INTEGRALITY OF RELATIVE BPS STATE COUNTS OF TORIC DEL PEZZO SURFACES

MICHEL VAN GARREL, TONY W. H. WONG, AND GJERGJI ZAIMI

ABSTRACT. Relative BPS state counts for log Calabi-Yau surface pairs were introduced by Gross-Pandharipande-Siebert in [GPS10] and conjectured by the authors to be integers. For toric Del Pezzo surfaces, we provide an arithmetic proof of this conjecture, by relating these invariants to the local BPS state counts of the surfaces. The latter were shown to be integers by Peng in [Pen07]; and more generally for toric Calabi-Yau threefolds by Konishi in [Kon06]. Local BPS state counts were computed by Chiang-Klemm-Yau-Zaslow in [CKYZ99] via local mirror symmetry. Analogously, relative BPS state counts are related to log mirror symmetry, which for the projective plane was developed by Takahashi in [Tak01]. Relative BPS state counts are an intrinsic (virtual) extension of the $A$-model invariants considered by Takahashi. The relative BPS state counts satisfy an adapted log mirror symmetry conjecture by Takahashi: they are linearly related to the local BPS state counts and are thus calculated by periods of the mirror family.

1. INTRODUCTION

1.1. Local BPS state counts. For Calabi-Yau threefolds, BPS invariants were defined by Gopakumar-Vafa in [GVa, GVb] using an $M$-theory construction. Their definition and the related conjectures were extended to all threefolds by Pandharipande in [Pan99, Pan02]. Presently, genus 0 invariants are considered, also called BPS state counts. For the local Calabi-Yau geometries relevant below, the terminology local BPS state counts is used. Let $S$ be a smooth Del Pezzo surface and denote by $E$ a smooth effective anticanonical divisor on it, that is, an elliptic curve. Denote furthermore by $K_S$ the non-compact local Calabi-Yau threefold given as the total space
of the canonical bundle $\mathcal{O}_S(-E)$ on $S$. For a curve class $\beta \in \text{H}_2(S, \mathbb{Z})$, denote by $n_\beta$ the local BPS state count in class $\beta$, whose definition we will give shortly. From a physics point of view, $n_\beta$ is a count of $D$-branes supported on genus 0 curves of class $\beta$. This definition does not rest on rigorous mathematical foundations, so alternative definitions are used. Perhaps the most common one, which is used in this paper, aims at extracting multiple cover contributions from Gromov-Witten invariants and is as follows. Denote by $\mathcal{M}_{0,0}(K_S, \beta)$ the moduli stack of stable maps $f : C \to K_S$ from genus 0 curves with no marked points to $K_S$ such that $f_*([C]) = \beta$. Then $\mathcal{M}_{0,0}(K_S, \beta)$ is of virtual dimension 0 and the degree of its virtual fundamental class,

$$I_{K_S}(\beta) := \int_{[\mathcal{M}_{0,0}(K_S, \beta)]_{\text{vir}}} 1 \in \mathbb{Q},$$

is called the genus 0 local Gromov-Witten invariant of degree $\beta$ of $S$. The definition of the associated BPS state counts is modeled on the following ideal (rarely satisfied) situation: Suppose that $K_S$ only contained a finite number of genus 0 degree $\beta$ curves. Then $n_\beta$ should be the number of such curves. Suppose moreover that all these curves are rigidly embedded in $K_S$, i.e. with normal bundle isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Let $\tilde{C} \subset K_S$ be such a rigid rational curve of degree $\beta$. According to the Aspinwall-Morrison formula proven by Manin in [Man95], a degree $k$ stable map

$$C \to \text{Tot}(\mathcal{O}_{\tilde{C}}(-1) \oplus \mathcal{O}_{\tilde{C}}(-1))$$

contributes a factor of $\frac{1}{k^3}$ to the Gromov-Witten invariant $I_{K_S}(k\beta)$. For $k \in \mathbb{N}$, we write $k|\beta$ to mean that there is $\beta' \in \text{H}_2(K_S, \mathbb{Z})$ such that $k\beta' = \beta$. In this ideal situation, the following equality would hold:

$$(1) \quad I_{K_S}(\beta) = \sum_{k|\beta} \frac{1}{k^3} n_{\beta/k}.$$

In general, the described geometric conditions are not satisfied, and so the $n_\beta$ do not count curves in general. They can nonetheless be defined via equation (1), or alternatively, via generating functions as follows.

**Definition 1.** (Stated as a formula by Gopakumar-Vafa in [GVa, GVb]; stated as a definition by Bryan-Pandharipande in [BP01]). Assume $\beta$ to be primitive. Then the local BPS state counts $n_{d\beta}$, for $d \geq 1$, are defined as

It is believed that a (non-algebraic) deformation of $K_S$ exhibits these conditions.
rational numbers via the formula

\[ \sum_{l=1}^{\infty} I_{K_S}(l\beta) q^l = \sum_{d=1}^{\infty} n_d \sum_{k=1}^{\infty} \frac{1}{k^3} q^{dk}. \]

**Conjecture 2.** *(attributed to Gopakumar-Vafa; stated by Bryan-Pandharipande in [BP01]). For all curve classes \( \beta \in H_2(S,\mathbb{Z}) \),

\[ n_\beta \in \mathbb{Z}. \]

Conjecture 2 was proven by Peng in [Pen07] in the case of toric Del Pezzo surfaces, which are the Del Pezzo surfaces of degree \( \geq 6 \). More generally, a proof for toric Calabi-Yau threefolds was given by Konishi in [Kon06].

1.2. **Relative BPS state counts.** The definitions and conjectures relating to relative BPS state counts mirror the discussion of the previous section. These invariants were introduced by Gross-Pandharipande-Siebert in [GPS10]. Whereas the previous section is concerned with local Calabi-Yau threefolds, this one deals with open Calabi-Yau surfaces, which are examples of log Calabi-Yau surfaces.

**Definition 3.** *(See [GPS10]).* Let \( S \) be a smooth surface and let \( D \subset S \) a smooth divisor. Let furthermore \( \gamma \in H_2(S,\mathbb{Z}) \) be nonzero. The pair \((S,D)\) is called *log Calabi-Yau with respect to* \( \gamma \) if

\[ D \cdot \gamma = c_1(S) \cdot \gamma. \]

If equation (3) holds for all \( \gamma \in H_2(S,\mathbb{Z}) \), which is the situation considered below, we abbreviate and say that \((S,D)\) is a *log Calabi-Yau* surface pair. Sometimes the divisor \( D \) is excluded from the notation.

The discussion in [GPS10] is concerned with any log Calabi-Yau surface pair. For our purposes, we restrict to Del Pezzo surfaces. As in the previous section, let \( S \) be a Del Pezzo surface and denote by \( E \) be a smooth effective anticanonical divisor on it. Then the pair \((S,E)\) is log Calabi-Yau. This is the open Calabi-Yau geometry considered in this section. The fact that the canonical bundle of \( S \) is trivial away from \( E \) justifies this terminology. Let \( \beta \in H_2(S,\mathbb{Z}) \) be the class of a curve and set \( w = E \cdot \beta \). A generic curve representing \( \beta \) would meet \( E \) in \( w \) points of simple tangency. The moduli stack of such curves is of virtual dimension \( w - 1 \). Instead, we can impose that the curve meets \( E \) in fewer points with higher tangencies, cutting down the virtual dimension. We consider the maximal case. Denote
by $\overline{M}(S/E, w)$ the moduli stack of genus 0 relative stable maps $f : C \to S$ representing $\beta$ and such that the image of $C$ meets $E$ in one point of tangency $w$. Then $\overline{M}(S/E, w)$ is of virtual dimension 0 and the degree of its virtual fundamental class,

$$N_S[w] := \int_{[\overline{M}(S/E, w)]^{vir}} 1 \in \mathbb{Q},$$

is called the genus 0 relative Gromov-Witten invariant of degree $\beta$ and maximal tangency of $(S, E)$. Denote by $\iota : P \to S$ a rigid element of $\overline{M}(S/E, w)$. For $k \geq 1$, denote by $M_P[k]$ the contribution of $k$-fold multiple covers of $P$ to $N_S[kw]$ (see [GPS10] for precise definitions).

**Proposition 4.** (Proposition 6.1 in [GPS10]).

$$M_P[k] = \frac{1}{k^2} \binom{k(w - 1) - 1}{k - 1}.$$

Consequently:

**Definition 5.** (Paragraph 6.3 in [GPS10]). For $d \geq 1$, the relative BPS state counts $n_S(dw) \in \mathbb{Q}$ are defined by means of the equality

$$\sum_{l=1}^{\infty} N_S[lw] q^l = \sum_{d=1}^{\infty} n_S(dw) \sum_{k=1}^{\infty} \frac{1}{k^2} \binom{k(dw - 1) - 1}{k - 1} q^{dk}. \tag{4}$$

**Conjecture 6.** (Conjecture 6.2 in [GPS10]). Let $\beta \in H_2(S, \mathbb{Z})$ be an effective curve class and set $w = \beta \cdot E$. Then, for all $d \geq 1$,

$$n_S(dw) \in \mathbb{Z}.$$

1.3. **Main result.** Our main result is based on the following theorem, which was proved for $\mathbb{P}^2$ by Gathmann in [Gat03]. A proof for all Del Pezzo surfaces was announced by Graber-Hassett.

**Theorem 7.** (Gathmann for $\mathbb{P}^2$ in [Gat03], for general $S$ announced by Graber-Hassett). Let $S$ be a Del Pezzo surface and denote by $E$ a smooth effective anticanonical divisor on it. Let $\beta \in H_2(S, \mathbb{Z})$ be an effective curve class and set $w = \beta \cdot E$. Then the following identities of Gromov-Witten invariants hold:

$$N_S[w] = (-1)^{w+1} w I_\beta(K_S). \tag{5}$$

In the present paper, we prove the following theorem:
Theorem 8. Let $\beta \in H_2(S,\mathbb{Z})$ be an effective non-zero primitive curve class. Consider two sequences of rational numbers $\{N_d\beta(S,D)\}_{d \geq 1}$ and $\{I_d\beta(K_S)\}_{d \geq 1}$. Assume that they are related, for all $d \geq 1$, by equation (5). Define two sequences of rational numbers $\{n_S[dw]\}_{d \geq 1}$ and $\{n_d\beta\}_{d \geq 1}$ by means of the equations (2) and (4). Then, $n_S[dw] \in \mathbb{Z}$ for all $d \geq 1$ if and only if $dw \cdot n_d \beta \in \mathbb{Z}$ for all $d \geq 1$.

An immediate consequence is as follows:

Corollary 9. Conjecture 2 for $K_S$ implies conjecture 6 for $(S,D)$.

Using the integrality result of Peng in [Pen07] or of Konishi in [Kon06] yields:

Corollary 10. Conjecture 6 holds for toric Del Pezzo surfaces.

Remark 11. It follows from theorem 7 that the relative BPS state counts are related to the local BPS state counts (see lemma 13 below). The local invariants are calculated via mirror symmetry (see Chiang-Klemm-Yau-Zaslow in [CKYZ99]) and thus, it is expected that the relative BPS state counts are directly computed via mirror symmetry as well. It is not clear though what the $B$-model is. A mirror symmetry conjecture in this sense for the projective plane was formulated and explored by Takahashi in [Tak01]. Let us note though that Takahashi considers an alternative enumerative version of relative BPS state counts. There is at present no physical interpretation for these invariants.

1.4. Relationship to Takahashi’s work on log mirror symmetry.
Takahashi in [Tak01] develops logarithmic mirror symmetry for $\mathbb{P}^2$ relative to an elliptic curve $E$. The $A$-model invariants considered in [Tak01] are as follows. Let $d \geq 1$. A degree $d$ curve in $\mathbb{P}^2$ will meet $E$ in a $3d$-torsion point. Choose a group structure on $E$ such that the zero element $0 \in E$ is a flex point. Let $d \geq 1$ and choose $P \in E$ a point of order $3d$ for the chosen group structure. Then $m_d$ is defined as the number of rational degree $d$ curves in $\mathbb{P}^2$ meeting $E$ at only $P$. The relative BPS state counts $n_{\mathbb{P}^2}[3d]$ are a virtual extension of $m_d$ in the sense that the virtual rational curves counted by $n_{\mathbb{P}^2}[3d]$ are allowed to meet $E$ in any $3d$-torsion point, not just at $P$. The attribute virtual is justified since the $n_{\mathbb{P}^2}[3d]$ arise from Gromov-Witten invariants. Based on his calculations and on the work by Gathmann in [Gat03], Takahashi conjectures that the $m_d$ are related to the local BPS state counts $n_d$ of $\mathbb{P}^2$ as follows.
Conjecture 12. \((\text{Takahashi in [Tak01]}\)\)

\[ 3d m_d = (-1)^{d+1} n_d. \]

The above conjecture provides an enumerative interpretation of the invariants \(n_d\). We prove a result analogous to conjecture 12 in lemma 13 below. This lemma provides a linear relationship between the sets of invariants \(n_{p^2}[3d]\) and \(n_d\). It follows that the relative BPS state counts are calculated from the periods of the mirror family. This is more generally true for any Del Pezzo surface, since lemma 13 holds in that setting.

1.5. Outline. The proof of theorem 8 is split into two parts. In section 2, lemma 13 states the precise relationship between the local and relative BPS state counts that we consider. Each set of invariants is related to the other by means of an invertible matrix. In section 3, we analyze congruence classes relating to the entries of this matrix. We prove that each entry is integer valued, which proves theorem 8.

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2. Combinatorics

Let \(S\) be a Del Pezzo surface with smooth effective anticanonical divisor \(E\). Let \(\beta \in H_2(S, \mathbb{Z})\) be a non-zero effective primitive curve class. Consider two sequences of rational numbers \(\{N_{d\beta}(S, D)\}_{d \geq 1}\) and \(\{I_{d\beta}(K_S)\}_{d \geq 1}\), which we assume to be related, for all \(d \geq 1\), by equation (5). Define two sequences
of rational numbers \( \{n_S[dw]\}_{d \geq 1} \) and \( \{n_{d\beta}\}_{d \geq 1} \) by means of the equations (2) and (4). Note that formula (4) is equivalent to the set of equations:

\[
N_S[dw] = \sum_{k|d} \frac{1}{k^2} \left( \frac{k(\frac{d}{k} w - 1) - 1}{k - 1} \right) n_S[dw/k].
\]

Combining the formulas (5), (1) and (6) yields the following collection of formulas:

\[
\sum_{k|d} \frac{1}{k^2} \left( \frac{k(\frac{d}{k} w - 1) - 1}{k - 1} \right) n_S[dw/k] = (-1)^{dw+1} dw \sum_{k|d} \frac{1}{k^3} n_{d\beta/k}.
\]

Fix a positive integer \( N \) and consider the formulas of (7) for \( 1 \leq d \leq N \). Hence, in matrix form, this collection of formulas is expressed as

\[
R [n_S[dw]]_d = A \cdot L \cdot A^{-1} \left[ (-1)^{dw+1} dw n_{d\beta} \right]_d,
\]

where \( R, A \) and \( L \) are the lower triangular \( N \times N \) matrices given by

\[
R_{ij} := \begin{cases} 
\frac{1}{(j/i)^2} \left( \frac{i/j}{i/j-1} \right)^2 & \text{if } j|i, \\
0 & \text{else};
\end{cases}
\]

\[
A_{ij} := (-1)^{iw+1} iw \delta_{ij};
\]

\[
L_{ij} := \begin{cases} 
\frac{1}{(j/i)^3} & \text{if } j|i, \\
0 & \text{else}.
\end{cases}
\]

Note that \( A \cdot L \cdot A^{-1} \) has determinant 1.

**Notation.** For a square-free integer \( n \), denote by \( \#(n) \) the number of primes in the prime factorization of \( n \). For integers \( k \) and \( m \), write \( k \in I(m) \) to mean that \( k \) divides \( m \) and that \( m/k \) is square-free.

**Lemma 13.** Define the \( N \times N \) matrix \( C \) as follows. If \( t|s \), set

\[
C_{st} := \frac{(-1)^{sw}}{(s/t)^2} \sum_{k \in I(s/t)} (-1)^{\#(s/kt)} (-1)^{kw} \left( \frac{k(tw - 1) - 1}{k - 1} \right).
\]

If \( t \nmid s \), set \( C_{st} = 0 \). Then the invariants \( \{n_S[dw]\} \) and \( \{(-1)^{dw+1} dw n_{d\beta}\} \), for \( 1 \leq d \leq N \), are related via

\[
C \cdot [n_S[dw]]_d = \left[ (-1)^{dw+1} dw n_{d\beta} \right]_d.
\]

Moreover, \( C \) has determinant 1 and is lower triangular. It follows by Cramer’s rule that

\[
C \text{ integral } \iff C^{-1} \text{ integral}.
\]
Proof. We start by writing \( L = B \cdot \tilde{L} \cdot B^{-1} \), where
\[
\tilde{L}_{ij} = \begin{cases} 
1 & \text{if } j|i, \\
0 & \text{else}; 
\end{cases} 
\]
\[
B_{ij} = \frac{1}{i^3} \delta_{ij}.
\]
By Möbius inversion, the inverse of \( \tilde{L} \) is given by
\[
\tilde{L}^{-1}_{ij} = \begin{cases} 
(-1)^{(i/j)} & \text{if } j|i \text{ and } i/j \text{ is square-free}, \\
0 & \text{else}.
\end{cases}
\]
Then,
\[
(AB)_{ij} = (-1)^{iw+1} w \delta_{ij},
\]
and
\[
((AB)^{-1})_{ij} = (-1)^{iw+1} \frac{r^2}{w} \delta_{ij}.
\]
It thus follows from formula (8) that \( C \) is given by
\[
C = AB \cdot \tilde{L}^{-1} \cdot (AB)^{-1} \cdot R.
\]
A calculation yields
\[
(AB \cdot \tilde{L}^{-1})_{sr} = \begin{cases} 
(-1)^{sw+1} w \frac{r}{s^2} (-1)^{(s/r)} & \text{if } r|s \text{ and } s/r \text{ is square-free}, \\
0 & \text{else}; 
\end{cases}
\]
and
\[
((AB)^{-1} \cdot R)_{rt} = \begin{cases} 
(-1)^{rw+1} \frac{r^2}{w} \frac{1}{(r/t)^2} \left(\frac{r/t}{r/t-1}\right)^{-1} & \text{if } t|r, \\
0 & \text{else}.
\end{cases}
\]
If \( t \) does not divide \( s \), then there is no integer \( r \) such that \( t|r|s \), so that \( C_{st} = 0 \). If, however, \( t|s \), then
\[
C_{st} = \sum(-1)^{sw+1} (-1)^{(s/r)} (-1)^{rw+1} \frac{1}{(s/t)^2} \left(\frac{r/t}{r/t-1}\right)^{-1} = \frac{(-1)^{sw+1}}{(s/t)^2} \sum(-1)^{(s/r)} (-1)^{rw+1} \left(\frac{r/t}{r/t-1}\right)^{-1}.
\]
where the sum runs over all \( r \) such that \( t|r|s \) and such that \( s/r \) is square-free.

Set \( k = r/t \), so that, for \( t \) dividing \( s \),

\[
C_{st} = \frac{(-1)^{sw+1}}{(s/t)^2} \sum_{k \in I(s/t)} (-1)^{\#(s/kt)} (-1)^{kw+1} \frac{k(tw - 1) - 1}{k - 1}
\]

\[
= \frac{(-1)^{sw}}{(s/t)^2} \sum_{k \in I(s/t)} (-1)^{\#(s/kt)} (-1)^{kw} \frac{k(tw - 1) - 1}{k - 1},
\]

finishing the proof. \( \Box \)

Lemma 13 reduces theorem 8 to proving that the coefficients of the matrix \( C \) are integers. This is achieved in the lemmas 16 and 17 of the next section.

3. Integrality

We start by stating the following lemma, which follows directly form the proof of lemma 1.1 of [Pen07].

**Lemma 14.** (Peng) Let \( a, b \) and \( \alpha \) be positive integers and denote by \( p \) a prime number. If \( p = 2 \), assume furthermore that \( \alpha \geq 2 \). Then

\[
\begin{pmatrix} p^\alpha a - 1 \\ p^\alpha b - 1 \end{pmatrix} \equiv \begin{pmatrix} p^{\alpha - 1}a - 1 \\ p^{\alpha - 1}b - 1 \end{pmatrix} \mod (p^{2\alpha}).
\]

In the case that \( p = 2 \) and \( \alpha = 1 \), we have the following lemma:

**Lemma 15.** Let \( k \geq 1 \) be odd and let \( a \) be a positive integer. Then

\[
\begin{pmatrix} 2ka - 1 \\ 2k - 1 \end{pmatrix} \equiv (-1)^{a+1} \begin{pmatrix} ka - 1 \\ k - 1 \end{pmatrix} \mod (4).
\]

**Proof.** Note that

\[
\begin{pmatrix} 2ka - 1 \\ 2k - 1 \end{pmatrix} = \frac{2ka - 1}{2k - 1} \cdot \frac{2ka - 2}{2k - 2} \cdot \cdots \frac{2ka - 2k}{2k - 2k} \cdot \frac{2ka - 2k + 1}{2k - 2k + 1}
\]

\[
= \frac{2ka - 1}{2k - 1} \cdot \frac{ka - 1}{k - 1} \cdot \cdots \frac{ka - k + 1}{k - k + 1} \cdot \frac{2ka - 2k + 1}{2k - 2k + 1}
\]

\[
= \frac{(ka - 1)(ka - 2) \cdots (ka - k + 1)}{(k - 1)(k - 2) \cdots 1} \cdot \frac{(2ka - 1)(2ka - 3) \cdots (2ka - 2k + 1)}{(2k - 1)(2k - 3) \cdots 1}
\]

\[
= \frac{(ka - 1)}{k - 1} \cdot \frac{(2ka - 1)(2ka - 3) \cdots (2ka - 2k + 1)}{(2k - 1)(2k - 3) \cdots 1}.
\]
and hence
\[
\binom{2ka - 1}{2k - 1} + (-1)^a \binom{ka - 1}{k - 1} = \binom{ka - 1}{k - 1} \left( (-1)^a + \frac{(2ka - 1)(2ka - 3)(2ka - 2k + 1)}{(2k - 1)(2k - 3)\cdots 1} \right).
\]

It thus suffices to show that
\[
\left(\frac{2ka - 1)(2ka - 3)(2ka - 2k + 1)}{(2k - 1)(2k - 3)\cdots 1} \equiv (-1)^{a+1} \mod (4). \tag{11}
\]
Suppose first that \(a\) is even, so that the left-hand-side of (11) is congruent to
\[
\frac{(-1)(-3)\cdots(-2k + 3)(-2k - 1)}{1 \cdot 3 \cdot (2k - 3)(2k - 1)} \equiv (-1)^k \equiv (-1)^{a+1} \mod (4),
\]
where the last congruence follows from the fact that \(k\) is odd. Suppose now that \(a\) is odd. Then the left-hand-side of the expression (11) is congruent to
\[
\frac{2k(a - 1) + (2k - 1)}{2k - 1} \cdot \frac{2k(a - 1) + (2k - 3)}{2k - 3} \cdots \frac{2k(a - 1) + 1}{1} \equiv 1 \equiv (-1)^{a+1} \mod (4).
\]

We return to the proof of theorem 8. If \(s = t\), then \(C_{st} = 1\) is an integer.
We assume henceforth that \(t|s\), but \(t \neq s\). Let \(p\) be a prime number and \(\alpha\) a positive integer. For an integer \(n\), we use the notation
\[
p^\alpha | n,
\]
to mean that \(p^\alpha | n\), but \(p^{\alpha+1} \not| n\). In order to show that \(C_{st} \in \mathbb{Z}\), we show, for every prime number \(p\), that if
\[
p^\alpha | s/t,
\]
then
\[
p^{2\alpha} \mid \sum_{k \in I(s/t)} (-1)^{\#(s/kt)} (-1)^{ktw} \binom{k(tw - 1) - 1}{k - 1}.
\]
Fix a prime number \(p\) and a positive integer \(\alpha\) such that
\[
p^\alpha | s/t.
\]
Let $k \in I(s/t)$. For $s/kt$ to be square-free, it is necessary that $p^{\alpha-1}|k$. This splits into the two cases

$$ p^{\alpha-1}|k, $$

in which case $k \in I(s/pt)$; or

$$ p^\alpha|k, $$

so that $k = pl$ for $l \in I(s/pt)$. Regrouping the terms of the expression (9) accordingly yields

$$ \sum_{l \in I(s/pt)} \sum_{k \in \{l, pl\}} (-1)^{\#(s/kt)} (-1)^{ktw} \left( \frac{k(tw - 1) - 1}{k - 1} \right). $$

Thus, it suffices to prove that for all $l \in I(s/pt)$,

$$ f(l) := \sum_{k \in \{l, pl\}} (-1)^{\#(s/kt)} (-1)^{ktw} \left( \frac{k(tw - 1) - 1}{k - 1} \right) \equiv 0 \mod \left( p^{2\alpha} \right), $$

which we proceed in showing. There are two cases: either the sign $(-1)^{ktw}$ in the above sum changes or not. The only case where the sign does not change is when $p = 2, \alpha = 1$, and both $t$ and $w$ are odd.

**Lemma 16.** Assume that either $p \neq 2$ or, if $p = 2$, that $\alpha > 1$. Then

$$ f(l) \equiv 0 \mod \left( p^{2\alpha} \right). $$

**Proof.** In this case,

$$ f(l) = \pm \left( \left( \frac{pl(tw - 1) - 1}{k - 1} \right) - \left( \frac{l(tw - 1) - 1}{k - 1} \right) \right) $$

$$ \equiv 0 \mod \left( p^{2\alpha} \right), $$

by lemma 14. □

**Lemma 17.** Assume that $p = 2$ and that $\alpha = 1$. Then

$$ f(l) \equiv 0 \mod (4). $$

**Proof.** In this case,

$$ f(l) = \pm \left( \left( \frac{2l(tw - 1) - 1}{2l - 1} \right) + (-1)^{tw-1} \left( \frac{l(tw - 1) - 1}{l - 1} \right) \right) $$

$$ \equiv 0 \mod (4), $$

follows from lemma 15. □
This finishes the proof that the entries of $C$ are integers. Consequently, by lemma 13, the proof of theorem 8 is complete.

References


 Fields Institute, 222 College Street, Toronto, ON, M5T 3J1, Canada

E-mail address: mvangarr@fields.utoronto.ca

Department of Mathematics, Kutztown University of Pennsylvania, 15200 Kutztown Road, Kutztown, PA 19530, USA

E-mail address: wong@kutztown.edu

Department of Mathematics, California Institute of Technology, MC 253-37, 1200 East California Boulevard, Pasadena, CA 91125, USA

E-mail address: gzaimi@caltech.edu