Local BPS Invariants: Enumerative Aspects and Wall-Crossing

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We study the BPS invariants for local del Pezzo surfaces, which can be obtained as the signed Euler characteristic of the moduli spaces of stable one-dimensional sheaves on the surface S. We calculate the Poincaré polynomials of the moduli spaces for the curve classes β having arithmetic genus at most 2. We formulate a conjecture that these Poincaré polynomials are divisible by the Poincaré polynomials of $((-K_S).\beta - 1)$-dimensional projective space. This conjecture motivates the upcoming work on log BPS numbers [8].

1 Introduction

Given a Calabi–Yau threefold X, physical reasoning was used in [13, 14] to produce the Gopakumar–Vafa invariants from moduli spaces of one-dimensional sheaves on X.

There are mathematical definitions of these invariants $n_\beta^g \in \mathbb{Z}$ given in [19] for $g = 0$ and (consistently) in [27] for general g. In this paper, we refer to these invariants as the *BPS invariants* of X, conjectured to be related to other enumerative invariants of X as described in [14].

In this paper, we let X be a local del Pezzo surface, the total space of the canonical bundle K_S of a del Pezzo surface S. In this case, stable one-dimensional sheaves on X can be identified with stable one-dimensional sheaves on S.

While we include calculations of the refined BPS indices and the higher genus BPS invariants in Section 4.4, our main focus in this paper is on the genus 0 BPS invariants $n_\beta = n_\beta^0$. Henceforth, *local BPS invariants* shall mean the genus 0 BPS invariants, unless specified otherwise.

The moduli space of one-dimensional stable sheaves M_β of class $\beta \in H_2(S, \mathbb{Z})$ on X is equipped with a symmetric obstruction theory. It follows from [1] that $n_\beta = \deg[M_\beta]^{\mathrm{vir}}$. In the local del Pezzo surface case, M_β is smooth and hence the degree of the virtual cycle is given by the signed topological Euler characteristic of M_β. In the present paper, we calculate the Betti numbers of M_β. The results of our calculations, stated at the level of the local BPS invariants, are as follows.

Theorem 1.1 (Theorem 4.14).   Let β be a line class, a conic class, or a nef and big curve class on a del Pezzo surface S of arithmetic genus $p_a(\beta)$ at most 2. Let $w = (-K_S).\beta$ and let η be the maximum number of disjoint lines l such that $\beta.l = 0$. Then we have

(i)   if $p_a(\beta) = 0$, then $n_\beta = (-1)^{w-1} w$;

(ii)   if $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $n_\beta = (-1)^{w-1} w(e(S) - \eta)$;

(iii)   if $\beta = -K_{S_8}$, then $n_\beta = 12$;

(iv)   if $p_a(\beta) = 2$ and $\beta \neq -2K_{S_8}$, then $n_\beta = (-1)^{w-1} w \left(\binom{e(S) - \eta}{2} + 5 \right)$.

Here, $e(-)$ denotes the topological Euler characteristics.

The main observation is that n_β is divisible by w. Moreover, if we denote by $P_t(M)$ the Poincaré polynomial of a variety M, our calculations suggest the following.

Conjecture 1.2 (Conjecture 4.15).   Let β be a line class, a conic class, or a nef and big curve class on a del Pezzo surface S. Let $w = (-K_S).\beta$. Then the Poincaré polynomial $P_t(M_\beta)$ has a factor $P_t(\mathbb{P}^{w-1})$ and the quotient $P_t(M_\beta)/P_t(\mathbb{P}^{w-1})$ is a palindromic polynomial. Consequently, n_β is divisible by w.

When $S = \mathbb{P}^2$, Conjecture 1.2 is shown to be true whenever $P_t(M_\beta)$ or n_β is calculated. See, for example, [6, 7] for calculations of $P_t(M_\beta)$ up to degree 6 and [20, Section 8.3] for a calculation of n_β up to degree 10. In this paper, we prove that Conjecture 1.2 holds for all del Pezzo surfaces and β with $p_a(\beta) \leq 2$.

In [32, 33], it was observed that up to degree 8, the log BPS numbers for $\mathbb{P}^2$ agree with counts of rational curves of given degree, which intersect a fixed smooth elliptic curve E on $\mathbb{P}^2$ only at a given special point and are smooth at this point. In a sequel [8], we generalize this idea to give a rigorous direct definition for the log BPS numbers by using the log Gromov-Written (GW) theory for the pair of a del Pezzo surface S and a smooth anticanonical curve E on S. In a different direction, [3, Conjecture 44] stipulates a relationship, after a change of variable, of $P_t(M_\beta)$ with a generating function of certain higher genus log Gromov–Witten invariants. Combining [3] and [8] suggests a reconstruction result of higher genus log Gromov–Witten invariants in terms of genus 0 invariants.

Our strategy to prove Theorem 1.1 is as follows. We use the wall-crossing in the moduli space of δ-stable pairs. The same strategy is used in [6] to study M_β when $S = \mathbb{P}^2$. See Section 3.3 for a review of the δ-stable pair theory. When δ is sufficiently large, the space of δ-stable pairs is isomorphic to the space of pairs (C, Z) of curves C in class β and zero-dimensional subschemes $Z \subset C$. The latter space is a projective bundle over a Hilbert scheme provided β is sufficiently very ample (Proposition 3.16). More precisely, in our main situation of pairs with holomorphic Euler characteristic 1, the latter space is a projective bundle if β is very ample (resp. base point free) when the arithmetic genus of β is 2 (resp. 1).

It is known that β is very ample (resp. base point free) if and only if β has a positive (resp. nonnegative) intersection with all lines (i.e., (-1)-curves) on S and $\beta \neq -2K_{S_8}$ (resp. $\beta \neq -K_{S_8}$). We show that the moduli space M_β remains unchanged under blowing down S along a (-1)-curve and taking the pushforward of β. After blowing down all (-1)-curves l with $\beta.l = 0$, the moduli space of stable pairs can be computed and hence M_β can also be computed through wall-crossing.

The rest of this paper is organized as follows. In Section 2, we collect basic facts about the curve classes on del Pezzo surfaces. In Section 3, we start by reviewing the stability of one-dimensional sheaves and give a definition of local BPS invariants in Section 3.1. We prove the blowup property of BPS invariants in Section 3.2. We review the theory of δ-stable pairs and wall-crossing in Section 3.3. In Section 4, we compute the Poincaré polynomial of M_β and prove Theorem 1.1. Throughout this paper, we work over $\mathbb{C}$.

2 Preliminaries

In this section, we collect basic facts about curve classes on del Pezzo surfaces. Let S be a del Pezzo surface. Denote by S_r the blowup of $\mathbb{P}^2$ along r general points. Then S

is either S_r for $0 \leq r \leq 8$ or $\mathbb{P}^1 \times \mathbb{P}^1$. We will mainly consider the case $S = S_r$ and will make remarks for $\mathbb{P}^1 \times \mathbb{P}^1$ separately whenever needed. The results of this paper hold for $\mathbb{P}^1 \times \mathbb{P}^1$ as well.

Definition 2.1. A class $\beta \in H_2(S, \mathbb{Z})$ is a curve class if it can be represented by a nonempty subscheme of dimension 1. We often consider β as a divisor on S.

Since del Pezzo surfaces are rational, by Poincaré duality, $\text{Pic}(S) \cong H_2(S, \mathbb{Z})$. So when we write $|\mathcal{O}_S(\beta)|$ or simply $|\beta|$, we mean the complete linear system $|L|$ for the unique $L \in \text{Pic}(S)$ such that $c_1(L) = \beta$.

For S_r, let h be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let e_i for $1 \leq i \leq r$ be the exceptional divisors. The Picard group $\text{Pic}(S_r)$ is generated by h and the e_i's. We use the notation $(d; a_1, \cdots, a_r)$ for the divisor $dh - \sum a_i e_i$. When there are repetitions in the a_i's, we sometimes use superscripts to indicate the number of repetitions. For example, $(1; 1^2)$ means the class $h - e_1 - e_2$. The anticanonical divisor is $-K_{S_r} = (3; 1^r)$.

For $\mathbb{P}^1 \times \mathbb{P}^1$, we denote by h_1 and h_2 the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ from each factor. The anticanonical divisor is $-K_{S_r} = 2h_1 + 2h_2$.

Definition 2.2. A *line class* on S is a class $l \in \text{Pic}(S)$ such that $l^2 = -1$ and $(-K_S).l = 1$.

It is well known that each line class contains a unique irreducible line and there are only finitely many lines on S.

Example 2.3. By numerical calculation, we list all line classes up to permutation of the e_i's:

$$e_i, (1; 1^2), (2; 1^5), (3; 2, 1^6), (4; 2^3, 1^5), (5; 2^6, 1^2), (6; 3, 2^7).$$

Definition 2.4. A divisor D on S is said to be *nef* if $D.C \geq 0$ for any curve C. A nef divisor D is said to be *big* if in addition $D^2 > 0$.

Definition 2.5. A line bundle L on S is said to be *k-very ample* for an integer $k \geq 0$ if given any zero-dimensional subscheme Z of S of length $k + 1$, the restriction map $H^0(L) \to H^0(L|_Z)$ is surjective. A divisor D is said to be *k-very ample* if the associated line bundle is k-very ample.

Note that 0-very ample divisors are globally generated divisors and 1-very ample divisors are very ample divisors. Di Rocco in [11] found the following numerical criterion for k-very ampleness on a del Pezzo surface.

Lemma 2.6 ([11] and [2, (2.1.1)]).   Let $D \in \mathrm{Pic}(S)$ be a divisor and $k \geq 0$ be an integer. Suppose that $D \neq -kK_{S_8}$ and $D \neq -(k+1)K_{S_8}$ and that $D \neq -K_{S_7}$ when $k = 1$. Then $D \in \mathrm{Pic}(S)$ is k-very ample if and only if

  (i)   when $S = \mathbb{P}^2$, $D.h \geq k$,
  (ii)   when $S = \mathbb{P}^1 \times \mathbb{P}^1$, $D.h_i \geq k$ for $i = 1, 2$,
  (iii)   when $S = S_1$, $D.l \geq k$ for any line class l and $D.(h - e_1) \geq k$,
  (iv)   when $S = S_r$ for $r \geq 2$, $D.l \geq k$ for any line class l.

Lemma 2.7 ([11]).   An effective divisor $D \in \mathrm{Pic}(S)$ is nef if and only if it is 0-very ample, except for the case $D = -K_{S_8}$, which is nef but not 0-very ample.

Lemma 2.8 ([22, Section 2.3 (P5, P7)]).   If D is base point free, then $H^i(D) = 0$ for $i = 1, 2$. Furthermore, if D is nef and big, then a general member of $|D|$ is smooth and irreducible.

Definition 2.9.   For $\beta \in \mathrm{Pic}(S)$, we denote by $p_a(\beta)$ the arithmetic genus of β. By adjunction we have

$$p_a(\beta) = \frac{1}{2}\beta(\beta + K_S) + 1.$$

Definition 2.10.   A *conic class* on S is the class $D \in \mathrm{Pic}(S)$ such that $p_a(D) = 0$ and $(-K_S).D = 2$.

By the remark after Corollary 3.3 of [34], nef but non-big divisors on S are multiples of conic classes. The complete linear system of a conic class D has projective dimension 1, which gives a ruling $S \to \mathbb{P}^1$. The fiber class of this ruling is D.

Example 2.11.   The list of all conic classes is obtained by numerical calculation as follows (up to permutations of the e_i's). See also [20, Appendix A].

$$(1; 1), (2; 1^4), (3; 2, 1^5), (4; 2^3, 1^4), (4; 3, 1^7),$$
$$(5; 2^6, 1), (5; 3, 2^3, 1^4), (6; 3^2, 2^4, 1^2), (7; 3^4, 2^3, 1), (7; 4, 3, 2^6), \tag{2.1}$$
$$(8; 3^7, 1), (8; 4, 3^4, 2^3), (9; 4^2, 3^5, 2), (10; 4^4, 3^4), (11; 4^7, 3).$$

3   Local BPS Invariants and Pairs

3.1   BPS invariants

Let X be a Calabi–Yau threefold, and fix an ample line bundle L on X. The Hilbert polynomial of F is defined by $\chi(F \otimes L^m)$. In case $X = \mathrm{Tot}(K_S)$, we consider coherent

sheaves F on X set-theoretically supported on S, so that we can consider its homology class in $H_2(S, \mathbb{Z})$.

Definition 3.1. A sheaf F supported on a curve of class β is called *stable* if

  (a) F is pure, that is, F has no zero-dimensional subsheaves.
  (b) For any proper nonzero subsheaf G of F, we have

$$\frac{\chi(G)}{r(G)} < \frac{\chi(F)}{r(F)},$$

  where $r(F)$ is the linear coefficient of the Hilbert polynomial of F.

Definition 3.2. We denote by $M_{\beta,n}$ the moduli space of stable sheaves F on S with $\chi(F) = n$ and $[F](:= c_1(F)) = \beta$. When $n = 1$, we simply write $M_\beta = M_{\beta,1}$.

When X is a smooth projective variety, M_β is projective. The moduli space M_β carries a symmetric obstruction theory [19], and hence a virtual invariant is well defined. It is known that this virtual invariant is independent of the choice of the ample line bundle L. See, for example, [36, Lemma 4.8]. For a del Pezzo surface S, we take $L = -K_S$. Note that when $\chi(F) = 1$, a proper nonzero subsheaf G of F is destabilizing if and only if $\chi(G) \geq 1$. The following proposition is well known (see [24]).

Proposition 3.3.
  (i) Provided it is nonempty, M_β is smooth of dimension $\beta^2 + 1$.
  (ii) $n_\beta = (-1)^{\beta^2+1} e(M_\beta)$.

In Proposition 3.3 and below, β^2 denotes the self-intersection $\beta.\beta$.

Proof. The obstruction at $F \in M_\beta$ is given by $\mathrm{Ext}^2(F, F)$. By Serre duality, we have

$$\mathrm{Ext}^2(F, F) = \mathrm{Hom}(F, F \otimes K_S)^\vee.$$

The latter space is zero because F is stable with respect to $-K_S$ [18, Proposition 1.2.7]. Therefore, M_β is smooth. Moreover, by Riemann–Roch,

$$\chi(F, F) = 1 - \mathrm{ext}^1(F, F) = \int_S \mathrm{ch}^\vee(F)\mathrm{ch}(F)\mathrm{td}(S) = -\beta^2.$$

The dimension of M_β at F is $\mathrm{ext}^1(F, F) = \beta^2 + 1$.

Let X be the total space of K_S and let $M_\beta(X)$ be the moduli space of stable sheaves on X with the same numerical condition as M_β. It is elementary to show that $M_\beta(X)$ is in fact equal to M_β. For example, the proof of [31, Lemma 4.24] works under our assumption that F is a stable sheaf on $X = \mathrm{Tot}(K_S)$ when $-K_S$ is ample. It is well known that $M_\beta(X)$ is equipped with a symmetric obstruction theory and hence a virtual cycle $[M_\beta(X)]^{\mathrm{vir}} \in A_0(M_\beta(X))$. Toda in [36] proved that $n_\beta = \deg[M_\beta(X)]^{\mathrm{vir}}$ whenever the Gromov-Written/Pandharipande-Thomas (GW/PT) correspondence holds, which is the case for local del Pezzo surfaces. Since M_β is smooth of dimension $\beta^2 + 1$, we have $\deg[M_\beta(X)]^{\mathrm{vir}} = (-1)^{\beta^2+1} e(M_\beta)$. ∎

Remark 3.4. In Section 4.4, we discuss an $sl_2 \times sl_2$-action on the cohomology of M_β, which will allow us to refine the n_β^g.

3.2 Blowup property of the BPS invariant

Let $\pi : S_{r+1} \to S_r$ be the blowup at a general point $p \in S_r$. Let β be a divisor on S_r. In this section, we show that $M_\beta(S_r)$ and $M_{\pi^*\beta}(S_{r+1})$ are isomorphic. Consequently, $n_{\pi^*\beta}(S_{r+1}) = n_\beta(S_r)$.

Let $F \in M_\beta(S_r)$. We let $\mathrm{supp}(F) \in |\beta|$ be the support scheme defined by the Fitting ideal. Since F has pure dimension 1 we have a presentation of F as

$$0 \to E_1 \xrightarrow{\phi} E_2 \to F \to 0, \tag{3.1}$$

where E_1 and E_2 are locally free of the same rank. Then $\mathrm{supp}(F) \in |\beta|$ is the subscheme of S defined by the vanishing of $\det \phi$ and is well known to be independent of the choice of resolution. This defines the Chow morphism

$$M_\beta \to |\beta|.$$

Lemma 3.5. For a pure one-dimensional sheaf F on S_r, π^*F is pure and

$$\mathrm{supp}(\pi^*F) = \pi^*\mathrm{supp}(F).$$

Proof. Choose a presentation (3.1) of F. Applying π^* we get

$$0 \to \pi^*E_1 \xrightarrow{\pi^*(\phi)} \pi^*E_2 \to \pi^*F \to 0. \tag{3.2}$$

The sequence (3.2) is exact on the left because the kernel of $\pi^*(\phi)$ is zero away from the exceptional curve as $\pi^*(\phi)$ can be identified with ϕ, which implies that the kernel is zero everywhere since π^*E_1 is locally free.

Then π^*F is pure by the Auslander–Buchsbaum formula, and $\mathrm{supp}(\pi^*F)$ is the subscheme of S_{r+1} defined by the vanishing of $\det \pi^*(\phi) = \pi^* \det \phi$, which is equal to the scheme-theoretic inverse image $\pi^*(\mathrm{supp}(F))$. ∎

Thus, we can think of F and π^*F having "the same" support, after identifying $|\beta|$ with $|\pi^*\beta|$ via $C \mapsto \pi^*C$.

Lemma 3.6. For a pure one-dimensional sheaf F on S_r, $\pi_*\pi^*F \simeq F$, $R^1\pi_*\pi^*F = 0$, and $\chi(\pi^*F) = \chi(F)$.

Proof. We apply π_* to (3.2), noting by the projection formula that $\pi_*\pi^*E_i \simeq E_i$ and $R^j\pi_*\pi^*E_i = 0$ for $i = 1, 2$ and $j > 0$, since each E_i is locally free and $R^j\pi_*\mathcal{O}_{S_{r+1}} = 0$. We obtain

$$0 \to E_1 \xrightarrow{\phi} E_2 \to \pi_*\pi^*F \to 0 \tag{3.3}$$

and $R^1\pi_*\pi^*F = 0$. Comparing (3.3) and (3.1) we see that $\pi_*\pi^*F \simeq F$. Finally, by Leray

$$\chi(\pi^*F) = \chi(\pi_*\pi^*F) - \chi(R^1\pi_*\pi^*F) = \chi(F).$$

∎

Proposition 3.7. For a pure one-dimensional sheaf F on S_r with $\chi(F) = 1$, F is stable if and only if π^*F is stable.

Proof. Suppose π^*F is stable. Let G be a saturated subsheaf of F. Since F/G is pure, by (3.2), $L_1\pi^*(F/G) = 0$ and hence π^*G is a subsheaf of π^*F. By Lemma 3.6, if G destabilizes F, then π^*G destabilizes π^*F. Hence, F is also stable.

Conversely, suppose that F is stable and $G \subset \pi^*F$ is a subsheaf with $\chi(G) \geq 1$. Applying π_* we get that π_*G is a subsheaf of $\pi_*\pi^*F \simeq F$. Since $R^1\pi_*G$ is supported at p, we have $\chi(R^1\pi_*G) = h^0(R^1\pi_*G) \geq 0$ and

$$\chi(\pi_*G) = \chi(G) + \chi(R^1\pi_*G) \geq \chi(G) \geq 1$$

by Leray. Hence, π_*G destabilizes F, contradicting the stability of F. ∎

Lemma 3.8. Let $F \in M_{\pi^*\beta}(S_{r+1})$. Then π_*F is pure with $c_1(\pi_*F) = \beta$.

Proof. Clearly, π_*F can only have torsion at p. Suppose we had a skyscraper sheaf $\mathbb{C}_p \subset \pi_*F$ at p. This gives a global section s of π_*F annihilated by m_p. Then s corresponds

to a global section $\tilde{s}$ of F, which is then necessarily annihilated by $\pi^{-1}(m_p)\mathcal{O}_{S_{r+1}} = \mathcal{I}_E$. So $\tilde{s}$ induces a map $\mathcal{O}_{S_{r+1}}/\mathcal{I}_E \simeq \mathcal{O}_E \to F$, which is injective because $\mathcal{O}_E$ is pure. Since $\chi(\mathcal{O}_E) = 1$, this would violate the stability of F.

The class of π_*F is β since $\pi_*F|_{S_r-p}$ is identified with $F|_{S_{r+1}-E}$ via π and the restriction map

$$\mathrm{Pic}(S_{r+1}) \to \mathrm{Pic}(S_{r+1} - E) \simeq \mathrm{Pic}(S_r - p) \simeq \mathrm{Pic}(S_r)$$

is the left inverse of $\pi^* : \mathrm{Pic}(S_r) \to \mathrm{Pic}(S_{r+1})$. ∎

Now we consider the natural map $\rho : \pi^*\pi_*F \to F$.

Proposition 3.9. Let $F \in M_{\pi^*\beta}(S_{r+1})$. Then $\rho : \pi^*\pi_*F \to F$ is an isomorphism. Consequently, π_*F is stable and $\chi(\pi_*F) = 1$.

Proof. Let G be the kernel of ρ. Then π_*G is a subsheaf of $\pi_*\pi^*\pi_*F$. But $\pi_*\pi^*\pi_*F \simeq \pi_*F$ by the 1st statement of Lemma 3.6 applied to π_*F. But π_*G is supported at p, contradicting the purity of π_*F unless $\pi_*G = 0$.

By Lemmas 3.5 and 3.8 we see that $c_1(\pi^*\pi_*F) = \pi^*(\beta)$; so letting $Q = \mathrm{coker}(\rho)$, it follows that $c_1(G) = c_1(Q)$.

Finally, we compute the Euler characteristics of π_*F in two different ways. Using ρ, we see that $\chi(\pi^*\pi_*F) = 1 + \chi(G) - \chi(Q)$. By Lemma 3.6, we see that $R^1\pi_*(\pi^*\pi_*F) = 0$. So by Leray for π_*, we compute $\chi(\pi^*\pi_*F) = \chi(\pi_*F)$. But again by Leray, we have $\chi(\pi_*F) = \chi(F) + h^0(R^1\pi_*F) = 1 + h^0(R^1\pi_*F)$. From $\pi_*G = 0$ it follows that $\chi(G) \leq 0$; hence, $\chi(Q) \leq 0$. We must have $Q = 0$ since F is stable, and G is zero-dimensional by $c_1(G) = c_1(Q)$. Again by $\pi_*G = 0$ we have $G = 0$, so ρ is an isomorphism. ∎

Proposition 3.10. Let $\pi : S_{r+1} \to S_r$ be a blowup. Let β be a divisor on S_r. Then, $M_\beta(S_r)$ and $M_{\pi^*\beta}(S_{r+1})$ are isomorphic.

Proof. Let $\mathcal{F}$ be a universal family on $M_\beta(S_r) \times S_r$. The pullback $\mathcal{F}' = (id \times \pi)^*\mathcal{F}$ is a family on $M_\beta(S_r) \times S_{r+1}$, whose fibers are stable sheaves in $M_{\pi^*\beta}(S_{r+1})$ by Proposition 3.7. So $\mathcal{F}'$ induces the morphism $\pi^* : M_\beta(S_r) \to M_{\pi^*\beta}(S_{r+1})$. By Proposition 3.9, π^* is bijective and since π is an isomorphism away from the exceptional divisor, π^* is a birational morphism. By Proposition 3.3, the two moduli spaces $M_\beta(S_r)$ and $M_{\pi^*\beta}(S_{r+1})$ are smooth. Therefore, by Zariski's main theorem, π^* is an isomorphism. ∎

In Section 4, we will consider curve classes β of arithmetic genus at most 2. By Lemma 2.6 and Proposition 3.10, with a few exceptions it is enough to calculate the BPS numbers for very ample classes β by blowing down all (-1)-curves l such that $\beta.l = 0$.

Remark 3.11.   The isomorphism constructed above commutes with the Chow morphisms. Therefore, the higher genus BPS invariants as well as their $sl_2 \times sl_2$ refinements as defined in [4, 21, 27] remain unchanged as well.

3.3 δ-stable pairs and wall-crossing

Suppose that the BPS invariants $n_\beta^g(X) = n_\beta^g$ satisfy the Gopakumar–Vafa formula

$$\sum_{\beta,g} \mathcal{I}_\beta^g(X) q^\beta \lambda^{2g-2} = \sum_{\beta,g,k} n_\beta^g \frac{1}{k} \left(2 \sin\left(\frac{k\lambda}{2} \right)^{2g-2} q^{k\beta} \right), \tag{3.4}$$

where $\mathcal{I}_\beta^g(X)$ are the (local) Gromov–Witten invariants. Using the conjectured GW/PT correspondence, we would then have the following PT/BPS formula [19]:

$$Z_{PT} = \prod_\beta \left(\prod_{j=1}^\infty \left(1 + (-1)^{j+1} q^j Q^\beta \right)^{jn_\beta^0} \right. $$
$$\left. \cdot \prod_{g=1}^\infty \prod_{k=0}^{2g-2} \left(1 + (-1)^{g-k} q^{g-1-k} Q^\beta \right)^{(-1)^{k+g} n_\beta^g \binom{2g-2}{k}} \right), \tag{3.5}$$

where Z_{PT} is the generating function for the PT invariants. The GW/PT correspondence is proven when S (and hence X) is toric by combining the toric Gromov-Written/Donaldson-Thomas (GW/DT) correspondence [26] with the DT/PT correspondence [5]. The GW/PT correspondence for a general del Pezzo surface S reduces to the toric cases by taking a toric blowup of $\mathbb{P}^2$ and then using deformation invariance of the GW and PT invariants. See Definition 3.15 for PT-stable pairs.

In [20], Katz, Klemm, and Vafa developed a geometric computational technique for BPS invariants. Later in [9], the refined BPS invariants are defined from the refined PT invariants and the method is extended to compute the refined BPS indices.

As a consequence of the product formula (3.5), it was suggested in [20] that the genus 0 BPS invariant $n_\beta := n_\beta^0$ can be computed by

$$n_\beta = PT_{\beta,1} - PT_{\beta,-1} + \text{correction terms}, \tag{3.6}$$

where $PT_{\beta,n}$ is the stable pair invariant of degree β and Euler characteristic n. The correction terms are combinations of lower-degree PT invariants. In [9], the correction terms are interpreted as a wall-crossing contribution of the moduli spaces of stable pairs. After the wall-crossing, the moduli spaces of stable pairs are related to the moduli space of (Gieseker-)stable sheaves. We will formulate and prove a refined version of (3.6) in Proposition 3.17 below.

To compute the local BPS invariants, we will use Proposition 3.3 and compute the topological Euler characteristic of the moduli spaces M_β. More generally, we compute the Poincaré polynomials.

Definition 3.12. For a complex algebraic variety M, we let $E_M(u, v)$ be its E-polynomial. We define the *virtual Poincaré polynomial* of M as the polynomial $P_t(M) = E_M(-t^{1/2}, -t^{1/2})$ in $t^{1/2}$. The virtual Poincaré polynomial satisfies the properties

(i) $P_t(M) = \sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(M, \mathbb{Q}) \, t^{i/2}$ if M is nonsingular and projective.

(ii) $P_t(M) = P_t(M \setminus Y) + P_t(Y)$ for a closed algebraic subset Y of M.

In our cases, the odd cohomology groups of M_β vanish so that $P_t(M_\beta)$ is actually a polynomial in t and the topological Euler characteristic is given by $e(M) = P_1(M)$. Note that although the Poincaré polynomial is not motivic in general the virtual Poincaré polynomial is motivic, and since M_β is smooth, the virtual Poincaré polynomial agrees with the usual Poincaré polynomial in t with $t^{1/2}$ substituted for t.

We will freely use the following properties of the virtual Poincaré polynomial, which follow from the definition (see [12, Section 4.5]).

(iii) If M is a disjoint union of a finite number of locally closed subvarieties M_i, then $P_t(M) = \sum P_t(M_i)$.

(iv) If M is a Zariski locally trivial fibration over Y with fibers F, then $P_t(M) = P_t(F)P_t(Y)$.

To compute the Poincaré polynomial, we relate M_β birationally with the moduli spaces of δ-stable pairs by wall-crossing. This approach is taken in [6] to compute the Betti numbers for M_β when $S = \mathbb{P}^2$ and $\beta = 4$ and 5. See also [9].

Definition 3.13. A *pair* on X is a pair (s, F) of a coherent sheaf F on X of class β together with a nonzero section $s \in H^0(F)$. A morphism between pairs is a morphism of sheaves, which preserves the sections up to multiplication by a constant.

The topological data of (s, F) are defined to be those of the sheaf F. The notion of pairs originated in the work of Le Potier [24] on *coherent systems*. A coherent system is

a pair (V, F) of a coherent sheaf F with a subspace $V \subset H^0(F)$ of fixed dimension. So, our pairs are coherent systems of dimension 1. It is often convenient to consider a sheaf as a coherent system of dimension 0.

Le Potier [24] studied a series of stability conditions on coherent systems, which reads as follows for pairs. See also [16].

Definition 3.14.   Let $\delta \in \mathbb{Q}_+$. A pair (s, F) is δ-*stable* if

    (a) F is pure.

    (b)   For any proper nonzero subsheaf G of F, we have

$$\frac{\chi(G) + \epsilon(s, G)\delta}{r(G)} < \frac{\chi(F) + \delta}{r(F)},$$

        where $r(F)$ is the linear coefficient of the Hilbert polynomial of F and $\epsilon(s, G) = 1$ if s factors through G and $\epsilon(s, G) = 0$ otherwise.

When the equality is allowed in Condition (2), then the pair is δ-*semistable*.

As in Section 3.1, we use the ample line bundle $L = -K_S$ to define the Hilbert polynomial of a sheaf on S. So, $r(F) = (-K_S).[F]$. We denote by $M^\delta_{\beta, n}$ the moduli space of δ-stable pairs (s, F) on S with $[F] = \beta$ and $\chi(F) = n$. When there are no strictly semistable δ-stable pairs, $M^\delta_{\beta, n}$ is constructed as a projective scheme by Geometric Invariant Theory (GIT).

The values of δ, where there exist strictly δ-semistable pairs are called the *walls*. Then the moduli space $M^\delta_{\beta, n}$ changes only at the walls. We will see that in our cases there are only finitely many walls.

One special case is when δ is sufficiently large, which we denote by $\delta = \infty$. In this case, the δ-stability condition is equivalent to the stability condition on pairs of Pandharipande and Thomas [28].

Definition 3.15.   A pair (s, F) is PT-stable if

    (a) F is pure of dimension 1.

    (b)   The cokernel of $s : \mathcal{O}_X \to F$ is zero-dimensional.

We denote by $P_n(S, \beta)$ the moduli space of PT-stable pairs on S. In other words, $P_n(S, \beta) = M^\infty_{\beta, n}$. By condition (2) in Definition 3.15, it is straightforward to see that $P_n(S, \beta)$ is empty when $n < 1 - p_a(\beta)$. Pandharipande and Thomas [28] proved that $P_n(X, \beta)$ is equipped with a symmetric obstruction theory when X is a Calabi–Yau threefold.

In general, when $X = \mathrm{Tot}(K_S)$, $P_n(X, \beta)$ may not be equal to $P_n(S, \beta)$. However, we will only consider the wall-crossing of δ-stable pairs defined on S.

Proposition 3.16.   Let S be a del Pezzo surface. Let $p_a = p_a(\beta)$ and assume $n \geq 1 - p_a$. Recall that $w = (-K_S).\beta$.

   (i)   If β is a line class, then $P_n(S, \beta) \simeq \mathbb{P}^{n-1}$.
   (ii) Assume that β is base point free. If β is $(n - 2 + p_a)$-very ample, then $P_n(S, \beta)$ is a projective bundle of rank $w - n$ over the Hilbert scheme $\mathrm{Hilb}^{n-1+p_a}(S)$.

Proof.   The proof is essentially the same as that of [6, Lemma 2.3]. By [29, Proposition B.8], $P_n(S, \beta)$ is isomorphic to the space of pairs (C, Z), where C is a curve in class β and Z is a subscheme of C of length $n - 1 + p_a$. In particular, the assertion for a line class β is straightforward. Note that each line class contains a unique line.

Now, let $\mathcal{I}$ be the universal ideal sheaf on $\mathrm{Hilb}^{n-1+p_a}(S) \times S$ and let $p : \mathrm{Hilb}^{n-1+p_a}(S) \times S \to \mathrm{Hilb}^{n-1+p_a}(S)$ and $q : \mathrm{Hilb}^{n-1+p_a}(S) \times S \to S$ be the projections. Then $P_n(S, \beta)$ is the projective bundle $\mathbb{P}(p_*(\mathcal{I} \otimes q^* \mathcal{O}_S(\beta)))$ provided that $p_*(\mathcal{I} \otimes q^* \mathcal{O}_S(\beta))$ is locally free. Since β is $(n - 2 + p_a)$-very ample, we have $H^1(I_Z \otimes \mathcal{O}(\beta)) \simeq H^1(\mathcal{O}(\beta))$ for any subscheme Z of length $n - 1 + p_a$. The latter space vanishes by Lemma 2.8. By the semicontinuity theorem, $p_*(\mathcal{I} \otimes q^* \mathcal{O}_S(\beta))$ is locally free and hence $P_n(S, \beta)$ is a projective bundle.

Since $H^1(I_Z \otimes \mathcal{O}(\beta)) \simeq H^1(\mathcal{O}(\beta)) = 0$ for base point free β, the rank of the projective bundle $\mathbb{P}(p_*(\mathcal{I} \otimes q^* \mathcal{O}_S(\beta)))$ can be computed by Riemann–Roch.   ∎

On the other extreme when δ is sufficiently small, which we denote by $\delta = 0^+$, it is elementary to check that for $(s, F) \in M^{0^+}_{\beta, n}$, the sheaf F is a stable sheaf provided that $(-K_S).\beta$ and n are coprime. In this case, we have a forgetful map

$$\xi : M^{0^+}_{\beta, n} \to M_{\beta, n}.$$

In what follows, we only consider the case where n is either 1 or -1; so the coprime condition is always satisfied.

Proposition 3.17. $P_t(M_\beta) = P_t(M^{0^+}_{\beta, 1}) - t P_t(M^{0^+}_{\beta, -1})$.

Proof.   This formula is proven for $S = \mathbb{P}^2$ in [6, Lemma 5.1]. The same proof applies to general del Pezzo surfaces. We sketch the proof here.

Let n be either 1 or -1. Let $(M_{\beta,n})_k$ (resp. $(M^{0^+}_{\beta,n})_k$) denote the locus in $M_{\beta,n}$ (resp. $M^{0^+}_{\beta,n}$) defined by the condition $h^0(F) = k$. Then the forgetful map ξ restricted to $(M^{0^+}_{\beta,n})_k$ is a Zariski locally trivial $\mathbb{P}^{k-1}$-fibration since any nonzero section of F defines a 0^+-stable pair and an automorphism of a stable sheaf is given by scalar multiplication. Therefore, we have

$$P_t\big(M^{0^+}_{\beta,n}\big) = \sum_k P_t\big((M^{0^+}_{\beta,n})_k\big) = \sum_k P_t(\mathbb{P}^k)P_t((M_{\beta,n})_k).$$

For a sheaf $F \in M_\beta$, we define its dual by $F^D = \mathcal{E}xt^1(F, \omega_S)$. Since F is a pure one-dimensional sheaf, $F^{DD} \simeq F$ [18, Proposition 1.1.10]. The local-to-global spectral sequence $E_2^{pq} = H^p(\mathcal{E}xt^q(F, \omega_S))$ degenerates at level 2 and hence $h^i(F^D) = h^{1-i}(F)$ for $i = 0, 1$. Thus, the association $F \mapsto F^D$ induces an isomorphism between $(M_{\beta,1})_k$ and $(M_{\beta,-1})_{k-1}$. The fact that this association is a morphism of schemes is proved in [25] when $S = \mathbb{P}^2$; but the same proof applies to a general del Pezzo surface S.

Therefore, we have

$$P_t\big(M^{0^+}_{\beta,1}\big) - tP_t\big(M^{0^+}_{\beta,-1}\big) = \sum_k P_t(\mathbb{P}^{k-1})P_t((M_{\beta,1})_k) - tP_t(\mathbb{P}^{k-1})P_t((M_{\beta,-1})_k)$$

$$= \sum_k P_t(\mathbb{P}^{k-1})P_t((M_{\beta,1})_k) - tP_t(\mathbb{P}^{k-1})P_t((M_{\beta,1})_{k+1})$$

$$= \sum_k (P_t(\mathbb{P}^{k-1}) - tP_t(\mathbb{P}^{k-2}))P_t((M_{\beta,1})_k)$$

$$= \sum_k P_t((M_{\beta,1})_k) = P_t(M_\beta).$$

∎

Proposition 3.17 suggests that the correction terms in (3.6) come from wall-crossing on δ-stable pairs. More detail on the correspondence between wall-crossing terms and the correction terms can be found in [9, Section 9.3].

Now we study how the moduli space changes when we cross a wall. Let δ_0 be a wall and let δ_- and δ_+ be rational numbers sufficiently close to δ_0 such that $\delta_- < \delta_0 < \delta_+$ and there are no walls between δ_- and δ_+ other than δ_0. We want to compare $M^{\delta_+}_{\beta,n}$ and $M^{\delta_-}_{\beta,n}$.

Let (s, F) be a δ_+-stable pair that is not δ_--stable. Let F'' be a subsheaf of F such that s factors through F''. So $\frac{\chi(F'') + \delta_+}{r(F'')} < \frac{\chi(F) + \delta_+}{r(F)}$. Since $r(F'') \leq r(F)$, this implies

$\frac{\chi(F'')+\delta_-}{r(F'')} < \frac{\chi(F)+\delta_-}{r(F)}$. Therefore, for (s,F) to be not δ_--stable, there must be a subsheaf F'' of F such that

$$\frac{\chi(F'')}{r(F'')} > \frac{\chi(F)+\delta_-}{r(F)}.$$

Necessarily, the section s does not factor through F''. Thus, we have an exact sequence of pairs

$$0 \to (0,F'') \to (s,F) \to (s',F') \to 0, \tag{3.7}$$

where $F' = F/F''$ and s' is the section on F' induced by s. Here, $(0,F'')$ denotes the sheaf F'' considered as a coherent system of dimension 0.

On the other hand, if $(\tilde{s},\tilde{F})$ is a δ_--stable pair that is not δ_+-stable, by the same reasoning, we have an exact sequence

$$0 \to (s',F') \to (\tilde{s},\tilde{F}) \to (0,F'') \to 0. \tag{3.8}$$

The wall δ_0 is called a *simple wall* if (s',F') is δ_0-stable and F'' is stable (as a sheaf) so that there are no further decompositions to be considered. In this paper, we will only consider the cases where all walls are simple walls.

To denote a decomposition as in (3.7) and (3.8), we use the notation

$$(1,(\beta,n)) = (1,(\beta',n')) + (0,(\beta'',n'')), \tag{3.9}$$

where $\beta' = [F']$, $\beta'' = [F'']$, $n' = \chi(F')$, and $n'' = \chi(F'')$. So if there is a wall-crossing for $M_{\beta,n}^{\delta}$, we must have a decomposition (3.9) such that $M_{\beta'',n''}$ and $M_{\beta',n'}^{\delta_0}$ are nonempty, where

$$\frac{n+\delta_0}{(-K_S).\beta} = \frac{n'+\delta_0}{(-K_S).\beta'} = \frac{n''}{(-K_S).\beta''}.$$

In such a case, the pairs in $M_{\beta,n}^{\delta_+}$ of the form (3.7) parametrized by $\mathbb{P}(\mathrm{Ext}^1((s',F'),(0,F'')))$ are replaced with the pairs in $M_{\beta,n}^{\delta_-}$ of the form (3.8) parametrized by $\mathbb{P}(\mathrm{Ext}^1((0,F''),(s',F')))$. This wall-crossing phenomenon can be explained by *elementary modification* of pairs. See [35, Section 3], [16, Lemma 4.24], and [6]. Now each Ext group can be computed using the following proposition.

Proposition 3.18.   [16, Corollary 1.6] Let $\Lambda = (s, F)$ and $\Lambda' = (s', F')$ be pairs on X. Then there is a long exact sequence

$$0 \to \mathrm{Hom}(\Lambda, \Lambda') \to \mathrm{Hom}(F, F') \to H^0(F')/\langle s' \rangle$$

$$\to \mathrm{Ext}^1(\Lambda, \Lambda') \to \mathrm{Ext}^1(F, F') \to H^1(F')$$

$$\to \mathrm{Ext}^2(\Lambda, \Lambda') \to \mathrm{Ext}^2(F, F') \to H^2(F') \to \cdots .$$

4   Calculations of Local BPS Numbers

We calculate the local BPS numbers by applying the wall-crossing techniques described in the previous sections. In this section, we assume that β is a line class, a conic class, or a nef and big curve class so that there are smooth irreducible curves in class β.

When β is nef and big, we have $H^i(\beta + K_S) = 0$ for $i > 0$, which is due to Ramanujam [30] and [23, Theorem 4.3.1]. Therefore,

$$h^0(\beta + K_S) = \chi(\beta + K_S) = \frac{1}{2}(\beta + K_S)\beta + 1 = p_a(\beta). \tag{4.1}$$

4.1   Arithmetic genus 0

For a nef and big curve class of arithmetic genus 0 on $S = S_r$, $\beta + K_S$ is not nef since $(\beta + K_S)\beta = -2$. Hence, if $r \geq 2$, there is a line l on S with $(\beta + K_S).l < 0$, and $\beta.l = 0$ follows from the nefness of β. By blowing down such lines, we see that β is a pullback of the class (1) or (2) on $\mathbb{P}^2$, the class $(d; d-1)$ on S_1 with $d \geq 2$, or the class $(1, k)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with $k \geq 1$.

Proposition 4.1.   Let β be a curve class on S of arithmetic genus 0. If β is a line class, a conic class, or a nef and big curve class, then M_β is isomorphic to $\mathbb{P}^{w-1}$.

Proof.   In the nef and big case, we may assume that (S, β) is $(\mathbb{P}^2, (1))$, $(\mathbb{P}^2, (2))$, $(S_1, (d; d-1))$ with $d \geq 2$ or $(\mathbb{P}^1 \times \mathbb{P}^1, (1, k))$ with $k \geq 1$ by Proposition 3.10 and the preceding discussion.

Let F be a stable sheaf with $\chi(F) = 1$. Then there is a nonzero section $s \in H^0(F)$, which induces a morphism $i : \mathcal{O}_S \to F$. Let C' be the curve on S defined by the kernel of i. Put $\beta' = [C']$. Then if $\beta' \neq \beta$, stability is contradicted because $p_a(\beta') \leq 0$ as can be seen using the description of β in each case. We conclude that $\beta' = \beta$ and $F \simeq \mathcal{O}_C$, where C is in class β. Therefore, M_β is isomorphic to the complete linear system $|\mathcal{O}(\beta)| \simeq \mathbb{P}^{w-1}$.   ■

Corollary 4.2. $P_t(M_\beta) = \dfrac{1 - t^w}{1 - t}$ and $n_\beta = (-1)^{w-1}w$.

4.2   Arithmetic genus 1

By Proposition 3.10, blowing down all lines l with $\beta.l = 0$ does not change the moduli space of stable sheaves.

Lemma 4.3.   Let β be a nef and big curve class on a del Pezzo surface S of arithmetic genus 1 such that $\beta.l \geq 1$ for all line classes l. Then $\beta = -K_{S_r}$ for $0 \leq r \leq 8$ or $\beta = -K_{\mathbb{P}^1 \times \mathbb{P}^1}$.

Proof.   We have $\beta(\beta + K_S) = 2p_a(\beta) - 2 = 0$. By (4.1), we have $h^0(\beta + K_S) = 1$. Therefore, $\beta + K_S$ is effective. Hence, it is enough to show that β is ample.

If $S = S_r$ with $r \geq 2$, β is ample from the assumption that $\beta.l \geq 1$ holds for all line classes l. On $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, any nef and big class is ample. On S_1, $\beta = (d; a)$ satisfies $\beta.(h - e_1) = d - a \geq 1$. It follows that $d > a$, and β is ample.   ∎

Suppose two distinct lines l_1 and l_2 satisfy $\beta.l_1 = \beta.l_2 = 0$, then since $\beta(l_1 + l_2) = 0$, by the Hodge index theorem, $(l_1 + l_2)^2 < 0$, which implies $l_1.l_2 = 0$. Therefore, they are mutually disjoint and the number of them is at most r.

After blowing down all lines l with $\beta.l = 0$, we may assume that $\beta = -K_S$. When $0 \leq r \leq 7$, $\beta = -K_S$ is base point free. The case $\beta = -K_{S_8}$ is the only case where β is neither base point free nor can be blown down to a base point free curve class. We will study this exceptional case in Example 4.6.

Proposition 4.4.   Let β be a nef and big curve class on $S = S_r$ of arithmetic genus 1 and $\beta \neq -K_{S_8}$. Let η be the maximum number of disjoint lines l such that $\beta.l = 0$. Then

$$P_t(M_\beta) = \frac{1 - t^w}{1 - t}\left(1 + (e(S) - 2 - \eta)t + t^2\right)$$

and $n_\beta = (-1)^{w-1}w(e(S) - \eta)$.

Proof.   Let $\pi : S \to S'$ be the blowing down of all lines l such that $\beta.l = 0$. By Proposition 3.10, $M_\beta(S) \simeq M_{\pi_*\beta}(S')$. By the remark before Proposition 3.16, $P_{-1}(S', \pi_*\beta)$ is empty, which implies that the forgetful map $\xi : M^{0+}_{\pi_*\beta, 1}(S') \to M_{\pi_*\beta}(S')$ is an isomorphism by the proof of Proposition 3.17. Hence, $M_\beta(S)$ is isomorphic to $M^{0+}_{\pi_*\beta, 1}(S')$.

If $\beta \neq -K_{S_8}$, then by Lemma 4.3 S' is either $\mathbb{P}^1 \times \mathbb{P}^1$ or S_r with $0 \leq r \leq 8$ and $\pi_*\beta = -K_{S'}$, which is base point free. So, by Proposition 3.16, $P_1(S', \pi_*\beta)$ is a $\mathbb{P}^{w-1}$-bundle over S'. One can check that there is no wall-crossing for stable pairs in this case so that $M_{\pi_*\beta,1}^{0^+}(S')$ is isomorphic to $M_{\pi_*\beta,1}^{\infty}(S') = P_1(S', \pi_*\beta)$. Indeed, at a wall δ_0, we have a decomposition of the form

$$(1, (\beta, 1)) = (1, (\beta', n')) + (0, (\beta'', n'')), \tag{4.2}$$

where $\beta = \beta' + \beta''$, $n' + n'' = 1$, and $\delta_0 = \frac{w}{(-K)\beta''}n'' - 1$. Since δ_0 must be positive, we see that $n' = 0$ and $n'' = 1$. Now to have a nontrivial wall-crossing, there must be a sheaf F' with $[F'] = \beta'$ and $\chi(F') = 0$, which in addition has a nontrivial section. Consequently, $p_a(\beta') \geq 1$. But one can numerically check that when $\beta = -K_{S'}$, such a decomposition does not exist. See, for example, the list of curve classes in [20, Appendix A].

Therefore, $M_\beta(S)$ is isomorphic to a $\mathbb{P}^{w-1}$-bundle over S'; hence, the results follow. ∎

Remark 4.5. Blowing down in the proof of Proposition 4.4 corresponds to the wall-crossing in pairs. When β is nef and big, by Proposition 3.16, $P_1(S, \beta)$ is a $\mathbb{P}^{w-1}$-bundle over S. For each line l such that $\beta.l = 0$, we have a decomposition of the form

$$(1, (\beta, 1)) = (1, (\beta', 0)) + (0, (l, 1)), \tag{4.3}$$

where β' is a curve of arithmetic genus 1 and $\beta'.l = 1$. The corresponding wall is at $\delta_0 = w - 1 > 0$. A pair in $(1, (\beta', 0))$ is of the form $(s, \mathcal{O}_{\beta'})$ and a pair in $(0, (l, 1))$ is of the form $(0, \mathcal{O}_l)$.

By using Proposition 3.18, we have

$$\mathrm{Ext}^1((s, \mathcal{O}_{\beta'}), (0, \mathcal{O}_l)) \simeq \mathbb{C}^2,$$

$$\mathrm{Ext}^1((0, \mathcal{O}_l), (s, \mathcal{O}_{\beta'})) \simeq \mathbb{C}.$$

Hence, by wall-crossing at δ_0, η copies of a $\mathbb{P}^1$ bundle over $\mathbb{P}^{w-1} \times \mathbb{P}^0$ in $M_{\beta,1}^{\delta_+}$ are replaced with η copies of a $\mathbb{P}^0$ bundle over $\mathbb{P}^{w-1} \times \mathbb{P}^0$ in $M_{\beta,1}^{\delta_-}$. One can check that this wall-crossing is in fact a blowup $\rho : M_{\beta,1}^{\delta_+} \to M_{\beta,1}^{\delta_-}$ along the locus isomorphic to η copies of $\mathbb{P}^{w-1} \times \mathbb{P}^0$.

Therefore,

$$P_t(M_\beta) = \frac{1 - t^w}{1 - t} \left((1 + (e(S) - 2)t + t^2) - \eta(1 + t - 1) \right)$$

as required.

Example 4.6.   Let $\beta = -K_{S_8}$ on $S = S_8$. Then β is nef and big but not 0-very ample because the linear system $|-K_{S_8}|$ has a base point. So Proposition 4.4 does not apply. In fact by Lemma 2.6, this is the only case where $p_a(\beta) = 1$ and β is not 0-very ample while there is no line class l such that $\beta.l \leq 0$. In this case, we can directly calculate the local BPS number. Since $w = K_{S_8}^2 = 1$, there are no wall-crossings. Also, $P_{-1}(S, \beta)$ is empty. Hence, $M_\beta \simeq P_1(S_8, -K_{S_8})$. The moduli space $P_1(S_8, -K_{S_8})$ is the space of pairs (C, p) of a point p on $\mathbb{P}^2$ and a cubic curve C passing through p and the eight points of the blowup. Hence, it is the total space of the pencil of cubic curves and is isomorphic to $\mathbb{P}^2$ blown up at nine base points of the pencil. We see that $P_t(M_\beta) = 1 + 10t + t^2$ and $n_\beta = 12 = e(S_8) + 1$.

4.3   Arithmetic genus 2

Now we compute the local BPS invariants for curve classes $\beta \neq -2K_{S_8}$ with arithmetic genus 2. By Proposition 3.10, it suffices to consider very ample classes by blowing down all lines l with $\beta.l = 0$. The following lemma shows that there are only finitely many such classes.

Lemma 4.7.   If β is a very ample curve class on a del Pezzo surface S of arithmetic genus 2, then $\beta + K_S$ is effective with $p_a(\beta + K_S) = 0$ and $(-K_S)(\beta + K_S) = 2$. Hence, $\beta + K_S$ is a conic class as in (2.1).

Proof.   We have $\beta.(\beta + K_S) = 2p_a(\beta) - 2 = 2$. By (4.1), $h^0(\beta + K_S) = 2$. Therefore, $\beta + K_S$ is effective.

Let $\lambda = (-K_S)(\beta + K_S) > 0$. We have

$$p_a(\beta + K_S) = \frac{1}{2}(\beta + K_S)(\beta + K_S + K_S) + 1$$

$$= p_a(\beta) + K_S(\beta + K_S)$$

$$= 2 - \lambda$$

Thus, $p_a(\beta + K_S) < 2$ and $p_a(\beta + K_S) = (\beta + K_S)^2$. Suppose that $p_a(\beta + K_S) = 1$. Then $\lambda = (-K_S)(\beta + K_S) = 1$ and $(\beta + K_S)^2 = 1$. By applying the Hodge index theorem to the lattice generated by $-K_S$ and $\beta + K_S$, we see that this is possible only if $\beta = -2K_{S_8}$. But $-2K_{S_8}$ is not very ample.

Now suppose $p_a(\beta + K_S) < 0$. In this case, $\beta + K_S$ is not nef. Since all effective curve classes on $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ are nef, we have $S = S_r$ with $r \geq 1$. If $r \geq 2$, there is a line l

such that $(\beta + K_S)l < 0$. Then $\beta.l < (-K_S).l = 1$, which contradicts that β is very ample. If $r = 1$ and $\beta = (d; a)$, we have $d > a > 0$ by ampleness and $d \geq a + 2$ from $p_a(\beta) = 2$. Then $(\beta + K_S).E_1 = a - 1 \geq 0$ and $(\beta + K_S).(H - E_1) = d - a - 2 \geq 0$, so $\beta + K_S$ is nef, a contradiction.

Therefore, $p_a(\beta + K_S) = 0$ and $(-K_S)(\beta + K_S) = 2$.   ∎

Lemma 4.8.   Let β be a very ample curve class on S of arithmetic genus 2. Then nontrivial wall-crossings for $M_{\beta,n}^{\delta}$ arise if there is a decomposition

$$(1, (\beta, 1)) = (1, (\beta_1, 0)) + (0, (\beta_2, 1)),\qquad(4.4)$$

where β_1 and β_2 are one of the following:

(i) $p_a(\beta_1) = 1$, $p_a(\beta_2) = 0$, $-K_S.\beta_1 = w - 2$, $-K_S.\beta_2 = 2$, $\beta_2^2 = 0$, and $\beta_1.\beta_2 = 2$. There is a unique such pair (β_1, β_2). It corresponds to the wall $\delta_0 = \frac{1}{2}w - 1$.

(ii) $p_a(\beta_1) = 1$, $p_a(\beta_2) = 0$, $-K_S.\beta_1 = w - 1$, $-K_S.\beta_2 = 1$, $\beta_2^2 = -1$, and $\beta_1.\beta_2 = 2$. The number of such pairs (β_1, β_2) is $2e(S) - 8$. They correspond to the wall $\delta_0 = w - 1$.

Proof.   By the previous lemma, we have $(-K_S).\beta = K_S^2 + 2 > 2$. The list of all very ample classes of arithmetic genus 2 can be obtained by Lemma 4.7 and Example 2.11. We can check the assertions for each curve classes. For example, if $\beta = (4; 2, 1, 1, 1, 1)$, then the possible decompositions of β on S_5 are

- $(4; 2, 1, 1, 1, 1) = (3; 1, 1, 1, 1, 1) + (1; 1, 0, 0, 0, 0)$,
- $(4; 2, 1, 1, 1, 1) = (3; 1, 1, 1, 1, 0) + (1; 1, 0, 0, 0, 1)$ (four decompositions of this type),
- $(4; 2, 1, 1, 1, 1) = (4; 2, 2, 1, 1, 1) + E_2$ (four decompositions of this type).

The 1st decomposition is the case (i) of the statement and the remaining two correspond to the case (ii). The other cases can be checked similarly.   ∎

Proposition 4.9.   Let β be a curve class on $S = S_r$ of arithmetic genus 2, and assume that β is very ample. Then

$$P_t(M_\beta) = \frac{1 - t^w}{1 - t}\left(1 + (e(S) - 2)t + \left(\binom{e(S) - 2}{2} + 4\right)t^2 + (e(S) - 2)t^3 + t^4\right)$$

and $n_\beta = (-1)^{w-1}w\left(\binom{e(S)}{2} + 5\right)$.

Proof. By Proposition 3.16, $P_t(P_{\beta,1}) = P_t(\mathbb{P}^{w-1})P_t(\mathrm{Hilb}^2(S))$ and $P_t(P_{\beta,-1}) = P_t(\mathbb{P}^{w+1})$. We have a wall-crossing for each decomposition in Lemma 4.8. By the similar calculation as before we compute the wall-crossing. For the decomposition in Lemma 4.8(1),

$$\mathrm{Ext}^1((s,\mathcal{O}_{\beta_1}),(0,\mathcal{O}_{\beta_2})) \simeq \mathbb{C}^3,$$

$$\mathrm{Ext}^1((0,\mathcal{O}_{\beta_2}),(s,\mathcal{O}_{\beta_1})) \simeq \mathbb{C}^2.$$

Since $(s,\mathcal{O}_{\beta_1}) \in M^\infty_{\beta_1,0} \simeq \mathbb{P}^{w-2}$ and $(0,\mathcal{O}_{\beta_2}) \in M_{\beta_2,1} \simeq \mathbb{P}^1$, the correction term for the Poincaré polynomial in this case is $t^2 P_t(\mathbb{P}^{w-2})P_t(\mathbb{P}^1)$.

For the decomposition in Lemma 4.8(2),

$$\mathrm{Ext}^1((s,\mathcal{O}_{\beta_1}),(0,\mathcal{O}_{\beta_2})) \simeq \mathbb{C}^3,$$

$$\mathrm{Ext}^1((0,\mathcal{O}_{\beta_2}),(s,\mathcal{O}_{\beta_1})) \simeq \mathbb{C}^2.$$

In this case $(s,\mathcal{O}_{\beta_1}) \in M^\infty_{\beta_1,0} \simeq \mathbb{P}^{w-1}$ and $(0,\mathcal{O}_{\beta_2}) \in M_{\beta_2,1} \simeq \mathbb{P}^0$. So, the correction term in this case is $(2\,e(S) - 8)t^2 P_t(\mathbb{P}^{w-1})$.

Therefore, we have

$$P_t(M_\beta) = P_t(\mathbb{P}^{w-1})P_t(\mathrm{Hilb}^2(S)) - P_t(\mathbb{P}^{w+1}) - t^2 P_t(\mathbb{P}^{w-2})P_t(\mathbb{P}^1) - (2e(S) - 8)t^2 P_t(\mathbb{P}^{w-1}).$$

The Poincaré polynomial of the Hilbert scheme is well known [15]. For the Hilbert scheme of two points, we have

$$P_t(\mathrm{Hilb}^2(S)) = 1 + (e(S) - 1)t + \binom{e(S)}{2}t^2 + (e(S) - 1)t^3 + t^4.$$

Then the result follows from elementary calculations. ∎

Remark 4.10. Without the very-ampleness assumption, we can calculate $P_t(M_\beta)$ and n_β by using the blowup property. If β is nef and big but not very ample, then we may blow down all lines l with $\beta.l = 0$. Let $\pi : S \to S'$ be the blowdown. After blowdown, $\pi_*\beta$ is very ample unless $\beta = -2K_{S_8}$, since there are no (-1)-curves that do not intersect β. Therefore, we may apply Proposition 4.9 to calculate $n_{\pi_*\beta}$ on S'. Then by

Proposition 3.10, $M_\beta \simeq M_{\pi_*\beta}$. Hence, if we let η be the number of lines l such that $\beta.l = 0$ as before, we conclude that

$$P_t(M_\beta) = \frac{1-t^w}{1-t}\left(1 + (e(S) - 2 - \eta)t + \left(\binom{e(S)-2-\eta}{2} + 4\right)t^2 + (e(S)-2-\eta)t^3 + t^4\right)$$

and

$$n_\beta = (-1)^{w-1}w\left(\binom{e(S)-\eta}{2} + 5\right).$$

Remark 4.11. For $\mathbb{P}^1 \times \mathbb{P}^1$, we can check the only very ample classes with arithmetic genus 2 are $2h_1 + 3h_2$ and $3h_1 + 2h_2$. The same calculation works for these classes and we have

$$P_t(M_\beta) = \frac{1-t^{10}}{1-t}(1 + 2t + 5t^2 + 2t^3 + t^4),$$

which matches with the result of Proposition 4.9 as $e(\mathbb{P}^1 \times \mathbb{P}^1) = 4$.

For these cases, the geometry of M_β is studied in [10]. We remark that the Poincaré polynomial obtained in [10, Corollary 3.8] using a different birational method agrees with ours.

Remark 4.12. Let $\beta = -2K_{S_8} = (6, 2^8)$. This curve class is neither very ample nor contracted to a very ample divisor. So, it is not covered by Proposition 4.9.

In conclusion, we have the following formulas for the Poincaré polynomials and the local BPS invariants.

Theorem 4.13. Let β be a line class, a conic class, or a nef and big curve class on a del Pezzo surface S of arithmetic genus at most 2. Let $w = (-K_S).\beta$ and let η be the number of disjoint lines l such that $\beta.l = 0$. Then we have the following:

(i)   if $p_a(\beta) = 0$, then $P_t(M_\beta) = \dfrac{1-t^w}{1-t}$;

(ii) if $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $P_t(M_\beta) = \dfrac{1-t^w}{1-t}\left(1 + (e(S) - 2 - \eta)t + t^2\right)$;

(iii) if $\beta = -K_{S_8}$, then $P_t(M_\beta) = 1 + 10t + t^2$;

(iv) if $p_a(\beta) = 2$ and $\beta \neq -2K_{S_8}$, then

$$P_t(M_\beta) = \frac{1-t^w}{1-t}\left(1 + (e(S) - 2 - \eta)t + \left(\binom{e(S)-2-\eta}{2} + 4\right)t^2 + (e(S)-2-\eta)t^3 + t^4\right).$$

Theorem 4.14. In the situation as in Theorem 4.13, we have the following:

(i)   if $p_a(\beta) = 0$, then $n_\beta = (-1)^{w-1}w$;

(ii)   if $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $n_\beta = (-1)^{w-1} w(e(S) - \eta)$;

(iii)   if $\beta = -K_{S_8}$, then $n_\beta = 12$;

(iv)   if $p_a(\beta) = 2$ and $\beta \neq -2K_{S_8}$, then $n_\beta = (-1)^{w-1} w \left(\binom{e(S) - \eta}{2} + 5 \right)$.

In all cases studied in this paper, we see that $P_t(M_\beta)$ has a factor of $P_t(\mathbb{P}^{w-1}) = \dfrac{1 - t^w}{1 - t}$. This may suggest that M_β has a projective bundle structure. However, it is not true in general. The stable base locus decomposition of M_β when $S = \mathbb{P}^2$ is studied in [7]. It is shown there that M_β is not itself a projective bundle but is birational to a projective bundle. We formulate the following conjecture, which we proved for β of arithmetic genus at most 2.

Conjecture 4.15.   Let β be a line class, a conic class, or a nef and big curve class on a del Pezzo surface S. Let $w = (-K_S).\beta$. Then $P_t(M_\beta)$ has a factor of $P_t(\mathbb{P}^{w-1})$ and the quotient $P_t(M_\beta)/P_t(\mathbb{P}^{w-1})$ is a palindromic polynomial. Consequently, n_β is divisible by w.

This conjecture motivated the theory of log BPS numbers.

Definition 4.16.   We define the log BPS numbers by $m_\beta = (-1)^{w-1} n_\beta / w$.

Remark 4.17.   In a sequel [8], we give a more geometric approach to the log BPS numbers. We fix a smooth anticanonical divisor $E \in |-K_S|$. The set $E(\beta)$ of points P on E such that there is a curve in class β meeting E only at P is a finite set. Roughly speaking, given a point $P \in E(\beta)$, the log BPS number counts the virtual number of rational curves in class β, which meet E only at P and are smooth at P. In [8], we give a precise definition of log BPS numbers using the log Gromov–Witten theory and conjecture that it is constant along points $P \in E(\beta)$. When $P \in E(\beta)$ is β-*primitive*, which means that there are only reduced irreducible rational curves in class β meeting E only at P, this is an actual count of curves. In this case, we show that the log BPS numbers of Definition 4.16 agree with the number of such rational curves when β has arithmetic genus at most 2.

4.4   Refined BPS indices and higher genus BPS invariants

In [14], physical reasoning was used to assert an $sl_2 \times sl_2$-representation on the cohomology $H^*(M_\beta)$ of the moduli space M_β, which refines the Gopakumar–Vafa invariants. The left and the right sl_2-actions are given by the Lefschetz actions from the maps $M_\beta \to |\beta| \to pt$ respectively. A mathematical proposal for an $sl_2 \times sl_2$-representation was

given in [21]. While a counterexample to this proposal was found in [27]; the problem does not occur for smooth moduli spaces. We therefore can and will use the proposal of [21] as a precise mathematical definition.

A computational algorithm for such $sl_2 \times sl_2$-representations based on conjectures from physics was developed in [20] and generalized in [9] using the refined PT invariants. Adapting the notations in [9] we let $\left[\frac{k}{2}\right]$ denote the irreducible sl_2-representation of dimension $k+1$. Then we may write $H^*(M_\beta) = \sum_{j_L, j_R} N^\beta_{j_L, j_R} [j_L, j_R]$ as an $sl_2 \times sl_2$-representation, where $j_L, j_R \in \frac{1}{2}\mathbb{Z}$. The multiplicity $N^\beta_{j_L, j_R}$ is called the *refined BPS index*. There is a conjectural product formula for the generating function of the refined PT invariant in terms of the refined BPS indices, see [9, Section 8].

The computation of the previous sections can be applied to the calculation algorithms in [9] of the refined BPS indices. We present the results here omitting the details. In the following, we assume $\left[\frac{k}{2}\right] = 0$ for $k < 0$.

Let $r = e(S) - 3$ as before. (For $S = \mathbb{P}^1 \times \mathbb{P}^1$, $r = 1$.)

(i) If $p_a(\beta) = 0$, then $H^*(M_\beta) = \left[0, \frac{w-1}{2}\right]$.

(ii) If $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $H^*(M_\beta) = \left[\frac{1}{2}, \frac{w}{2}\right] + (r - \eta)\left[0, \frac{w-1}{2}\right] + \left[0, \frac{w-3}{2}\right]$.

(iii) If $\beta = -K_{S_8}$, then $H^*(M_\beta) = \left[\frac{1}{2}, \frac{1}{2}\right] + 8[0,0]$.

(iv) If $p_a(\beta) = 2$ and $\beta \neq -2K_{S_8}$, then $H^*(M_\beta) = \left[1, \frac{w+1}{2}\right] + (r-\eta)\left[\frac{1}{2}, \frac{w}{2}\right] + \left[\frac{1}{2}, \frac{w-2}{2}\right] + \left(\binom{r-\eta}{2} + 2\right)\left[0, \frac{w-1}{2}\right] + (r - \eta)\left[0, \frac{w-3}{2}\right] + \left[0, \frac{w-5}{2}\right]$.

These results are consistent with the refined BPS indices obtained by mirror symmetry in [17, Section 5]. We remark that $N^d_{j_L, j_R}$ in [17, Section 5] is $\sum_{(-K_S)\cdot\beta=d} N^\beta_{j_L, j_R}$.

Upon restricting to the representation $(H^*(M_\beta))_\Delta$ of the diagonal $(sl_2)_\Delta \subset sl_2 \times sl_2$, we recover the cohomology of M_β. By simple computation, we see that

(i) if $p_a(\beta) = 0$, then $(H^*(M_\beta))_\Delta = \left[\frac{w-1}{2}\right]$;

(ii) if $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $(H^*(M_\beta))_\Delta = \left[\frac{w-1}{2}\right]([1] + (r - \eta)[0])$;

(iii) if $\beta = -K_{S_8}$, then $(H^*(M_\beta))_\Delta = [1] + 9[0]$;

(iv) if $p_a(\beta) = 2$ and $\beta \neq -2K_{S_8}$, then $(H^*(M_\beta))_\Delta = \left[\frac{w-1}{2}\right]\left([2] + (r-\eta)[1] + \left(\binom{r-\eta}{2} + 3\right)[0]\right)$.

In each case, $(H^*(M_\beta))_\Delta$ is divisible by $\left[\frac{w-1}{2}\right]$, consistent with Conjecture 4.15, as $\left[\frac{w-1}{2}\right]$ is the Lefschetz representation of $\mathbb{P}^{w-1}$.

Acknowledgments

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