i \) and defining \( \varepsilon_i := \sum_{k=1}^{i-1} (n_k + 1) \), we have

\[
Q_{i,j+\varepsilon_i}^X = \begin{cases} (w_X)_j^{(i)} & 1 \leq j \leq n_i + 1, \\ 0 & \text{else}, \end{cases} \tag{2.1}
\]

independent of the \( G_i \). Let \( H_i := \text{pr}_i^* c_1 (O_{\mathbb{P}^{G_i}((w;i)(i))}) \) denote the pull-back to \( X \) of the (ori-) hyperplane class of the \( i \)-th factor of \( X \) and let \( H := H_1 \cdots H_{r_X} \). These generate the classical cohomology ring,

\[
\Pi^*(X, \mathbb{C}) = \mathbb{C}[H_1, \ldots, H_{r_X}] / \left< \left( H_i^{n_i+1} \right)^{r_X}_{i=1} \right>, \tag{2.2}
\]

which is independent of the \( G_i \), and we can take a homogeneous linear basis for \( H^*(X, \mathbb{C}) \) in the form \( \{H^l\}_{l \leq n_i} \). Notice, in particular, that

\[
[pt] = \prod_{i=1}^{r_X} |G_i| \prod_{i,j} (w_X)_j^{(i)} H^n X.
\]
Indeed, if $G_i$ is trivial, this follows from applying [16, Theorem 1] to each component in the product; and if $G_i$ is non-trivial, then the extra factor comes from the component-wise identification $H^i \mathbb{P}(w^{(i)}), \mathbb{C}) = H^i(\mathbb{P}(w^{(i)}), \mathbb{C})^{G_i}$. We will also write $d = (d_1, \ldots, d_{r_X})$ for the curve class $d_1H_1 + \cdots + d_rH_r$. We order the toric divisors $D_j$ of $X$, $j = 1, \ldots, |n_X| + r_X$ in such a way that that $Q^X_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0) \cdot D_j$ where the 1 is in the $i^{th}$ position. Finally, we define
\[ e^X_j(d) := \sum_i Q^X_{ij}d_i = d \cdot D_j, \quad e^X(d) := \sum_{j=1}^{l_D} e^X_j(d) = -d \cdot K_X. \tag{2.3} \]

2.2. Log Gromov-Witten invariants. Let $(X, D)$ be a nef toric pair and let $d$ be an effective curve class on $X$.\(^1\)

For the definition of log Gromov-Witten invariants, we endow\(^2\) $X$ with the divisorial log structure coming from $D$, and view $(X, D)$ as a log smooth variety. The log structure is used to impose tangency conditions along the components $D_j$ of $D$: in this paper we consider genus 0 stable maps into $X$ of class $d$ that meet each component $D_j$ in one point of maximal tangency $d \cdot D_j$. The appropriate moduli space $\overline{M}_{0,m}^{log}(X, D, d)$ of genus 0 $m$-marked maximally tangent stable log maps was constructed (in all generality) in [1,5,15]. There is a virtual fundamental class
\[ [\overline{M}_{0,m}^{log}(X, D, d)]^{vir} \in H_{2\overline{\nu}dim_{log}^{(X, D, d)}}(\overline{M}_{0,m}^{log}(X, D, d)), \]

where
\[ \overline{\nu}dim_{log}^{(X, D, d)} = -d \cdot K_X + \dim X - 3 + m - \sum_{j=1}^{l_D} (d \cdot D_j - 1) \]
\[ = n_X + m + l_D - 3 = 2n_X + r_X + m - 3. \]

Evaluating at the marked points $p_i$ yields the evaluation maps
\[ ev_i : \overline{M}_{0,m}^{log}(X, D, d) \rightarrow X. \]

For $L_i$ the $i^{th}$ tautological line bundle on $\overline{M}_{0,m}^{log}(X, D, d)$, whose fiber at $[f : (C, p_1, \ldots, p_m) \rightarrow X]$ is the cotangent line of $C$ at $p_i$, there are tautological classes $v_i := c_1(L_i)$. We are interested in the calculation of the genus 0 log Gromov–Witten invariants of maximal tangency of $(X, D)$ with 1 or 2 point insertions and $\psi$-class insertions at one point, defined as follows:
\[ R^X_{p_d} := \int_{[\overline{M}_{0,m}^{log}(X, D, d)]^{vir}} ev^*_1([pt]) \cup \psi^{n_X + r_X - 2}_1, \tag{2.4} \]
\[ R^X_{q_d} := \int_{[\overline{M}_{0,m}^{log}(X, D, d)]^{vir}} ev^*_1([pt]) \cup ev^*_2([pt]) \cup \psi^{r_X - 1}_2. \tag{2.5} \]

The invariant $R^X_{p_d}$ (resp. $R^X_{q_d}$) is a virtual count of rational curves in $X$ of degree $d = (d_1, \ldots, d_r)$ that meet each toric divisor $D_j$ in one point of maximal tangency $d \cdot D_j = \sum_{i=1}^r d_iQ^X_{ij} = e^X_j(d)$ and that pass through one point in the interior with $\psi^{n_X + r_X - 2}$ condition (resp. two points in the interior, one of which with a $\psi^{r_X - 1}$ condition).

\(^1\)Note that unlike in Conjecture 1.1 we do not require that $d \cdot D_j > 0$ for all $1 \leq j \leq l_D$.

\(^2\)We refer to [13] for an introduction to log geometry.
Remark 2.2. Having a point condition on $X$ cuts down the dimension of the moduli space by $n_X$. Therefore, the only generally non-vanishing genus 0 log invariants with point insertions and descendent insertions at one point are (2.4) and (2.5).

### 2.3. Local Gromov-Witten invariants

Let $(X,D)$ be a nef toric pair as in Definition 2.1 and write $X^\text{loc}_D := \text{Tot}(\bigoplus_i O_X(-D_i))$ for the target space of the local theory. By Proposition 2.1, we can view $X$ and $X^\text{loc}_D$ as the coarse moduli schemes of smooth Deligne–Mumford stacks $\mathcal{X}$ and $\mathcal{X}^\text{loc}_D$ over $\mathbb{C}$, where

$$
\mathcal{X} := \bigotimes_{i=1}^{r_X} \left( (\mathbb{C}^{(n_X)} \setminus \{0\}) / (\mathbb{C}^* \times G_i) \right),
$$

$$
\mathcal{X}^\text{loc}_D := \bigotimes_{i=1}^{r_X} \left( (\mathbb{C}^{(n_X)} \setminus \{0\}) \times (\mathbb{C}^{(n_X)}+1) / (\mathbb{C}^* \times G_i) \right).
$$

Even though $X^\text{loc}_D$ is not proper and may be singular, the locution “Gromov–Witten theory of $X^\text{loc}_D$” receives a meaning in terms of the orbifold Gromov–Witten theory of $\mathcal{X}$ twisted by $\bigoplus_i O_X(-D_i)$ [2, 6] and restricted over its non-stacky part, and we refer the reader in particular to [2] for the relevant background on the Gromov–Witten theory of Deligne–Mumford stacks. Let $\overline{M}_{0,m}(\mathcal{X},d)$ be the moduli stack of twisted genus 0 $m$-marked stable maps $[f: C \to \mathcal{X}]$ with $f_*([C]) = d \in H_2(\mathcal{X},\mathbb{Q})$, where $C$ is an $m$-pointed twisted curve [2], and write $\overline{M}_{0,m}(X,d)$ for the substack of twisted stable maps such that the image of all evaluation maps is contained in the zero-age component of the (rigidified, cyclotomic) inertia stack of $\mathcal{X}$. The stack $\overline{M}_{0,m}(X,d)$ can be equipped with a virtual fundamental class [2, Section 4.5], which induces a virtual fundamental class of pure homological degree over the stack $\overline{M}_{0,m}(X,d)$ of stable maps to the coarse moduli space,

$$
[\overline{M}_{0,m}(X,d)]^\text{vir} \in H_{2\mathbf{vdim}(X,D,d)}(\overline{M}_{0,m}(X,d),\mathbb{Q}),
$$

where

$$
\mathbf{vdim}(X,D,d) := -K_X \cdot d + \dim X + m - 3 = \mathbf{vdim}(X,D,d) + e_X(d) - l_D.
$$

Let now $d$ be such that $d \cdot D_j > 0$ for all $1 \leq j \leq l_D$. Then $H^0(C,f^* \bigoplus_{j=1}^{l_D} O_X(-D_j)) = 0$ for every twisted stable map $[f: C \to \mathcal{X}]$ with $f_*([C]) = d$, and so $\text{Ob}_D := R^1 \pi_* f^* \left( \bigoplus_{j=1}^{l_D} O_X(-D_j) \right)$ is a vector bundle on $\overline{M}_{0,m}(\mathcal{X},d)$, which is of rank $\sum_{j=1}^{l_D} (d \cdot D_j - 1)$ and has fibre $H^1(C,f^* \bigoplus_{j=1}^{l_D} O_X(-D_j))$ at a stable map $[f: C \to \mathcal{X}]$. Restricting to the zero-age component defines the virtual fundamental class

$$
[\overline{M}_{0,m}(X^\text{loc}_D,d)]^\text{vir} := [\overline{M}_{0,m}(X,d)]^\text{vir} \cap c_{\text{top}}(\text{Ob}_D) \in H_{2(\mathbf{vdim}(X,D,d) + l_D - e_X(d))}(\overline{M}_{0,m}(X,d),\mathbb{Q}),
$$

and we have

$$
\mathbf{vdim} \overline{M}_{0,m}(X^\text{loc}_D,d) = \mathbf{vdim}(X,D,d) - e_X(d) + l_D = \mathbf{vdim}(X,D,d).
$$

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3This means that the coarse moduli space of $\mathcal{C}$ is a pre-stable curve in the ordinary sense, with cyclic-quotient stackiness allowed at special points, and satisfying kissing (balancing) conditions for the stacky structures at the nodes. See [2, Section 4] for more details.
The restriction to the untwisted sector gives well-defined evaluation maps \( \text{ev}_i : \overline{M}_{0,n}(X, d) \rightarrow X \), and there are tautological classes \( \psi_i := c_1(L_i) \), where the fibre of \( L_i \) at a stable map \( [f : C \rightarrow X] \) is given by the cotangent line to the coarse moduli space of \( C \) at the \( i \)th point. The (untwisted) local Gromov–Witten invariants of \( (X, D) \) are then caps of pull-backs of classes in \( H^\bullet(X, \mathbb{C}) \) via the evaluation maps against the virtual fundamental class (2.7). In particular, the local counterparts of (2.4) and (2.5) are defined by

\[
\begin{align*}
p^X_d &:= \int_{[\overline{M}_{0,1}(X^\text{loc}_d))]^{\text{vir}}} \text{ev}^*_1([\text{pt}]) \cup \psi^n_X r^{-2}_X, \\
qu^X_d &:= \int_{[\overline{M}_{0,2}(X^\text{loc}_d))]^{\text{vir}}} \text{ev}^*_1([\text{pt}]) \cup \text{ev}^*_2([\text{pt}]) \cup \psi^{r-1}_X.
\end{align*}
\]

3. Main results

The following Theorems 3.1 and 3.2 compute the log and local Gromov-Witten invariants defined in Sections 2.2 and 2.3 in all degrees for a nef toric pair \( (X, D) \).

**Theorem 3.1.** Let \( (X, D) \) be a nef toric pair and let \( d \) be an effective curve class on \( X \). If there is \( j \) such that \( d \cdot D_j = 0 \), then \( R_{p^X_d} = R_{q^X_d} = 0 \). If \( d \cdot D_j > 0 \) for all \( 1 \leq j \leq l_D \), then we have

\[
\begin{align*}
R_{p^X_d} &= 1, \\
R_{q^X_d} &= \prod_{i=1}^{r_X} |G_i| \left( \prod_{i,j} (w_X)^{(i)}_j \right) d^n_X. 
\end{align*}
\]

We write \( \prod_{j \in \{1, \ldots, |n_X| + r_X \} \mid e_j^X(d) \neq 0} \) to mean the product of \( e_j^X(d) \) over \( j \in \{1, \ldots, |n_X| + r_X \} \).  

**Theorem 3.2.** Let \( (X, D) \) be a nef toric pair and let \( d \) be an effective curve class on \( X \). Then

\[
\begin{align*}
p^X_d &= \frac{(-1)^{e^X(d) - n_X - r_X}}{\prod_{j \in \{1, \ldots, |n_X| + r_X \} \mid e_j^X(d) \neq 0}}, \\
qu^X_d &= \prod_{i=1}^{r_X} |G_i| \left( \prod_{i,j} (w_X)^{(i)}_j \right) d^n_X p^X_d \\
&= \prod_{i=1}^{r_X} |G_i| \left( \prod_{i,j} (w_X)^{(i)}_j \right) d^n_X \frac{(-1)^{e^X(d) - n_X - r_X}}{\prod_{j \in \{1, \ldots, |n_X| + r_X \} \mid e_j^X(d) \neq 0}}. 
\end{align*}
\]

We deduce from these the log-local principle proved in the present paper.

**Theorem 3.3.** The log-local principle holds for nef toric pairs \( (X, D) \) for up to two point insertions and descendent insertions at one point, and with no assumptions on \( d \cdot D_j \). That is, for every effective curve class \( d \), and denoting

\[
g^X_d := \prod_{j=1}^{l_D} (-1)^{d \cdot D_j + 1} d \cdot D_j = (-1)^{e^X(d) - |n_X| - r_X} \prod_{j=1}^{|n_X| + r_X} e_j^X(d),
\]

7
the following relations between log and local Gromov-Witten invariants hold:

\[ N^X_d \cdot p^X_d = R p^X_d, \quad N^X_d \cdot q^X_d = R q^X_d. \]

Theorem 3.3 is a direct corollary of the combination of Theorem 3.1 and Theorem 3.2. We will prove Theorem 3.1 using a tropical correspondence principle, and Theorem 3.2 using an equivariant mirror theorem. We review these technical tools in Section 4, and explain how to apply them to the proofs of Theorems 3.1 and 3.2 in Sections 5 and 6 respectively.

4. Computational methods

4.1. The log side: tropical curve counts. Let \( X = \prod_{i=1}^{rX} \mathbb{P}^G_i(w^{(i)}) \) as in Section 2.1 and let \( \Sigma \subset N^R \) be the fan of \( X = X_\Sigma \); here \( N \simeq \mathbb{Z}^{\rho X} \) and \( N^R := N \otimes \mathbb{Z} \mathbb{R} \). Define furthermore \( N_Q := N \otimes \mathbb{Z} \mathbb{Q} \) and let \( M := \text{Hom}(N, \mathbb{Z}) \) be the dual of \( N \). Denote by \([D_1], \ldots, [D_{\rho X} + r_X]\) the rays of \( \Sigma \) corresponding to the irreducible effective toric divisors of \( X \). We use correspondence results with tropical curve counts as developed in [18,20,22] (see [13] for an introduction) and state them in the generality needed for our purposes.

Denote by \( \Gamma \) the topological realisation of a finite connected graph and by \( \Gamma \) the complement of a subset of 1-valent vertices. We require that \( \Gamma \) has no univalent and no bivalent vertices. The set of its vertices, edges, non-compact edges and compact edges is denoted by \( \Gamma^{[0]}, \Gamma^{[1]}, \Gamma^{[1]}_{\infty} \) and \( \Gamma^{[1]}_c \) respectively. \( \Gamma \) comes with a weight function \( w : \Gamma^{[1]} \to \mathbb{Z}_{\geq 0} \). The non-compact edges come with markings. Weight 0, resp. positive weight, non-compact edges are interior, resp. exterior, markings. There will be 1 or 2 interior point markings, which we denote by \( P_1 \) and \( P_2 \), and \( |n_X| + r_X \) exterior markings corresponding to the toric divisors, which we denote by \([D_1], \ldots, [D_{|n_X| + r_X}]\) as well.

Definition 4.1. A genus 0 degree \( d \) maximally tangent parametrised marked tropical curve in \( X \) consists of \( \Gamma \) as above and a proper continuous map \( h : \Gamma \to N^R \) satisfying

1. For \( E \in \Gamma^{[1]} \), \( h|_E \) is constant if and only if \( w(E) = 0 \). Otherwise, \( h|_E \) is an embedding into an affine line with rational slope.

2. Let \( V \in \Gamma^{[0]} \) with \( h(V) \in N_Q \). For edges \( E \ni V \), denote by \( u_{(V,E)} \) the primitive integral vector at \( h(V) \) into the direction \( h(E) \) (and set \( u_{(V,E)} = 0 \) if \( w(E) = 0 \)). The balancing condition holds:

\[ \sum_{E \ni V} w(E) u_{(V,E)} = 0. \]

3. For each exterior marking \( D_j \), \( h|_{D_j} \) is parallel to the ray \([D_j]\) and \( w(D_j) = d \cdot D_j \).

4. The first Betti number \( b_1(\Gamma) = 0 \).

If \((\Gamma', h')\) is another such parametrised tropical curve, then an isomorphism between the two is given by a homeomorphism \( \Phi : \Gamma \to \Gamma' \) respecting the discrete data and such that \( h = h' \circ \Phi \). A genus 0 degree \( d \) maximally tangent marked tropical curve then is an isomorphism class of such.
Moreover, we say that an interior marking \( E \) satisfies a \( \psi^k \)-condition if \( h(E) \) is a \( k + 2 \)-valent vertex.

Denote by \( T(p)^\chi_d \) the (moduli) space of genus 0 degree \( d \) maximally tangent tropical curves in \( X \) with the interior marking equipped with a \( \psi^{\chi_d + r_X - 2} \)-condition passing through a fixed general point in \( \mathbb{R}^{\chi_d + r_X} \). Denote by \( T(q)^\chi_d \) the moduli space of genus 0 degree \( d \) maximally tangent tropical curves in \( X \) with the two interior markings \( P_1 \) and \( P_2 \) mapping to two fixed general points in \( \mathbb{R}^{\chi_d + r_X} \) and such that \( P_2 \) has a \( \psi^{r_X - 1} \)-condition. We will see in Propositions 5.1 and 5.4 that each of \( T(p)^\chi_d \) and \( T(q)^\chi_d \) consist of one element. Since \( T(p)^\chi_d \) and \( T(q)^\chi_d \) are finite hence, their elements are rigid [19, Definition 2.5].

Counts of tropical curves are weighted with appropriate multiplicities. There are a number of ways of defining the multiplicity \( \text{Mult}(\Gamma) \) of \( \Gamma \). The version we use was formulated (for \( X \) smooth) in [19, Theorem 1.2]. We state it for our setting. Set \( A := \mathbb{Z}[N] \otimes_{\mathbb{Z}} \Lambda^s M \). For \( n \in \mathbb{N} \) and \( \alpha \in \Lambda^s M \), write \( z^n \alpha \) for \( z^n \otimes \alpha \) and \( \iota_n \alpha \) for the contraction of \( \alpha \) by \( n \). Recall that if \( \alpha \in \Lambda^s M \), then \( \iota_n \alpha \in \Lambda^{s-1} M \). For \( k \geq 1 \), define \( \ell_k : A^{\otimes k} \to A \) via

\[
\ell_k(z^{n_1} \alpha_1 \otimes \cdots \otimes z^{n_k} \alpha_k) := z^{n_1 + \cdots + n_k} \iota_{n_1 + \cdots + n_k} (\alpha_1 \wedge \cdots \wedge \alpha_k).
\]

Let now \( h : \Gamma \to N_{\mathbb{R}} \) be in \( T(p)^\chi_d \) or \( T(q)^\chi_d \) and choose a vertex \( V_\infty \) of \( \Gamma \). Consider the flow on \( \Gamma \) with sink vertex \( V_\infty \). To each edge \( E \) of \( \Gamma \), we inductively associate an element \( \zeta_E = z^{n_E} \alpha_E \in A \), well-defined up to sign:

- For the exterior markings, set \( \zeta_{D_j} = z^{w(D_j)} \Delta(j) \), where \( \Delta(j) \) is the primitive generator of \([D_j]\).
- For an interior marking \( P \), set \( \zeta_P \) to be one of the two generators of \( \Lambda^{\chi_d} M \).
- If \( E_1, \ldots, E_k \) are the edges flowing into a vertex \( V \neq V_\infty \) and \( E_{\text{out}} \) is the edge flowing out, set \( \zeta_{E_{\text{out}}} = \ell_k(\zeta_{E_1} \otimes \cdots \otimes \zeta_{E_k}) \).

By [19, Theorem 1.2], \( \zeta_\Gamma := \prod_{E \ni V_\infty} \zeta_E \in z^0 \otimes \Lambda^{\chi_d} M \) and \( \text{Mult}(\Gamma) \) is the index of \( \zeta_\Gamma \) in \( \Lambda^{\chi_d} M \). It then follows from [18, Theorem 1.1] that \( R(p)^\chi_d \) is the number of \( \Gamma \) in \( T(p)^\chi_d \) counted with multiplicity \( \text{Mult}(\Gamma) \), and \( R(q)^\chi_d \) is the weighted cardinality of \( \{ \Gamma \in T(q)^\chi_d \} \), each weighted by \( \text{Mult}(\Gamma) \).

**Remark 4.1.** Note that a priori [18, Theorem 1.1] is stated for smooth varieties; in the cases of interest to us, however, the curves never meet the deeper toric strata and the arguments of [18] carry through.

### 4.2. The local side: mirror symmetry for toric stacks.

The second technical result we will use for the calculation of local Gromov-Witten invariants is Theorem 4.2 below. Consider a torus \( T \simeq \mathbb{C}^* \) acting on \( X_D^{\text{loc}} := \text{Tot}(\bigoplus_i \mathcal{O}_X(-D_i)) \) transitively on the fibres and covering the trivial action on the image of the zero section. We will denote by \( \lambda := c_1(\mathcal{O}_X(1)) \) the polynomial generator of the \( T \)-equivariant cohomology of a point, \( H_T(\text{pt}) = H(BT) \simeq \mathbb{C}[\lambda] \). The basis elements \( H^1 \) of Section 2.1 for the cohomology of \( X \) have canonical \( T \)-equivariant lifts, which by a slight abuse of notation we denote with the same symbol, to cohomology classes in \( X_D^{\text{loc}} \) forming a \( \mathbb{C}(\lambda) \) basis of
$H_T(X^\text{loc}_D)$, where as usual $\mathbb{C}(\lambda)$ is the field of fractions of $H_T(pt)$. The $T$-equivariant cohomology $H_T(X^\text{loc}_D)$ is furthermore endowed with a non-degenerate, symmetric bilinear form given by the restriction of the $T$-equivariant Chen–Ruan [10, Section 2.1] pairing on the untwisted component of the inertia stack of $X^\text{loc}_D$, 

$$\eta_{lm} := (H^1, H^m)_{X^\text{loc}_D} := \int_X \frac{H^1 \cup H^m}{\cup_i e_T(O_X(-D_i))},$$

where $e_T$ denotes the $T$-equivariant Euler class.

Let now $\tau \in H_T(X^\text{loc}_D)$. The \textit{equivariant big $J$-function} of $X^\text{loc}_D$ is the formal power series 

$$J_{\text{big}}^{X^\text{loc}_D}(\tau, z) := z + \tau + \sum_{d \in \text{NE}(X)} \sum_{n \in \mathbb{Z}^+} \sum_{m \leq n_X} \frac{1}{n!} \left\langle \tau_1, \ldots, \tau_t, \frac{H^1}{z - \psi} \right\rangle_{0,n,1,d}^X H^m \eta_{lm}^{z},$$

where we employed the usual correlator notation for Gromov-Witten invariants, 

$$\left\langle \tau_1^{\psi_1^{k_1}}, \ldots, \tau_n^{\psi_n^{k_n}} \right\rangle_{0,n,d}^{X^\text{loc}_D} := \int_{(\text{vir}^{X^\text{loc}_D}, d)} \psi_i^* (\tau_i) \psi_i^{k_i},$$

and $\eta_{lm} := (\eta^{-1})_{lm}$. Restriction to $t = t_0$ yields $J_{\text{big}}$ and use of the Divisor Axiom leads to the \textit{equivariant small $J$-function} of $X^\text{loc}_D$,

$$J_{\text{small}}^{X^\text{loc}_D}(t, z) := z e^{\sum_{i \phi_i / z}} \left( 1 + \sum_{d \in \text{NE}(X)} \sum_{m \leq n_X} e^{\sum_{i \phi_i}} \left\langle \frac{H^1}{z - \psi_1} \right\rangle_{0,1,d}^{X^\text{loc}_D} H^m \eta_{lm}^{z} \right).$$

The $n$-pointed genus zero Gromov–Witten invariants with one marked descendant insertion (respectively, the 1-pointed genus zero descendant invariants) of $X^\text{loc}_D$, and no twisted insertions, can thus be read off from the formal Taylor series expansion of $J_{\text{big}}$ (resp., $J_{\text{small}}$) at $z = \infty$.

The following theorem provides an explicit hypergeometric presentation of $J_{\text{small}}^{X^\text{loc}_D}(t, z)$. Let $\kappa_j := c_1(O(-D_j))$ be the $T$-equivariant first Chern class of $O(D_j)$ and $y_i \in \text{Spec} \mathbb{C}[[t]]$, $i = 1, \ldots, r_X$ be variables in a formal disk around the origin. Writing $(x)_n := \Gamma(x+n)/\Gamma(x)$ for the Pochhammer symbol of $(x, n)$ with $n \in \mathbb{Z}$, the $T$-equivariant I-\textit{functions} of $X$ and $X^\text{loc}_D$ are defined as the $H_T(X)$ and $H_T(X^\text{loc}_D)$ valued Laurent series 

$$I^X(y, z) := z 1_H(X) + \prod_i y_i^{H_i/z} \sum_{d \in \text{NE}(X)} \prod_i y_i^{d_i} z^{d-K_X} \frac{1}{\prod_j (\frac{\kappa_j}{z} + 1)^{d-D_j}},$$

$$I^{X^\text{loc}_D}(y, z) := z 1_H(X) + \prod_i y_i^{H_i/z} \sum_{d \in \text{NE}(X)} \prod_i y_i^{d_i} z^{d(K_X+D) - 1} \frac{1}{\prod_j (\frac{\kappa_j}{z} + 1)^{d-D_j}},$$

and their \textit{mirror maps} as their formal $O(z^0)$ coefficient,

$$\tilde{t}^i_X(y) := [z^0 1_H] I^X(y, z),$$

$$\tilde{t}^i_{X^\text{loc}_D}(y) := [z^0 1_H] I^{X^\text{loc}_D}(y, z).$$

Note that $X$ and $X^\text{loc}_D$ are smooth toric Deligne–Mumford stacks with coarse moduli schemes $X$ and $X^\text{loc}_D$ that are projective over their affinisation, and at this level of generality a result of [10]
can be applied to provide a Givental-style equivariant mirror statement for them, as follows. In the language of [10], the $I$-functions (4.5) and (4.6) are the stacky $I$-functions of [10, Definition 28 and 29] for $X$ and $\mathcal{A}_D^{\text{loc}}$ respectively, restricted to insertions in the zero-age sector of their inertia stack. The main result of [10] identifies the small $J$-function of a semi-projective toric Deligne–Mumford stack to its stacky $I$-function, up to a change-of-variables given by its $\mathcal{O}(z^0)$ term as in (4.7). In particular, the following statement is a projection to the untwisted sector of $X$ and $\mathcal{A}_D^{\text{loc}}$ of [10, Theorem 31 and Corollary 32].

**Theorem 4.2** ([10]). We have

\begin{align}
J^X_{\text{small}}(\tilde{t}_X(y), z) &= I^X(y, z), \\
J^{X,D}_{\text{small}}(\tilde{t}_X(y) + \tilde{t}_{X,D}^{\text{loc}}(y) - \log y, z) &= I^{X,D}_{\text{loc}}(y, z).
\end{align}

(4.8)

5. The log side: proof of Theorem 3.1

If there is $j$ such that $d \cdot D_j = 0$, then $R\mathfrak{p}_d^X = R\mathfrak{q}_d^X = 0$ because the virtual dimension of the moduli problem is negative in that case. For the remainder of this section, we therefore assume that $d \cdot D_j > 0$ for all $1 \leq j \leq l_D$.

Recall that $X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i}(w^{(i)})$ is given by the fan $\Sigma \subset N_\mathbb{R}$ where $N \simeq \mathbb{Z}^{nj}$ and $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$. Then $\Sigma$ is the product fan of the fans $\Sigma_i \subset (N_i)_\mathbb{R}$ of $\mathbb{P}^{G_i}(w^{(i)})$, where $N_i \simeq \mathbb{Z}^{n_i}$. Writing $\varepsilon_i := \sum_{k=1}^{i-1} (n_k + 1)$, the rays of $\Sigma_i$ are $[D_{\varepsilon_i + 1}], \ldots, [D_{\varepsilon_i + n_i + 1}]$ and have primitive generators $\Delta(\varepsilon_i + 1), \ldots, \Delta(\varepsilon_i + n_i + 1)$, which satisfy

$$w^{(i)}_{\varepsilon_i + 1}\Delta(\varepsilon_i + 1) + \cdots + w^{(i)}_{\varepsilon_i + n_i + 1}\Delta(\varepsilon_i + n_i + 1) = 0.$$ 

Write $L_i$ for the sublattice of $N_i$ generated by the $[\Delta(\varepsilon_i + j)]$ and write $B_i$ for the change of basis matrix from a $\mathbb{Z}$-basis of $N_i$ to a $\mathbb{Z}$-basis of $L_i$. Then

$$|\det B_i| = |N_i/L_i| = |G_i|.$$

Let $L$ be the sublattice of $N$ generated by the $L_i$ and let $B$ be the change of basis matrix from $N$ to $L$ given by the $B_i$. We have that $|\det B| = \prod_{i=1}^{r_X} |\det B^i| = \prod_{i=1}^{r_X} |G_i|$.

**Proposition 5.1.** The set $T(\mathfrak{p})_d^X$ has an unique element $\Gamma$ of multiplicity 1.

**Proof.** Each element $\Gamma$ of $T(\mathfrak{p})_d^X$ has $|n_X| + r_X$ exterior markings (=rays) parallel to the rays $[D_1], \ldots, [D_{|n_X| + r_X}]$ and one vertex (=unique interior marking) with valency $|n| + r_X$. Thus the only possibility is that $\Gamma$ is the translate of the rays of the fan of $X$. Write $\zeta$ for one of the two generators of $\Lambda^{|n_X|}M$. Then $\text{Mult}(\Gamma)$ is given by the index of

$$\zeta \prod_{j=1}^{[n_X]+r_X} z^{e_X(d)\Delta(j)} = \zeta \in \Lambda^{|n_X|}M$$

in $\Lambda^{|n_X|}M$, which equals 1. \qed
It follows from Proposition 5.1 and the correspondence result of [18] that
\[ R\varphi_d^X = 1. \]

We calculate the multiplicity of the element of \( T(q)_d^X \) in three steps of increasing generality.

**Proposition 5.2.** Assume that \( X \) is the fake weighted projective plane \( \mathbb{P}^G(w_1, w_2, w_3) \), where we assumed that \( \gcd(w_1, w_2, w_3) = 1 \). Then \( T(q)_d^X \) has an unique element of multiplicity \( |G| w_1 w_2 w_3 d^2 \).

**Proof.** From \( w_1 \Delta(1) + w_2 \Delta(2) + w_3 \Delta(3) = 0 \), it follows that \( |\Delta(1) \wedge \Delta(2)| = w_3 \det B \). Choose the rays \( \{\Delta(1), \Delta(2)\} \) of \( N_\mathbb{R} \). In this basis, choose \( P_1 \) to be \((1,0)\) and \( P_2 \) to be \((0,1)\). Then the unique genus \( 0 \) degree \( d \) maximally tangent tropical curve passing through \( P_1 \) and \( P_2 \) consists of the rays \( [D_1], [D_2], [D_3] \), meeting at \( 0 = (0,0) \), and with weights \( w_j d \) on \([D_j]\).

Choose \( 0 \) to be the sink vertex and let \( E_1 \), resp. \( E_2 \), be the edge connecting \( 0 \) with \( P_1 \), resp. \( P_2 \). Choose moreover \( \{e_1, e_2\} \) to be a \( Z \)-basis of \( M \) with dual basis \( \{e_1^*, e_2^*\} \). Then
\[
\zeta_{E_1} = \ell_2(\zeta_{D_1} \otimes \zeta_{P_1}) = \ell_2(\mathbb{Z} w_1 d \Delta(1) \otimes (e_1^* \wedge e_2^*)) = \mathbb{Z} w_1 d \Delta(1)_{w_1 d \Delta(1)}(e_1^* \wedge e_2^*)
\]
\[
= \mathbb{Z} w_1 d \Delta(1) ((\ell_{w_1 d \Delta(1)} e_1^*) \wedge e_2^* - e_1^* \wedge \ell_{w_1 d \Delta(1)}(e_2^*)) = \mathbb{Z} w_1 d \Delta(1) e_1^* (w_1 d \Delta(1)) e_2^* - e_2^* (w_1 d \Delta(1)) e_1^*.
\]
Similarly
\[
\zeta_{E_2} = \mathbb{Z} w_2 d \Delta(2) (e_1^* (w_2 d \Delta(2)) e_2^* - e_2^* (w_2 d \Delta(2)) e_1^*),
\]
and
\[
\zeta_{r} = \zeta_{D_3} \zeta_{E_1} \zeta_{E_2} = -e_1^* (w_1 d \Delta(1)) e_2^* (w_2 d \Delta(2)) e_2^* \wedge e_1^* - e_2^* (w_1 d \Delta(1)) e_1^* (w_2 d \Delta(2)) e_1^* \wedge e_2^*
\]
\[
= w_1 w_2 d^2 (e_1^* (\Delta(1)) e_2^* (\Delta(2)) - e_2^* (\Delta(1)) e_1^* (\Delta(2))) e_1^* \wedge e_2^*
\]
\[
= w_1 w_2 d^2 |\Delta(1) \wedge \Delta(2)| e_1^* \wedge e_2^* = w_1 w_2 w_3 d^2 |\det B| e_1^* \wedge e_2^*,
\]
which is indeed of index \( |G| w_1 w_2 w_3 d^2 \) in \( \Lambda^2 M \). \( \square \)

**Proposition 5.3.** Assume that \( r_X = 1 \), i.e. \( X = \mathbb{P}^G(w_1, \ldots, w_{n+1}) \) and that \( n \geq 3 \). Then the set \( T(q)_d^X \) has an unique element \( \Gamma \) of multiplicity \( |G| \prod_{j=1}^{n+1} w_j d^n \).

**Proof.** We choose as basis of \( N_\mathbb{R} \) the basis \( \{\Delta(1), \ldots, \Delta(n)\} \). We choose our second point (interior marking) \( P_2 \) to have coordinate \( (a_1, \ldots, a_n) \) for \( a_i < 0 \) and general. We choose our first marked point \( P_1 \) to have coordinate \( (b,0,\ldots,0) \) for \( b > 0 \) large enough so that restricted to the halfspace \( \{(x_1,\ldots,x_n)|x_1 > b\} \), any \( h \in T(q)_d^X \) is affine linear with image \( (b,0,\ldots,0)+\mathbb{R}_{>0} \Delta(1) \) and weight \( e_1(d) \).

For \( 1 < j \leq n \), write \( w_{1j} \Delta(1) := -w_1 \Delta(1) - w_j \Delta(j) \) with \( \Delta(1j) \) primitive and \( w_{1j} \in \mathbb{N} \). Consider the finite abelian group
\[
G^j := ((\langle \Delta(1), \Delta(2) \rangle_\mathbb{R} \cap N) / \langle \Delta(1), \Delta(2), \Delta(1j) \rangle).
\]
Given \( \Gamma \in T(q)_d^X \), projecting to the plane \( \langle \Delta(1), \Delta(2) \rangle_\mathbb{R} \) leads to a genus \( 0 \) maximally tangent tropical curve in \( \mathbb{P}^{G^j}(w_1, w_j, w_{1j}) \) passing through \( 2 \) general points. By Proposition 5.2, there is
only one such curve (and it has multiplicity \(|G^2| w_1 w_j w_{1j} d^2\)). These curves lift to a unique maximally tangent curve \(h : \Gamma \to N_R\).

Choose \(P_2\) to be the sink vertex and consider the associated flow. Since the \(a_i\) are chosen to be general, on the set \(\{(x_i) | x_i < a_i\}\), \(h\) is affine linear with slope parallel to \(\Delta(n + 1)\). We reorder the \(\Delta(j)\) such that following the flow from \(P_2\), the rays that are added to \(\Gamma\) are successively translates of \([D_n], [D_{n-1}], \ldots, [D_2]\). Note that all vertices are 3-valent since \(P_1\) and \(P_2\) are in general position. Starting at \(P_1\) and following the flow, we label the compact edges successively \(E_1, \ldots, E_n\). Choose a \(\mathbb{Z}\)-basis \(e_1, \ldots, e_n\) of \(N\). Then

\[
\zeta_{E_1} = \ell_2(z^{w_1d\Delta(1)} \otimes e_1^* \wedge \cdots \wedge e_n^*) = z^{w_1d\Delta(1)} \ell_{w_1d\Delta(1)}(e_1^* \wedge \cdots \wedge e_n^*).
\]

At the next step,

\[
\zeta_{E_2} = z^{w_1d\Delta(1)+w_2d\Delta(2)} \ell_{w_1d\Delta(1)+w_2d\Delta(2)} \circ \ell_{w_1d\Delta(1)}(e_1^* \wedge \cdots \wedge e_n^*)
\]

\[
= z^{w_1d\Delta(1)+w_2d\Delta(2)} \ell_{w_1d\Delta(1)+w_2d\Delta(2)}(e_1^* \wedge \cdots \wedge e_n^*)
\]

\[
= z^{w_1d\Delta(1)+w_2d\Delta(2)} \ell_{w_1w_2d\Delta(1),\Delta(2)}(e_1^* \wedge \cdots \wedge e_n^*).
\]

Iterating this process, we obtain that

\[
\zeta_{E_n} = z^{w_1d\Delta(1)+\cdots+w_n d\Delta(n)} \ell_{w_1,\ldots w_n d\Delta(n)}(e_1^* \wedge \cdots \wedge e_n^*).
\]

Since \(w_1 \Delta(1) + \cdots + w_{n+1} \Delta(n+1) = 0\), \(|\Delta(1) \wedge \cdots \wedge \Delta(n)| = w_{n+1} \det B\) and hence

\[
\zeta_\Gamma = \ell_{w_1,\ldots w_n d\Delta(n)}(e_1^* \wedge \cdots \wedge e_n^*) e_1^* \wedge \cdots \wedge e_n^*
\]

\[
= w_1 \cdots w_n d^n |\Delta(1) \wedge \cdots \wedge \Delta(n)| e_1^* \wedge \cdots \wedge e_n^*
\]

\[
= w_1 \cdots w_n w_{n+1} d^n |\det B| e_1^* \wedge \cdots \wedge e_n^*,
\]

which is indeed of index \(|G| w_1 \cdots w_n w_{n+1} d^n\) in \(\Lambda^n M\).

\[\square\]

**Proposition 5.4.** Let \(X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i} (w^{(i)})\) be the product of fake weighted projective spaces. Then the set \(T(q)_d^X\) has a unique element \(\Gamma\) of multiplicity \(\prod_{i=1}^{r_X} |G_i| \left( \prod_{i,j} (w_X)^{i+1} \right) d^n X\).

**Proof.** Label the last \(r_X\) divisors \(D_{|\alpha|+1}, \ldots, D_{|\alpha|+r_X}\) to be coming from distinct components of \(X\). Then \([D_1], \ldots, [D_{|\alpha|}] \in \Sigma^{[1]}\) form a \(\mathbb{R}\)-basis of \(N_R\). To calculate \(Rq^X_d\), we choose the marking \(P_2\) with the \(\psi^{r_X-1}\) condition to be the origin 0. We choose the marking \(P_1\) a general point that has positive coordinates with respect to the above basis. Then the \(r_X + 1\) incoming rays at \(P_2\) are necessarily \(D_{|\alpha|+1}, \ldots, D_{|\alpha|+r_X}\) with weights \(e_j^X(d)\) and a primitive vector in direction \(-e_{|\alpha|+1}(d)D_{|\alpha|+1} - \cdots - e_{|\alpha|+r_X}(d)D_{|\alpha|+r_X}\) with appropriate weight.

There is only one way to make a maximally tangent tropical curve \(\Gamma\) passing through \(P_1\) out of it. To see this, for each \(i\), consider the map of fans \(\Sigma \to \Sigma_i\) corresponding to the projection to the \(i\)th component. In \(\Sigma_i\) the tropical curve becomes straight at 0 and hence we are looking at maximally tangent curves of degree \(d_i\) passing through two general points. By Proposition 5.3, there is only one such. Moreover, the curve in \(N_R\) is uniquely determined by these projections.
Choose \( P_2 \) to be the sink vertex. Then the multiplicity of \( \Gamma \) is calculated as in Proposition 5.3 to be \( \prod_{i=1}^{r_X} |G_i| \left( \prod_{i,j} (w_X)_{ij}^{(i)} \right) d^{n_X} \).

The correspondence principle of [18] then entails that

\[
Rq^X_{d} = \prod_{i=1}^{r_X} |G_i| \left( \prod_{i,j} (w_X)_{ij}^{(i)} \right) d^{n_X},
\]

concluding the calculations of the logarithmic invariants of Theorem 3.1.

6. The local side: proof of Theorem 3.2

6.1. The Poincaré pairing. As in Section 4.2, we consider the scalar \( T \simeq \mathbb{C}^* \) action on \( X_D^{\text{loc}} \) that covers the trivial action on the base \( X \), and denote \( \lambda = c_1(O_{BC*}(1)) \) for the corresponding equivariant parameter. Notice that for any \( l \leq n_X \), the Gram matrix \( \eta_{lm} \) for the restriction to the untwisted sector of the \( T \)-equivariant Chen-Ruan pairing (4.1) of \( X_D^{\text{loc}} \) satisfies

\[
\eta_{n_X} = \int \frac{H^{1+n_X}}{e_T(N_X/X_D^{\text{loc}})} = \int \frac{H^{1+n_X}}{e_T(N_X/X_D^{\text{loc}})[0]} = \int \frac{H^{1+n_X}}{\prod_{i=1}^{n_X} \eta_{l,i}^{(i)}} \lambda^{n_X+r_X} \lambda^{-1} = \begin{cases} 1, & \text{if } l = 0, \\ \lambda^{n_X+r_X} / \lambda^{n_X-1}, & \text{else,} \end{cases}
\]

for degree reasons. Also, \( \eta_{lm} = 0 \) if \( |l| + |m| > |n_X| \) for the same reason: this means that \( \eta_{lm} \) is upper anti-triangular, and \( \eta_{lm}^* := (\eta^{-1})_{lm} \) is lower anti-triangular with anti-diagonal elements \( \eta_{l,n_X-l}^* = 1/\eta_{l,n_X-l} \).

6.2. One pointed descendants. In the following, let \( y = y_1 \cdots y_{r_X} \) and \( Q = e^{t_1 \cdots + t_{r_X}} \). From (4.4), we have

\[
J_{D}^{X_{n_X}}(t, z) := \prod_{i=1}^{r_X} e^{t_i H_i/z} \left[ 1 + \sum_{d,a,l,m} Q^d z^{-a-2} \left( H^l \psi^a \right)^{X_D^{\text{loc}}}_{1,0,1} \eta_{lm} H^m \right] = \sum_m (J_{D}^{X_{n_X}})^m H^m. \quad (6.2)
\]

Using (6.1), we get that the component of the small, twisted \( J \)-function along the identity class is

\[
(J_{D}^{X_{n_X}})^{[0]} := z \left[ 1 + \sum_{d,a,l} Q^d z^{-a-2} \left( H^l \psi^a \right)^{X_D^{\text{loc}}}_{1,0,1} \eta_{l0} \right],
\]

\[
= z \left[ 1 + \sum_{d,a} \frac{\lambda^{n_X+r_X}}{\prod_i |G_i| \prod_{i,j} (w_X)_{ij}^{(i)}} \sum_{d,a} Q^d z^{-a-2} \left( X_D^{\text{loc}} \right)^{X_D^{\text{loc}}}_{0,1,1} \right],
\]

\[
= z \left[ 1 + \lambda^{n_X+r_X} \sum_{d,a} Q^d z^{-a-2} \left( X_D^{\text{loc}} \right)^{X_D^{\text{loc}}}_{0,1,1} \right]. \quad (6.3)
\]

Therefore our first set of invariants (2.8) can be computed from (6.3) as

\[
p_{d}^{X} := \left( \left[ pt \right] \psi^{n_X} + r_X - 2 \right)_{0,1,1} = \frac{1}{\lambda^{n_X+r_X}} \left[ z^{-n_X-r_X} \left( J_{D}^{X_{n_X}} \right)^{[0]} \right]. \quad (6.4)
\]
To compute the r.h.s. we use Theorem 4.2. For quantities $a(j)$ depending on $e_j^X(d)$, the notation $\prod_j a(j)$ refers to the product of $a(j)$ over $j \in \{1, \ldots, |n_X| + r_X \mid e_j^X(d) \neq 0\}$. From (4.5) and (4.6), the $I$-functions of $X$ and $X^\text{loc}_D$ are

$$I^X(y, z) := z \sum_d r^X \prod_{i=1}^{r^X} y_i^{H_i/z + d_i} \frac{1}{\prod_{m_j=1}^{e_j^X(d)} (m_jz + \sum_i Q_j^X H_i)} = \sum_m (I^X_{\text{loc}})^{|m|} H^m,$$  

(6.5)

$$I^{X^\text{loc}}_{\text{loc}}(y, z) := z \sum_d r^X \prod_{i=1}^{r^X} y_i^{H_i/z + d_i} \frac{1}{\prod_{m_j=1}^{e_j^X(d)} (m_jz + \sum_i Q_j^X H_i)} = \sum_m (I^{X^\text{loc}}_{\text{loc}})^{|m|} H^m.$$  

(6.6)

**Lemma 6.1.** The mirror maps of $X$ and $X^\text{loc}_D$ are trivial,

$$\tilde{t}_i^X(y) = \tilde{t}_i^{X^\text{loc}}(y) = \log y_i.$$  

(6.7)

**Proof.** This is a straightforward calculation from (6.5) and (6.6). Keeping track of the powers of $z$ in the general summands entails that $I^X(y, z) = z + \sum_i \log y_i H_i + \mathcal{O}(1/z) = I^{X^\text{loc}}_{\text{loc}}(y, z)$, from which the claim follows. \hfill \square

By the previous Lemma and (6.4), to compute $p_d^X$ we just need to evaluate the component of the $I$-function of $X^\text{loc}_D$ along the identity, divide by $\lambda^{\sum_i \log y_i H_i}$, and isolate the coefficient of $\mathcal{O}(z^{-|n_x| - r_x})$. We have

$$(I^{X^\text{loc}}_{\text{loc}})^{|0|} = z \sum_d y^d \prod_j \prod_{m_j=0}^{e_j^X(d)-1} (\lambda - m_j z) = z \sum_d y^d \prod_j \prod_{m_j=0}^{e_j^X(d)-1} (\lambda - m_j z).$$  

(6.8)

The numerator in the general summand of (6.8) is divisible by $\lambda^{|n_x| + r_x}$ (corresponding to setting all $m_j = 0$ in the product):

$$\prod_j \prod_{m_j=0}^{e_j^X(d)-1} (\lambda - m_j z) = \lambda^{|n_x| + r_x} \prod_j \prod_{m_j=1}^{e_j^X(d)-1} (\lambda - m_j z),$$  

(6.9)

hence dividing by $\lambda^{|n_x| + r_x}$ we get

$$\prod_j \prod_{m_j=1}^{e_j^X(d)-1} (\lambda - m_j z) = (-z)^{e_j^X(d)-|n_x|-r_x} \left( \prod_j (e_j^X(d) - 1)! + \mathcal{O}(1/z) \right).$$  

(6.10)

In particular this implies that

$$p_d^X = \left[ \frac{1}{\lambda^{\sum_i \log y_i H_i}} \left( z^{-|n_x| - r_x} y^d \right) \right]_{0, 1, d} = \frac{1}{\lambda^{\sum_i \log y_i H_i}} \left( z^{-|n_x| - r_x} y^d \right) (I^{X^\text{loc}}_{\text{loc}})^{|0|} = \frac{(-1)^{e_j^X(d)-|n_x|-r_x}}{\prod_j (e_j^X(d) - 1)!}.$$  

(6.11)
proving the first part of Theorem 3.3.

6.2.1. Two pointed descendents. Let us now turn to the computation of \( \mathfrak{q}_d^X \). We start with the following observation: from (6.6), we have

\[
I^{X_{DL}}(y, z) := z + \sum_{l \leq n_X} \frac{1}{z^{|l|+1}} \left[ \prod_{i=1}^{r_X} \log y_i \frac{H_i^l}{l_1!} \right] + O \left( \frac{1}{z} \right). \tag{6.12}
\]

This follows immediately from the fact that

\[
\prod_{j=1}^{n_X} \prod_{m_j=0}^{r_X} e^{X_j(d)} - 1 \left( \lambda - m_j z - \sum_j Q_{ij}^X H_i \right) = O \left( z - \sum_j \theta(c_j^X(d)) \right), \tag{6.13}
\]

where \( \theta(x) = 0 \) (resp. \( \theta(x) = 1 \)) for \( x = 0 \) (resp. \( x > 0 \)).

From this we deduce the following Lemma. For \( t \in H_T(X_{DL}^X) \), let \( \hat{s}_t \) denote the big quantum cohomology product,

\[
H^1 \hat{s}_t H^m := \sum_{d \in NE(X)} \sum_{n \in \mathbb{N}} \sum_{l \leq n_X} \langle H^1, H^m, H^t, \ldots, t \rangle_{0, \lambda, m, d} \eta^k H^k \tag{6.14}
\]

and \( \hat{s}_y \) its restriction to small quantum cohomology at \( t = \sum_i \log y_i H_i \),

\[
H^1 \hat{s}_y H^m := \sum_k c^k_{lm}(y) H^k := \sum_{d \in NE(X)} \sum_{l, k} \langle H^1, H^m, H^t \rangle_{0, \lambda, m, d} \eta^k H^k. \tag{6.15}
\]

Write \( t = \sum_{l \leq n_X} t_i H_i \). In the following we denote \( \nabla_H := \partial_t \) and, for any function \( f : H_T(X_{DL}^X) \rightarrow \mathbb{C}(\lambda) \), \( f|_{\text{sqc}} \) indicates its restriction to small quantum cohomology, \( t \rightarrow \sum_{i=1}^{r_X} t_i H_i \).

**Lemma 6.2.** For \( l < n_X \) we have

\[
(*)_X^{l+1} H_i^{l+1} = \bigcup_{i=1}^{r_X} H_i^{l} \quad \text{and} \quad H_i^{l+1} = : H_i^l. \tag{6.16}
\]

Moreover,

\[
z \nabla_H \hat{s}_t I^{X_{DL}}(y, z)|_{\text{sqc}} = \prod_i \left( z y_i \partial y_i \right)^{t_i} I^{X_{DL}}(y, z). \tag{6.17}
\]

**Remark 6.3.** This proposition is a variation of the well-known statement that for \( \mathbb{P}^n \) the small quantum product is the same as the cup product for all degrees up to and excluding \( n \).

**Proof.** Recall that the components of \( J_{\text{big}}^{X_{DL}}(t, z) \) are a set of flat coordinates for the Dubrovin connection in big quantum cohomology,

\[
z \nabla_H \nabla_H \hat{s}_t I^{X_{DL}}(y, z)|_{\text{sqc}} = \nabla_H \hat{s}_t I^{X_{DL}}(y, z). \tag{6.18}
\]

Write now \( (\mathcal{I})_{[k]} := [z^{-k}] \mathcal{I} \) for any Laurent series \( \mathcal{I} \in \mathbb{C}((z)) \) and suppose \( |l| = |m| = 1 \). We have that

\[
\nabla_H \nabla_H (J_{\text{big}}^{X_{DL}}(t))|_{\text{sqc}} = \nabla_H \hat{s}_t I^{X_{DL}}(y, z)|_{[s-1]} = \sum_k c_{lm}^k(y) \nabla_k (J_{\text{big}}^{X_{DL}}(t)|_{[s-1]} \tag{6.19}
\]
from (6.12) and the fact that $(J^X_{\text{big}})^{[k]}$ is the $[k]$-component of the gradient of the genus-0 Gromov-Witten potential. Then,

$$
(y_l \partial y_l)(y_m \partial y_m)J^X_{\text{big}}(y) = \nabla_H \nabla H^m(J^X_{\text{big}}(t))_{[s]} = \sum_k c^k_l(0) \nabla_k (J^X_{\text{big}}(t))_{[s-1]} \bigg|_{\text{sqc}}.
$$

(6.21)

Now, for $|m| = 1$ and by induction on $1 \leq |l| < |n_X|$ we have, from (6.12), that

$$
c^k_l(y) = \nabla_H \nabla H^m(J^X_{\text{big}}(t))_{[s]} = \left(\prod_i y_i \partial y_i\right) (I^X_{\text{big}}(y))_{[s]} = \delta_{l+m} = c^k_l(0),
$$

(6.22)

and for $s \geq |l|,$

$$
\left(\prod_i y_i \partial y_i\right) (y_m \partial y_m) (I^X_{\text{big}}(y))_{[s]} = \nabla_H \nabla H^m(J^X_{\text{big}}(t))_{[s-|l|]} \bigg|_{\text{sqc}} = \sum_k c^k_l(0) \nabla_k (J^X_{\text{big}}(t))_{[s-|l|+1]} \bigg|_{\text{sqc}}.
$$

(6.23)

**Corollary 6.4.** We have

$$
q^X_d = \prod_i |G_i| \left(\prod_{i,j} (w_X)^{(i)}_j \right) d^{n_X} p^X_d.
$$

(6.24)

**Proof.** From the previous Lemma we have, in particular, that

$$
\nabla_H^{n_X} (J^X_{\text{big}}(t))_{[s]} \bigg|_{\text{sqc}} = \left(\prod_i (y_i \partial y_i)^{n_i} \right) (I^X_{\text{big}}(y))_{[s+|n_X|]-1}.
$$

(6.25)

From (2.9) and (6.25) we have that

$$
q^X_d = |y^d| \eta_{n_X} \nabla H^{n_X} (J^X_{\text{big}}(t))_{[r_X+1]} \bigg|_{\text{sqc}} = \prod_i |G_i| \prod_{i,j} (w_X)^{(i)}_j \prod_i (y_i \partial y_i)^{n_i} \left( (I^X_{\text{big}}(y))_{[s+|n_X|]-1} \right)
$$

(6.26)

concluding the proof.
Remark 6.5. The statement of Lemma 6.2 also immediately reconstructs explicitly two-point descendent invariants where the powers of $\psi$-classes are distributed among the two marked points by standard structure results about $g = 0$ Gromov-Witten theory (namely the symplecticity of the $S$-matrix, which is a consequence of WDVV and the string equation: this is [17, Lemma 17]). Their agreement with the corresponding log invariants is an easy exercise left to the reader.

REFERENCES

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