LOG BPS NUMBERS OF LOG CALABI-YAU SURFACES

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Abstract. Let \((S, E)\) be a log Calabi-Yau surface pair with \(E\) a smooth divisor. We define new conjecturally integer-valued counts of \(A^1\)-curves in \((S, E)\). These log BPS numbers are derived from genus 0 log Gromov-Witten invariants of maximal tangency along \(E\) via a formula analogous to the multiple cover formula for disk counts. A conjectural relationship to genus 0 local BPS numbers is described and verified for del Pezzo surfaces and curve classes of arithmetic genus up to 2. We state a number of conjectures and provide computational evidence.

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1. Introduction

Let \((S, E)\) be a log Calabi-Yau surface with a smooth divisor, by which we shall mean that \(S\) is a smooth projective surface and \(E\) is a smooth effective anticanonical divisor on it. By the adjunction formula, each connected component of \(E\) has genus 1. It is expected that \(S \setminus E\) admits a Strominger-Yau-Zaslow (SYZ) special Lagrangian torus fibration [33], with singular fibers away from \(E\). We are interested in counts of holomorphic disks in \(S \setminus E\) with boundary ending on a SYZ fiber near \(E\). In analogy to [32, Theorem 3.4] and [49, 50], these are in turn predicted to correspond to counts of tropical disks in the relevant scattering diagrams of [9]. The latter describe the wall-crossing automorphisms, cf. [9, Proposition 5.4].

The SYZ mirror conjecture is successfully implemented in algebraic geometry in the Gross-Siebert program [34], see also [33, 29, 31, 36, 28, 21]. Assume that \(S\) is del Pezzo. The construction of the mirror for \((S, E)\) proceeds via the relevant scattering diagram as detailed in [9]. In particular, the superpotential on the mirror is constructed via summing the monomials

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attached to broken lines. In [9, §5.4] the wall-crossing functions are expressed as generating functions of Maslov index 0 tropical disks. They in turn are expected to be expressible as counts of $A^1$-curves on $(S, E)$, which are rational curves in $S$ meeting $E$ in one point of maximal tangency. Their virtual definition is as the (Q-valued) genus 0 log or relative Gromov-Witten invariants \[23, 17, 48, 35, 1, 10\] of maximal tangency. They are virtually counting rational curves in $S$ meeting $E$ in one point of maximal tangency.

In this paper, we explore how to obtain $\mathbb{N}$-valued invariants out of these log Gromov-Witten (GW) invariants, which should be the underlying counts of (immersed) $A^1$-curves. In accordance with standard definitions, we call them log BPS numbers. The relationship between the log Gromov-Witten invariants of maximal tangency and the log BPS numbers is the formula of Definition 1.1. It is analogous to the multiple cover formula for genus 0 open Gromov-Witten invariants [16] and generalized DT invariants [39]. In [32, Proposition 6.1] (see (6.1)), the authors compute the contribution of multiple covers over rigid relative maps to the relative GW invariants. This leads them to define relative BPS state counts, which are shown to be integers for toric del Pezzo surfaces in [22]. One motivation for this work is that the enumerative meaning of relative BPS state counts are not clear in the context of a smooth divisor. We make the connection of log BPS numbers with relative BPS state counts and loop quiver DT invariants in §6.

Apart from the connection to the Gross-Siebert program, the advantages of log BPS state counts are twofold. Firstly, generically they are weighted counts of curves (Proposition 1.7). Secondly, they are conjecturally independent of the point of contact (Conjecture 1.3).

In this paper, we set up the theory of log BPS state counts. We state a number of conjectures, some of which we prove for arithmetic genus up to 2 in the case of del Pezzo surfaces. This paper is a continuation of [11]. A B-model analogue of log GW invariants of maximal tangency will be developed in [19].

1.1. Summary of results and conjectures. We will always denote by $S$ a smooth projective surface and by $E$ a smooth divisor on $S$, and require additional conditions on $(S, E)$ as needed. Let $\beta \in H_2(S, \mathbb{Z})$ be a curve class, by which we shall mean that $\beta$ can be represented by a nonempty one-dimensional subscheme, and assume that $w := \beta . E > 0$. If the triple $(S, E, \beta)$ satisfies $(K_S + E) . \beta = 0$, following [32], we will say that $(S, E)$ is log Calabi-Yau with respect to $\beta$. Denote by $N_\beta(S, E)$ the log Gromov-Witten invariant of rational curves in $S$ with maximal tangency along $E$, as introduced in Section 2. It can be non-zero only if $(K_S + E) . \beta = 0$. We define the total log BPS numbers for any such $(S, E, \beta)$.

Often we will additionally assume that $S$ is a regular surface, i.e. a surface with irregularity $h^1(\mathcal{O}_S) = 0$, and that $E$ is elliptic. In this case $\beta$ determines a unique Chow class, and the set $E(\beta)$ (see Definition 2.13) of possible points of contact of maximally tangent curves with $E$ is finite of order $w^2$. Then log BPS numbers are obtained by fixing the point of contact with $E$. Note that for the pair of an irregular surface with elliptic divisor, we can fix a finite set of possible points of contact if we refine the invariants by using a Chow class $\beta$. 
If $S$ is regular and $E$ is an elliptic curve, Proposition 2.18 states that under certain conditions $S$ has to be rational for log BPS numbers to be non-zero. Accordingly, in this introduction we mostly assume $S$ is rational hereafter. Note that if $S$ is rational and $E$ smooth anticanonical, then $E$ is necessarily an elliptic curve (after choosing a zero element).

Whenever $E(\beta)$ is finite, $\mathcal{N}_\beta(S, E)$ decomposes as a finite sum

$$\mathcal{N}_\beta(S, E) = \sum_{P \in E(\beta)} \mathcal{N}_{\beta}^P(S, E).$$

As is described in Corollary 2.10, see also Figure 2.1 and Section 6.1, the relevant moduli space contains both multiple covers and reducible curves.

The next definition and conjecture constitute the principal novelty to curve counts of the present paper. They are motivated from discussions with Pierrick Bousseau, through considerations of open Gromov-Witten invariants. In the case of $S = \mathbb{P}^2$, they were already formulated in [65, Remark 4.11].

**Definition 1.1** (See Definition 3.3). Let $(S, E)$ be a log Calabi-Yau surface with respect to $\beta \in H_2(S, \mathbb{Z})$. The total log BPS number $m^\text{tot}_\beta$ is defined implicitly via

$$\mathcal{N}_\beta(S, E) = \sum_{k | \beta} (-1)^{(k-1)w/k} m^\text{tot}_{\beta/k}.$$ 

Assume that $S$ is rational, $E$ anticanonical and let $P \in E(\beta)$. The log BPS number at $P$, $m^P_\beta$, is defined implicitly via

$$\mathcal{N}_{\beta}^P(S, E) = \sum_{k | \beta} (-1)^{(k-1)w/k} m^P_{\beta/k},$$

where we set $m^P_{\beta/k} = 0$ if $P \not\in E(\beta')$. We note that there is an inclusion $E(\beta/k) \subseteq E(\beta)$, so that $m^\text{tot}_{\beta/k} = \sum_{P \in E(\beta)} m^P_{\beta/k}$ holds.

Note that the number of non-zero terms in the above formula depends on the arithmetic properties of $P$. If $S = \mathbb{P}^2$ for example, at points of maximal order there will only be one term, whereas at points of lower order there will be many: see Section 6.1.

**Conjecture 1.2.** Let $(S, E)$ be a log Calabi-Yau surface with respect to $\beta \in H_2(S, \mathbb{Z})$. Then $m^\text{tot}_\beta \in \mathbb{N}$. Additionally, if $S$ is rational and $E$ is elliptic, then $m^P_\beta \in \mathbb{N}$ for $P \in E(\beta)$.

For a del Pezzo surface and an anticanonical divisor, the above conjecture is a consequence of Conjecture 1.3, Proposition 1.7 (1), Proposition 4.10 and deformation invariance of $\mathcal{N}_{\beta}^P(S, E)$. Note that in general, for $P \neq P' \in E(\beta)$, $\mathcal{N}_{\beta}^P(S, E) \neq \mathcal{N}_{\beta}^{P'}(S, E)$.

**Conjecture 1.3** (See Conjecture 3.4). Let $(S, E)$ be a rational log Calabi-Yau surface with smooth divisor and let $\beta \in H_2(S, \mathbb{Z})$. For all $P, P' \in E(\beta)$,

$$m^P_\beta = m^{P'}_\beta.$$
Equivalently, for all $P \in E(\beta)$,
\[ m_{\beta}^{\text{tot}} = w^2 m_{\beta}^P. \]

Our first main result, proven in Section 6 is:

**Theorem 1.4.** Conjecture 1.3 holds for $S = \mathbb{P}^2$ and degree $\leq 4$.

These new invariants exhibit a surprising connection to the local BPS invariants $n_{\beta}$ \cite{25, 26, 56, 57, 8, 40}. For the definition in general, see (3.1). In the case of del Pezzo surfaces, there is the following equivalent definition from \cite{40}.

**Definition 1.5.** Assume that $S$ is a del Pezzo surface. Denote by $M_{\beta}$ the moduli space of one-dimensional stable (with respect to $-K_S$) sheaves $F$ on $S$ with holomorphic Euler characteristic $\chi(F) = 1$ and $[F] = \beta$. The genus 0 local BPS invariant $n_{\beta} = n_{\beta}^0 \in \mathbb{Z}$ of class $\beta$ is
\[ n_{\beta} := (-1)^{w-1} e(M_{\beta}), \]
where $e(\cdot)$ denotes the (topological) Euler characteristic.

Remarkably, see (3.6), if $S$ is rational and $E$ is anticanonical and nef, Conjecture 1.3 has the equivalent characterization that for all $P \in E(\beta)$,
\[ n_{\beta} = (-1)^{w-1} w m_{\beta}^P. \]

We introduce in Definition 4.6 the notion of $\beta$-primitive points for del Pezzo surfaces. A weaker conjecture is as follows:

**Conjecture 1.6.** Assume that $S$ is del Pezzo and $E$ anticanonical, and let $P \in E(\beta)$ be $\beta$-primitive. Then
\[ n_{\beta} = (-1)^{w-1} w m_{\beta}^P. \]

Viewing $m_{\beta}^P$ as counts of disks in $S \setminus E$ bounding a SYZ fiber near $E$ and viewing $n_{\beta}$ as counts of rational curves in a (hypothetic) deformation of Tot$(K_S)$, Conjecture 1.6 can be seen as an instance of open-closed duality, see §1.2.

If $S$ is del Pezzo, $E$ is anticanonical and $P \in E(\beta)$ is $\beta$-primitive, then by definition, each curve of class $\beta$ (and of any genus) which meets $E$ only at $P$ is irreducible. Among those, denote by $M_{\beta,P}$ the finite set of irreducible rational curves of class $\beta$ maximally tangent to $E$ at $P$.

We remark that a maximally tangent curve in $(S,E)$, by definition (Definition 2.15), has only 1 analytic branch on $E$. For an (irreducible) rational curve $C$, recall that the compactified Jacobian $\text{Pic}^0(C)$ of $C$ is the scheme parametrizing torsion free, rank 1, degree 0 sheaves on $C$.

\[ ^{1} \text{The sign is usually given as } (-1)^{\beta,\beta+1}, \text{ which equals } (-1)^{w-1} \text{ by the adjunction formula.} \]
Proposition 1.7 (See Proposition 4.20). Let $(S, E)$ be a regular log Calabi-Yau surface with smooth divisor and let $\beta \in H_2(S, \mathbb{Z})$ be a curve class. Let $C$ be an (irreducible) rational curve of class $\beta$ maximally tangent to $E$ at $P$, and denote the normalization map by $n : \mathbb{P}^1 \to C$. Then:

(1) The map $n$ gives an isolated point of $\overline{\mathcal{M}}_\beta(S, E)$, and contributes a positive integer to $N_\beta(S, E)$.
(2) If $C$ is immersed outside $P$, $[n]$ contributes 1 to $N_\beta(S, E)$.

Assume that $S$ is del Pezzo.

(3) If $n$ is smooth at $P$, then it contributes $\eta(\text{Pic}^0(C))$ to $N_\beta(S, E)$.
(4) If $C$ is smooth at $P$ and has arithmetic genus 1, then $[n]$ contributes $\eta(C)$ to $N_\beta^P(S, E)$.
(5) Let $P \in E(\beta)$ be $\beta$-primitive. Assume that all $C \in M_{\beta, P}$ are smooth at $P$. Then

$$m_{\beta}^P = \sum_{C \in M_{\beta, P}} \eta(\text{Pic}^0(C)).$$

In the case of $\mathbb{P}^2$, Conjecture 1.6 was originally stated for $\mathbb{P}^2$ in [66, Remark 2.2]. Using calculations from [64], it was verified that it holds for $\mathbb{P}^2$ and degree $\leq 6$, as well as for degree 7 and 8 under certain technical hypotheses. The combination of Conjecture 1.6 and Proposition 1.7 yields an enumerative interpretation of the $n_\beta$. For an explanation of the factor $w$, see [1.2].

Recall that $p_a(\beta) := \frac{1}{2} \beta(\beta + K_S) + 1$ is the arithmetic genus of $\beta$. Recall that line classes on $S$ are the classes $l \in \text{Pic}(S)$ such that $l^2 = -1$ and $-K_S.l = 1$. Conic classes are the classes $D \in \text{Pic}(S)$ such that $p_a(D) = 0$ and $-K_S.D = 2$. Our main result is as follows:

Theorem 1.8. Let $S$ be a del Pezzo surface, $E$ a smooth anticanonical divisor and $\beta$ a curve class on $S$. Assume that $\beta$ is a line class, a conic class or a nef and big class. Then Conjecture 1.6 holds if $p_a(\beta) = 0$ or 1. Assuming that $(S, E)$ is general, it holds for classes $\beta$ of arithmetic genus 2 as well.

The proof of Theorem 1.8 proceeds by calculation of both of the relevant log and local BPS numbers. On the log side we find in Section 5 all rational curves of a given class in $S$ that are maximally tangent to $E$ at a primitive point. On the local side, we carried out the calculation in [11] via wall-crossing with stable pairs.

Theorem 1.9 (Theorem 1.1 in [11]). Assume that $S$ is a del Pezzo surface, that $E$ is a smooth anticanonical divisor and that $\beta$ is a line class, a conic class or a nef and big curve class. Let $\eta$ be the number of line classes $l$ such that $\beta.l = 0$. We denote by $S_8$ the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in 8 general points.

(1) If $p_a(\beta) = 0$, then $n_\beta = (-1)^{w-1}w$.
(2) If $p_a(\beta) = 1$ and $\beta \neq -K_{S_8}$, then $n_\beta = (-1)^{w-1}w(e(S) - \eta)$.
(3) If $\beta = -K_{S_8}$, then $n_\beta = 12$.
(4) If $p_a(\beta) = 2$ and $\beta \neq -2K_{S_8}$, then $n_\beta = (-1)^{w-1}w \left( \left( \frac{e(S)}{2} - \eta \right) + 5 \right)$. 
Remark 1.10. Note that for $\beta = -2K_{S_8}$, the log BPS number is calculated to be 66 in [61]. This matches with the corresponding (physics) computation carried out in [38]. Note that in [38, Section 5.1], the refined BPS index is calculated for the combined classes with $w = 2$, which suffices for the verification since we know the local and log BPS numbers for all the other $w = 2$ classes.

Remark 1.11. Note that in the situation of Theorem 1.9, the numbers $(-1)^{w-1}n_{\beta}/w$ only depend on the arithmetic genus of $\beta$ and on topological numbers of $S$. This structure bears similarities to the works [27, 45, 46, 51, 67, 68] on Severi degrees and extensions thereof. One might expect there to be some universal polynomials that calculate the $(-1)^{w-1}n_{\beta}/w$.

Assuming Conjecture 1.6, we have $w|n_{\beta}$. This is also a consequence of the following conjecture on the cohomology of $M_{\beta}$. For line, conic or nef and big classes of arithmetic genus up to 2 it is proven in [11] except in the case $\beta = -2K_{S_8}$.

Conjecture 1.12 (Conjecture 1.2 in [11]). The Poincaré polynomial $P_{t}(M_{\beta})$ has a factor of $P_{t}([P^{w-1}])$. Consequently, $n_{\beta}$ is divisible by $w$.

In a somewhat orthogonal direction, note Conjecture 44 of [6], which stipulates a relationship, after a change of variable, of $P_{t}(M_{\beta})$ with a generating function of certain higher genus log Gromov-Witten invariants. Combining the two suggests a reconstruction result of higher genus log GW invariants in terms of genus 0 invariants.

There is a further unexpected connection that arises from of our calculations. Proposition 6.4 states that certain multiple cover contributions to the log BPS numbers are given by loop quiver Donaldson-Thomas invariants. We believe that this is a manifestation of a more general phenomenon.

1.2. Open-closed duality. We describe a heuristic that views Conjecture 1.6 as an instance of open-closed duality, i.e. a correspondence between disk and curve counts. Note that this correspondence is expected to hold as the BPS version of the (algebro-geometric) proof of the main result of [20].

Assume that $S$ is del Pezzo and $E$ anticanonical. Let $P \in E(\beta)$ be $\beta$-primitive and let $C \in M_{\beta,P}$. We first explain how $C$ should be viewed as a holomorphic disk $\Sigma \to S \setminus E$ with boundary mapping to a special Lagrangian $L$ near $E$. After that, we describe a procedure that associates to this disk (modulo multiplicity) a rational curve in $\text{Tot}(K_S)$.

The complement $S \setminus E$ is symplectomorphic to $V \setminus \partial V$, where $V$ is a Liouville manifold with boundary $\partial V$ a circle bundle in $N_{E/S}$. Moreover, a symplectic tubular neighborhood $U$ of $E$ in $S$ is an open disk bundle in $N_{E/S}$. In particular, $\partial U \to E$ is a circle bundle. Gluing $V$ and $U$ along their boundaries yields the symplectic sum $M$. 

According to \[13\] Lemma 2.7, \(C\) corresponds to a holomorphic map from an open disk to \(V \setminus \partial V\) with boundary converging to a \(w\)-multiple orbit of the fiber of \(\partial V\) over \(P\)\(^2\). According to \[3\] Conjecture 7.3, special Lagrangian SYZ-fibers \(L\) of \(\overline{U} \cap (S \setminus E)\) are (a perturbation of) circle bundles \(L \to L_E\), with fibers in \(N_{S/E}\) and where \(L_E\) is a special Lagrangian in \(E\).

Combining this, one expects there to be a special Lagrangian \(S \setminus E \supset \partial U \supset L \to L_E \subset E\) such that \(C\) correspond to a closed disk \(\Sigma \to S \setminus E\) with boundary \(\partial \Sigma\) mapping (in degree \(w\)) to a vanishing cycle of \(L\). Note that after perturbation, it is expected that the same \(L\) works for all \(C \in M_{\beta,P}\).

Next, we join the image of \(\Sigma\) with a \(w\)-multiple of the fiber of \(U \to E\) over \(P\) to obtain a holomorphic map \(\mathbb{P}^1 \to M\). The choice of multiple cover introduces a factor of \((-1)^{w-1}/w^2\), cf. \[20\]. We add a suitable line bundle on \(M\) (see \[20\] for details) such that (1) it does not affect the count of disks in \(S \setminus E\) and (2) the total space of the line bundle deforms to \(\text{Tot}(K_S)\).

In this deformation, \(\mathbb{P}^1 \to M\) deforms to a rational curve \(\tilde{C}\) in \(\text{Tot}(K_S)\) which is counted \((-1)^{w-1}/w^2\) times. Furthermore, the deformation formula, cf. \[20\], introduces a factor \(w\) in that \(w\) local curves are expected to degenerate to the same log curve. Finally we need to take into account that \(|E(\beta)| = w^2\) (assuming as in Conjecture 1.3 that each point contributes the same). Putting this together yields the expectation that

\[n\beta = |E(\beta)| \cdot w \cdot \frac{(-1)^{w-1}}{w^2} \cdot m_{\beta}^{\text{tot}},\]

which simplifies to Conjecture 1.6. Finally, note that the contribution of \(C\) to \(m_{[C]}^P\) and that of \(\tilde{C}\) to \(n_{[\tilde{C}]}\) are both expected to be given by the Euler characteristic of the compactified Jacobian.

### 1.3. Pairs of maximal boundary

Starting with a smooth rational surface \(S\), there are two fundamentally different ways of obtaining a log Calabi-Yau surface. In the present paper, we proceed by choosing a smooth anticanonical divisor. This can be viewed as the tail of a type II semistable degeneration of a K3 surface. We conjecture that the resulting invariants are related to the local geometry of \(S\), the total space \(\text{Tot}(K_S)\) of the canonical bundle on \(S\).

The other way proceeds by choosing a singular normal crossings divisor. This means that the divisor is either a nodal genus 1 curve or a cycle of rational curves. This case is usually referred to as being of maximal boundary, which means that there is a toroidal structure near the divisor, and is treated in \[32, 29, 5\]. One example thereof concerns components of maximally unipotent type III degenerations of K3 surfaces. This is described in the introduction of \[33\] and the full mirror symmetry picture will be detailed in the upcoming work \[31\].

Considering refined contributions in the maximal boundary toric case lead Bousseau in \[7\] to a remarkable result stating that generating series of higher genus log GW invariants of toric surfaces are, after a suitable change of variables, refined Block-Gütsche counts of

\[^2\text{Note that }13\text{ imposes restrictive conditions on the pairs considered, in order to guarantee transversality of the relevant moduli spaces. In our case, the log BPS invariants are weighted counts of curves, so no virtual techniques are required and the proof of }13\text{ Lemma 2.7 carries through.}\]
tropical curves. The result of [7] and our conjectures have the same origin. Namely, that the log GW/log BPS numbers very much behave like an algebraic version of open GW invariants, see [36].

1.4. Analogies with K3 surfaces. Log BPS numbers feature some striking parallels with counts of rational curves in K3 surfaces. On the one hand, denote by \((S,E)\) a log Calabi-Yau surface pair with \(S\) del Pezzo. Let \(\beta \in H_2(S,\mathbb{Z})\) be a curve class and let \(P \in E(\beta)\) be \(\beta\)-primitive. Consider the \(p_a(\beta)\)-dimensional linear system \(|O_S(\beta,P)|\) (Definition 4.15) of curves of class \(\beta\) with maximal intersection multiplicity with \(E\) at \(P\). On the other hand, denote by \(X\) a K3 surface, let \(g \geq 0\) and consider a \(g\)-dimensional complete linear system \(L\) of curves of genus \(g\) on \(X\).

Firstly, by Proposition 1.7, a rational curve \(C \in |O_S(\beta,P)|\), smooth at \(P\), contributes \(e(S_C)\) to \(m_P\beta\). In parallel, the contribution of a rational curve \(C \in L\) to the BPS count of the Yau-Zaslow formula [71] is given (see [3, 15]) by \(e(Pic^0(C))\).

Secondly, choosing \(P\) to be \(\beta\)-primitive amounts to choosing a point on \(E\) of maximal order (for a suitable choice of zero element, see Lemma 4.9). This is analogous to taking the reduced class for K3 surfaces. The moduli space of stable maps (of genus 0 and given class) in \(X\) is of virtual dimension \(-1\) and hence all Gromov-Witten invariants are zero. This is because there is a non-algebraic deformation of \(X\) in which there are no rational curves. This issue is resolved by restricting the deformation space to algebraic deformations (the Noether-Lefschetz locus). Similarly, assume that \((S,E)\) occurs as the tail of a type II (algebraic) degeneration of \(X\). To simplify, assume that the degeneration consists of another component \((S',E')\) and that \(S\) and \(S'\) are glued together along \(E \simeq E'\). Denote by \(N\) and \(N'\) the respective normal bundles of the components of the central fiber. For the deformation to be algebraic, \(N|_E = N'|_E\) must hold. Consider a maximally tangent rational curve \(C\) in \((S,E)\), resp. \(C'\) in \((S',E')\), each meeting \(E\) in \(P\) in tangency \(w = [C].E = [C'].E\). In order to algebraically deform \(C \cup P C'\) to a rational curve in \(X\), \(P\) should have the same order with respect to \([C]\) and \([C']\), for example by requiring \(P\) to be both \([C]\)- and \([C']\)-primitive.

The third analogy is that \(m_P\beta\) also seems to depend on intersection numbers only. The BPS number of rational curves in \(|L|\) depends only on the intersection number \(2g - 2\) (conjectured in [54, 41], proven in [43] in genus 0, in [55] in all genus and primitive classes and in [58] without the primitivity assumption). In our calculations (for \(p_a(\beta) \leq 2\), \(m_P\beta\) depends only on \(e(S) - \eta\), where \(\eta\) is the number of line classes \(l\) such that \(\beta.l = 0\).

For K3 surfaces, [49, 50] prove an equivalence between counts of Maslov index 0 disks with boundary on a SYZ-fiber and tropical curves in the base. In the log setting of this paper, the analogous correspondence is expected to hold between the log BPS numbers and tropical curves in the scattering diagrams of [9].

We end this introduction by mentioning that (combinations of) the log BPS numbers introduced here are numbers that occur in other contexts such as in [62]. Throughout this paper, we work over \(\mathbb{C}\).
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2. Maximally tangent stable log maps

Let $(S, E)$ be a log Calabi-Yau surface with a smooth divisor, i.e. $S$ is a smooth projective surface and $E$ a smooth anticanonical divisor, in this case an elliptic curve or a disjoint union of elliptic curves. Log BPS numbers are derived from counting rational curves in $S$ with a single point of maximal tangency along $E$. This can be done in several ways. One may consider relative Gromov-Witten invariants which are defined for smooth very ample divisors in [23] and for any smooth divisors in [47, 48]. In our setting this approach is taken in [65]. For our purpose, cf. the proof of Proposition 1.7, the appropriate setting is the further generalization to log Gromov-Witten invariants [35], where instead of a divisor only a log structure on the target is prescribed. Note also the analogous construction [1, 10], as well as, for smooth divisors, the treatment in [42]. All these invariants agree in the case at hand, cf. [2].

We describe the moduli space of basic stable log maps of [35], in the setting of importance to us, namely in genus 0 with one condition of maximal tangency along $E$. Note that by Proposition 2.12 below, the virtual dimension is zero and hence we need not consider insertions.
2.1. Genus 0 stable log maps of maximal tangency. The schemes in this section will be endowed with a log structure. We do not distinguish notationally when we consider their underlying schemes as it will be clear from the context. We write $x$ to denote the marked point or a node. When we wish to emphasize that $x$ is the marked point, resp. a node, we sometimes write $x_1$, resp. $q$.

Let $X$ be a smooth variety and $D$ a smooth divisor on $X$. We view $X$ as the log scheme $(X, M_X)$ given by the divisorial log structure $M_X = M_{(X,D)}$. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class.

Definition 2.1. Let $(C/W, \{x_1\})$ be a 1-marked pre-stable log curve ([35, Def. 1.3]) over a log point $W = (\text{Spec } \kappa, Q)$ where $\kappa$ is an algebraically closed field over $\mathbb{C}$, and $(C/W, \{x_1\}, f)$ a stable log map (i.e., $f : C \to X$ is a log morphism over $\text{Spec } \mathbb{C}$ and $f$ is a stable map, see [35, Def. 1.6]).

It is called a stable log map of maximal tangency of genus 0 and class $\beta$ if the following hold:

(i) $C$ is of arithmetic genus 0, $f_*[C] = \beta$.

(ii) the natural map $\Gamma(X, \overline{M}_X) \cong \mathbb{N} \rightarrow \overline{M}_{C,x_1} \cong Q \oplus \mathbb{N} \overset{pr_2}{\rightarrow} \mathbb{N}$ is given by $1 \mapsto D.\beta$.

We will later see (Proposition 2.9) that (ii) follows from other conditions.

In the language of [35, Def. 3.1], this is the case $g = 0$, $k = 1$, the condition $A$ provided by $\beta$, $Z_1 = D$ and $s_1 \in \Gamma(D, (\overline{M}_D^{gp})^*)$ given by $\overline{M}_D^{gp} \cong Z_D \rightarrow \mathbb{Z}_D$, $1 \mapsto D.\beta$.

By [35, Prop. 1.24], a stable log map $f$ as above is induced from a basic stable log map over $(\text{Spec } \kappa, Q^{\text{basic}})$. Since $Q^{\text{basic}}$ is a toric monoid, we can take a local homomorphism $Q^{\text{basic}} \rightarrow \mathbb{N}$ and consider the induced stable log map over the standard log point $(\text{Spec } \kappa, \mathbb{N})$. Hence, to study the underlying morphism of schemes, we may consider stable log maps over the standard log point.

To a stable log map, one can associate its graph, type, “tropical data” and “$\tau$-rays”. We explain this in the case of a 1-marked genus 0 stable log map $f$ over $(\text{Spec } \kappa, \mathbb{N})$ to $(X, D)$. See [35, §§1.4] for details. Let us write $\beta = f_*[C]$. Note that we do not assume the maximal tangency condition (ii) here.

Notation: If $\eta$ is the generic point of an irreducible component of $C$, then $C_\eta$ denotes this irreducible component, the closure $\overline{\eta}$ of $\eta$.

Definition 2.2. (Graph) The dual graph $\Gamma$ of $C$ consists of the following data. The vertex set $V(\Gamma)$ is the set of irreducible components of $C$. The edge set $E(\Gamma)$ consists of one unbounded edge in addition to a number of bounded edges. The unbounded edge is attached to the vertex corresponding to the irreducible component of $C$ containing the marked point. There is a bounded edge for each node, connecting the vertices (possibly the same) corresponding to the irreducible components containing each of the two local analytic branches at the node.
The following lemma gives some clue about how \( f \) meets \( D \).

**Lemma 2.3.** For any generic point \( \eta \) of \( C \), \( f^{-1}(D) \cap C_\eta \) is either \( C_\eta \) or consists of nodes and marked points.

**Proof.** This follows from [35, Remark 1.9], which says that the stalk of \( f^{-1}M_X \) jumps (i.e. the generization map is not an isomorphism) only at nodes and marked points. \( \square \)

**Definition 2.4.** (Tropical data) Since the genus is 0, a node \( q \) is the intersection of two components \( C_{\eta_1} \) and \( C_{\eta_2} \). We denote by \( e_q \in \mathbb{N}_{>0} \) the positive integer such that the following holds: \( M_{C,q} \) is embedded as \( \langle (e_q,0),(1,1),(0,e_q) \rangle \subseteq \mathbb{N} \oplus \mathbb{N} \), where \( (1,1) \) corresponds to the image of \( 1 \in \mathbb{N} \cong M_{(\text{Spec} \kappa, \mathbb{N})} \) and the projections are identified with the generization maps to \( M_{C,q}, 1) \cong N \).

Let \( \varphi : f^{-1}M_X \to M_C \) be the morphism induced from \( f \). We define

\[
V_\eta := \varphi(1) \in \mathbb{N}_{>0}
\]

if \( f(\eta) \in D \), where \( f^{-1}(M_X)_{\eta} \) is identified with \( \mathbb{N} \); otherwise \( V_\eta := 0 \).

The tuple \( ((V_\eta), (e_q)_N) \) is called the **tropical data**.

**Definition 2.5.** (Type) For a node as above, write \( \{i,j\} = \{1,2\} \) and let

\[
u_{\eta,q} := \frac{V_\eta - V_0}{e_q} \in \mathbb{Z},
\]

which is 0 unless \( f(q) \in D \). (Note that this notation avoids the issue of ordering as in [35]. This is possible because a node is the intersection of two components in this case.)

For the marked point \( x_1 \), we define

\[
u_{\eta,x_1} := (\varphi_{x_1}(1) \mod M_{(\text{Spec} \kappa, \mathbb{N})}) \in \mathbb{N}
\]

if \( f(x_1) \in D \), where \( f^{-1}(M_X)_{x_1} \) is identified with \( \mathbb{N} \) and \( M_{C,x_1}/M_{(\text{Spec} \kappa, \mathbb{N})} \) is identified with \( \mathbb{N}_{x_1} \); otherwise \( \nu_{\eta,x_1} := 0 \).

The data \( ((\nu_{\eta,q}, \nu_{\eta,x_1}) \) is called the **type**.

As is seen from the definition, the type is determined by the tropical data, except for \( \nu_{\eta,x_1} \). We will see in Proposition 2.9 that \( \nu_{\eta,x_1} \) must be \( D,\beta \).

**Definition 2.6.** (\( \tau \)-rays) For each \( \eta \), let \( N_\eta := \Gamma(C_\eta, f^{-1}M_X^\text{gp})^* \). If \( \Sigma_\eta \) denotes the set of nodes and marked points of \( C_\eta \), then

\[
N_\eta \cong \bigoplus_{x \in \Sigma_\eta} f^{-1}(D)Z \quad \text{if } f(\eta) \not\in D,
\]

\[
N_\eta \cong \left( \bigoplus_{x \in \Sigma_\eta} Z \right)/H \cong \mathbb{Z} \quad \text{if } f(\eta) \in D,
\]

where \( H \) is the subgroup generated by the differences of basic vectors and the isomorphism to \( \mathbb{Z} \) is given by \( (a_x) \mapsto \sum a_x \). Write the class of \( (a_x) \) by \( [(a_x)] \).

Then, if \( f(\eta) \not\in D \), \( \tau_\eta = (\tau_x) \) is given by \( \tau_x := -\mu_x((f|_{C_\eta})^*D) \), the multiplicity at \( x \); if \( f(\eta) \in D \), \( \tau_\eta := -\deg(f|_{C_\eta})^*D \). (These are the \( \tau_\eta \) of [35, §3.4].)
Now these data satisfy the balancing condition.

**Proposition 2.7.** ([35] Prop. 1.15) For each \( \eta \),

\[
\tau_\eta + \left[ (u_{q,x}) \right] = 0
\]

holds in \( N_\eta \).

In Proposition 2.9 we will give a description of \( 1 \)-marked genus 0 stable log maps to \((X,D)\).

**Definition 2.8.** (1) Let \( \eta_1, \eta_2 \in V(\Gamma) \) correspond to the component on which the marked point lies. For \( \eta_1, \eta_2 \in V(\Gamma) \), we write \( \eta_1 \preceq \eta_2 \) if \( \eta_2 \) is on the unique simple path connecting \( \eta_1 \) and \( \eta_0 \).

Note that this is a partial ordering since \( \Gamma \) is a tree, and that \( \eta_0 \) is the largest element.

(2) We write \( \eta_1 \mapsto \eta_2 \) if \( \eta_1 \) and \( \eta_2 \) are adjacent and \( \eta_1 \prec \eta_2 \). (Then \( \preceq \) is the partial ordering generated by \( \mapsto \).) In this case, if \( q \) is the edge connecting \( \eta_1 \) and \( \eta_2 \), we write \( \eta_1 \mapsto q \) and \( q \mapsto \eta_2 \). We also write \( \eta_0 \mapsto x_1 \).

(3) For any \( \eta \), there is a unique edge \( q \) with \( \eta \leftrightarrow q \). We denote this by \( q(\eta) \).

(4) For \( \eta \in V(\Gamma) \), let \( C_{\leq \eta} = \bigcup_{\eta' \preceq \eta} C_{\eta'} \).

**Proposition 2.9.** Assume that \( D.f_*[C_{\eta}] \geq 0 \) holds for any \( \eta \).

(1) For each \( \eta \), the inverse image \( f^{-1}(D) \cap C_{\leq \eta} \) of \( D \) on \( C_{\leq \eta} \) is either empty, \( \{ q(\eta) \} \), or \( \bigcup_{\eta' \in V(\Gamma') \cap C_{\eta'}} \) for a subtree \( \Gamma' \) containing \( \eta \).

Thus, if \( f^{-1}(D) \cap C_{\eta} \neq \emptyset \), we can identify \( N_\eta \) with \( \mathbb{Z} \) and \( \tau_\eta \) with \( -D.f_*[C_{\eta}] \).

(2) For each \( \eta \), we have \( u_{\eta,q(\eta)} = D.f_*[C_{\leq \eta}] \).

In particular, \( f \) is of maximal tangency.

Note that, if \( f(C) \not\subseteq D \), we can write \( V_\eta \) in terms of \( e_q \) and \( D.f_*[C_{\eta}] \), which also leads to relations between these data.

**Proof.** We prove the assertions by induction on the maximum length of simple paths from \( \eta \) to minimal vertices.

Note that, if the first half of (1) is proven for \( \eta \), the second half is easy to see from the description of \( N_\eta \) and \( \tau_\eta \) in **Definition 2.6**.

If \( \eta \) is itself minimal, it has exactly one special point, \( q(\eta) \). By **Lemma 2.3**, \( f^{-1}(D) \) is either empty, \( \{ q(\eta) \} \) or \( C_{\eta} \), so (1) holds. Thus \( N_\eta \) can be identified with \( \mathbb{Z} \) or \( \{ 0 \} \), and by **Proposition 2.7** we have \( u_{\eta,q(\eta)} = -\tau_\eta = D.f_*[C_{\eta}] \), so (2) holds.

Now assume that the assertions hold for each \( \eta' \) with \( \eta' \mapsto \eta \). If \( f^{-1}(D) \cap C_{\leq \eta'} \) is empty for all such \( \eta' \), then \( f^{-1}(D) \cap C_{\leq \eta} \) is empty or \( \{ q(\eta) \} \) by **Lemma 2.3**.

If \( f^{-1}(D) \cap C_{\leq \eta'} \) is nonempty for some \( \eta' \), then it contains \( q := q(\eta') = C_{\eta'} \cap C_{\eta} \) and we have

\[
V_\eta = V_{\eta'} + u_{\eta',q} e_q.
\]
If \( f^{-1}(D) \cap C \preceq q' \) consists of \( q \), then \( u_{q',q} \geq D.f_s[C_{q'}] > 0 \). Otherwise, it contains \( C_{q'} \) and \( V_{q'} > 0 \) holds. In either case, we have \( V_q > 0 \) and \( C_q \) is contained in the inverse image of \( D \), and the assertion (1) follows.

Then we have
\[
\begin{align*}
{u}_{\eta,q(\eta)} &= -\tau_\eta - \sum_{\eta' \leftarrow \eta} u_{\eta',q(\eta')} \\
&= D.f_s[C_\eta] + \sum_{\eta' \leftarrow \eta} u_{\eta',q(\eta')} \\
&= D.f_s[C_\eta] + \sum_{\eta' \leftarrow \eta} D.f_x[C_{\preceq q'}],
\end{align*}
\]
so (2) follows.

Corollary 2.10. Let \( X \) be a divisorial log scheme given by a smooth variety \( X \) and a smooth divisor \( D \).

For a genus 0 stable log map \( f : (C, x_1) \to X \), assume the following:
- \( w := D.f_s[C] > 0 \) and \( w_i := D.f_s[C_i] \geq 0 \) for any irreducible component \( C_i \) of \( C \).
- If \( C_i \) is an irreducible component of \( C \) that is not collapsed by \( f \), then \( f(C_i) \not\subseteq D \).

Then it is of maximal tangency, and the following holds.

1. \( f(C) \cap D \) consists of one point \( P \).
2. If there is only 1 non-collapsed component, then \( C \cong \mathbb{P}^1 \) and \( f^*(D) = wx_1 \).
3. If there are at least 2 non-collapsed components, and \( D.f_s[C_i] > 0 \) holds for non-collapsed components, then \( C \) is given by adding \( C_i = \mathbb{P}^1 \) as leaves to a tree \( C' \) of \( \mathbb{P}^1 \) collapsed to \( P \), with maps \( f_i : C_i \to S \) satisfying \( f_i^*(D) = w_i (C_i \cap C') \).

Proof. (1) By the assumption, \( f(C) \) meets \( D \). Applying Proposition 2.9, the inverse image of \( D \) is the marked point or a tree of irreducible components, which are collapsed by the assumption. Hence its image consists of one point.

(2) A minimal vertex \( \eta \) has only 1 special point (a node or the marked point), so it is non-collapsed by stability. By assumption we have only one minimal vertex, so the graph is a chain. By stability it has only one vertex and the assertion follows.

(3) Again each minimal vertex \( \eta \) is non-collapsed, and it meets \( f^{-1}(D) \) by assumption.

We return to the setting of a log Calabi-Yau surface \((S, E)\). Let \( \beta \in H_2(S, \mathbb{Z}) \) be a curve class. Denote by \( \overline{M}_\beta(S, E) \) the moduli space of maximally tangent genus 0 basic stable log maps to \((S, E)\) of degree \( \beta \). Cf. Corollary 2.10, generic elements of the various strata of \( \overline{M}_\beta(S, E) \) are as in Figure 2.1, where components \( C_6, C_7 \) and \( C_8 \) are collapsed.

We state a result of [70] in this setting. Note that in genus 0, the relevant log scheme of morphisms is already saturated, cf. its proof.
Proposition 2.11 (Corollary 1.2 in [70]). The forgetful morphism injects $\overline{M}_\beta(S, E)$ into the moduli space of genus 0 stable maps $\overline{M}_{0,0}(S, \beta)$.

The conditions of genus 0, degree $\beta$ and one marked point mapping in maximal tangency result in a finite number of types and hence make up a combinatorially finite class, cf. [35, Definition 3.3]. Hence $\overline{M}_\beta(S, E)$ admits a perfect obstruction theory, which is of virtual dimension 0, and yields a virtual fundamental class, as well as corresponding log Gromov-Witten invariants

$$N_\beta(S, E) := \int_{[\overline{M}_\beta(S, E)]^{vir}} 1 \in \mathbb{Q}.$$  

We remark that much of this section could have been conveniently expressed in the language of tropical curves. (For example, under the assumption of the preceding corollary, the ordering of vertices is compatible with the ordering given by $V_\eta$.)

2.2. Rational curves of maximal tangency. We state a proposition from [64] and provide a proof for convenience.

Proposition 2.12 (Proposition 1.1 in [64]). Let $D \subset X$ be a smooth hypersurface of a smooth $n$-dimensional projective variety $X$. Assume that $|K_X + D| \neq \emptyset$. Let $\beta \in H^2(X, \mathbb{Z})$ be a curve class. Consider the set $U$ consisting of the union of all rational curves $C \subset X$ of degree $\beta$ having only one point of intersection with $D$. Then $U$ is contained in a proper Zariski-closed subset of $X$.

Proof. Assume that $U$ is not contained in a proper Zariski-closed subset of $X$. Then we can find a diagram
\[ Y := M \times \mathbb{P}^1 \xrightarrow{f} X \]

such that

- \( M \) is a smooth variety of dimension \( n - 1 \),
- \( f \) is dominant,
- \( M_i \) are disjoint sections of the projection map \( p \),
- \( f^*(D) = \sum a_i M_i \) for \( a_i \) positive integers.

Indeed, start with the (noncompact) moduli space of birational morphisms \( \mathbb{P}^1 \to X \) whose image meets \( D \) in one point and consider a \((n - 1)\)-dimensional subspace whose universal curve maps dominantly to \( X \). After possibly taking a quasi-finite cover, this is the space \( M \) with \( f^* D = \sum a_i M_i \) for \( a_i \) positive integers and \( M \) disjoint sections of \( p \).

By assumption, there is a non-zero \( n \)-form \( \omega \) on \( X \), which is regular away from \( D \) and has at most logarithmic poles along \( D \). The pullback \( f^* \omega \) is a non-zero \( n \)-form on \( M \times \mathbb{P}^1 \) and has at most a logarithmic pole along \( \bigcup M_i \). Hence \( \text{res}_{\bigcup M_i} f^* \omega = a_i (f|_{M_i})^* \text{res}_D \omega \). After possibly shrinking \( M \), we take a non-vanishing holomorphic \((n - 1)\)-form \( \omega' \) on \( M \) and write

\[ f^* \omega = \omega'' \wedge p^* \omega', \]

for some relative 1-form \( \omega'' \). Over a general fiber of \( p \), \( \omega'' \) restricts to a non-zero 1-form on \( \mathbb{P}^1 \) with at most logarithmic poles along which residues are positive integers times a certain complex number, which is a contradiction.

\[ \square \]

2.3. Point-dependence. Assume in this section that \( S \) is a regular surface (which we defined to mean \( h^1(O_S) = 0 \)) and \( E \) is an elliptic curve on \( S \). Let \( \beta \in H^2(S, \mathbb{Z}) \) be a curve class and recall that \( w = \beta.E \), and assume that \( w > 0 \). Then there is a unique \( L \in \text{Pic}(S) \) such that \( c_1(L) = \beta^{\text{PD}} \). We use the notation \( \beta|_E := L|_E \in \text{Pic}^w(E) \) for the induced class. Denote by \( \text{Pic}^0(E)[w] \) the \( w \)-torsion points of \( \text{Pic}^0(E) \). For \( P \in E \), we write \( \beta|_E \sim wP \) to indicate that \( \beta|_E = [wP] \) in \( \text{Pic}^w(E) \).

**Definition 2.13.**

\[ E(\beta) := \{ P \in E \mid \beta|_E \sim wP \}. \]

**Lemma 2.14.** \( E(\beta) \) is a torsor for \( \text{Pic}^0(E)[w] \simeq \mathbb{Z}/w \times \mathbb{Z}/w \).

**Proof.** Since \( E \simeq \text{Pic}^1(E) \) is a torsor for \( \text{Pic}^0(E) \), \( E \) admits an effective action of \( \text{Pic}^0(E)[w] \). In addition, the latter acts on \( E(\beta) \) transitively, since for \( P_1, P_2 \in E(\beta) \), \( wP_1 - wP_2 \sim 0 \) so \( P_1 - P_2 \in \text{Pic}^0(E)[w] \). Hence \( E(\beta) \) is a torsor for \( \text{Pic}^0(E)[w] \). \[ \square \]

We consider the images of log stable maps.
Definition 2.15. Let $M_\beta$ be the set of image cycles of genus 0 stable log maps to $(S, E)$ of class $\beta$ of maximal tangency.

For $P \in E$, let $M_{\beta, P} := \{ D \in M_\beta \mid \text{Supp}(D) \cap E = P \}$.

We also define a notion of maximal tangency for curves.

Definition 2.16. If $(S, E)$ is a pair consisting of a variety and a divisor, a curve $D$ on $S$ is said to be maximally tangent to $E$ if $D$ meets $E$ at only 1 point and has only 1 branch there; or equivalently, if the inverse image of $E$ on the normalization of $D$ consists of 1 point.

An irreducible proper rational curve $D$ maximally tangent to $E$ can also be considered as an $\mathbb{A}^1$-curve on $S \setminus E$. The following proposition shows how elements of $M_\beta$ are related to such curves.

Proposition 2.17. Let $S$ be a smooth surface, $E$ an elliptic curve on $S$ and $\beta \in H_2(S, \mathbb{Z})$ a curve class. Assume that $w := E . \beta > 0$.

Consider a 1-marked genus 0 stable log map to $(S, E)$ of class $\beta$. Then it is of maximal tangency, and the underlying stable map $f : (C, x) \to S$ satisfies the following.

1. $f(C) \cap E$ consists of one point $P$. If $S$ is regular, then $P \in E(\beta)$ and consequently $M_\beta = \bigsqcup_{P \in E(\beta)} M_{\beta, P}$.

2. If the image cycle is irreducible and reduced, then $C \cong \mathbb{P}^1$ and $f^*(E) = w x$. In this case, $C' := f(C)$ is a rational curve maximally tangent to $E$, and $f$ is the normalization map. Conversely, if $C'$ is a rational curve maximally tangent to $E$, then the normalization map of $C'$ lifts to a (unique) genus 0 stable log map of maximal tangency with image cycle $C'$.

3. If $E$ is ample, any element of $M_\beta$ is a sum of rational curves maximally tangent to $E$, meeting $E$ at the same point.

Proof. Since $E$ is an elliptic curve and any component of $C$ is rational, Corollary 2.10 applies. Thus it is of maximal tangency, and the image meets $E$ at one point, which necessarily belongs to $E(\beta)$ if $S$ is regular, which proves (1).

(2) If the image cycle is irreducible, then $C$ has only 1 non-collapsed component mapped birationally. Thus the first half follows from Corollary 2.10 (2).

The latter half can be easily checked.

(3) follows from Corollary 2.10 (3). $\square$

Because $E(\beta)$ is finite, we have the decomposition

$$\overline{M}_\beta(S, E) = \bigsqcup_{P \in E(\beta)} \overline{M}_{\beta, P}(S, E).$$

Moreover, as the obstruction theory of a disjoint union is the sum of the obstruction theories of each component, we obtain the finite decomposition

$$N_\beta(S, E) = \sum_{P \in E(\beta)} N_{\beta, P}(S, E),$$

(2.1)
where 

\[ \mathcal{N}_\beta^P(S, E) := \int_{[M^P_\beta(S, E)]^\text{vir}} 1 \in \mathbb{Q} \]

are the genus 0 log GW invariants of \((S, E)\) of degree \(\beta\) maximally tangent to \(E\) at \(P\).

So far we have worked mainly with regular surfaces and elliptic curves on it. In the following proposition we illustrate what our surfaces are like. It appears that interesting things happen mostly on rational surfaces in the context of log Gromov-Witten theory of genus 0 maximally tangent curves.

**Proposition 2.18.** Let \(S\) be a regular surface.

1. If there exists a non-zero effective divisor \(E\) such that \(K_S + E \sim 0\), then \(S\) is rational.
2. Let \(E\) be a curve on \(S\) and \(\beta\) a curve class with \(E.\beta > 0\) and \(\text{vdim } \overline{M}_\beta(S, E) \geq 0\).
   
   (a) If \(\beta\) is nef, then \(S\) is rational,
   
   (b) If \(S\) is not rational and \(\beta\) contains an irreducible (not necessarily reduced) member, then \(|\beta| = \{mC\}\) where \(C\) is a \((-1)\)-curve with \(E.C = 1\).

3. If \(\beta\) is an ample class of arithmetic genus 1 on \(S\) containing an irreducible reduced member, then \(\beta = -K_S\) and hence \(S\) is del Pezzo.

   In particular, if \(S\) contains an ample elliptic curve \(E\), then \(S\) is del Pezzo and \(E\) is anticanonical.

**Proof.** (1) Since \(|mK_S| = \emptyset\) for any \(m > 0\), \(S\) is birationally ruled. From regularity it follows that \(S\) is rational.

(2) The virtual dimension of the moduli space of stable maps \(\overline{M}_{0,0}(S, \beta)\) (i.e. before imposing the tangency condition) is \(-K_S.\beta - 1\). Imposing the condition of maximal tangency with \(E\) cuts down the dimension by \(E.\beta - 1\). Therefore

\[ \text{vdim } \overline{M}_\beta(S, E) = (-K_S - E).\beta. \]

Thus \(-K_S.\beta \geq E.\beta > 0\) holds.

If \(S\) is regular but not rational, by the Enriques-Kodaira classification, \(S\) is birationally non-ruled. Let \(\pi : S \to S'\) be the minimal model of \(S\). Then \(K_{S'}\) is nef and \(K_S = \pi^*K_{S'} + \sum a_iF_i\), where the \(F_i\) are the exceptional curves and \(a_i > 0\). Since \(K_{S'}\) is nef, \((\sum a_iF_i).\beta \leq K_S.\beta < 0\) and \(\beta\) is not nef. If \(\beta\) has an irreducible member, it is a multiple of one of \(F_i\). From \(K_S.\beta < 0\) it follows that \(K_S.F_i < 0\), and \(F_i\) is a \((-1)\)-curve. Then \(0 < E.\beta \leq -K_S.\beta\) implies \(E.F_i = 1\).

(3) Let \(C\) be an irreducible reduced member of \(|\beta|\). By regularity we have \(h^1(\mathcal{O}_S(K_S)) = h^1(\mathcal{O}_S) = 0\), so there is an exact sequence

\[ 0 \to H^0(\mathcal{O}_S(K_S)) \to H^0(\mathcal{O}_S(K_S + \beta)) \to H^0(\mathcal{O}(K_S + \beta)|_C) \to 0. \]

Since \(\mathcal{O}(K_S + \beta)|_C \cong \omega_C \cong \mathcal{O}_C\), it follows that there exists an element \(D \in |K_S + \beta|\) such that \(D \cap C = \emptyset\). Since \(C\) is ample, \(D\) must be 0, hence \(\beta \sim -K_S\). \(\square\)
3. LOCAL AND LOG BPS NUMBERS

Let \((S, E)\) be a log Calabi-Yau surface with smooth divisor. We start by reviewing the (original) definition of local BPS invariants obtained by removing multiple cover contributions from the local Gromov-Witten invariants of \(S\) and yielding integer invariants. The local Gromov-Witten invariants of \(S\) are the ordinary Gromov-Witten invariants of the noncompact Calabi-Yau threefold \(X = \text{Tot}(K_S)\). The Gromov-Witten invariants of \(X\) are well-defined even though \(X\) is noncompact since the moduli spaces of stable maps to \(X\) are compact.

For \(m \geq 0\), denote by \(\overline{M}_{0,m}(S, \beta)\) the Deligne-Mumford moduli stacks of (isomorphism classes of) stable maps \([f : C \to S]\) of genus 0, with \(m\) marked points and such that \(f^*(\mathcal{C}) = \beta\). There is a forgetful morphism

\[
\pi : \overline{M}_{0,1}(S, \beta) \to \overline{M}_{0,0}(S, \beta),
\]

which is the universal curve over \(\overline{M}_{0,0}(S, \beta)\). The latter carries a virtual fundamental class

\[
\left[\overline{M}_{0,0}(S, \beta)\right]_{\text{vir}}^\ast \in H_{2vdim}(\overline{M}_{0,0}(S, \beta), \mathbb{Z})
\]

determines the obstruction bundle

\[
\text{Ob} := R^1\pi^*ev^*K_S,
\]

which is of rank \(vdim\) and has fiber \(H^1(C, f^*K_S)\) over a stable map \([f : C \to S]\).

Definition 3.1. The genus 0 degree \(\beta\) local Gromov-Witten invariant \(GW_{\beta}(X)\) of \(S\) is defined as

\[
GW_{\beta}(X) := \int_{[\overline{M}_{0,0}(S, \beta)]_{\text{vir}}}^\ast c_{vdim}(\text{Ob}) \in \mathbb{Q}.
\]

We will make use of the following correspondence theorem. Its initial form for \(\mathbb{P}^2\) was conjectured in [66] and proven in [24]. Then it was generalized in [20] to any smooth projective variety of any dimension with maximal tangency condition along a smooth nef divisor (the most general statement is at the level of virtual fundamental classes).

Theorem 3.2 (See [66, 24, 20]). Assume that \(E\) is nef. Then

\[
(-1)^{w-1}w \, GW_{\beta}(X) = N_\beta(S, E).
\]

The relation between the \(n_\beta\) and the \(GW_{\beta}(X)\) is given by the multiple cover formula (Aspinwall-Morrison formula):

\[
GW_{\beta}(X) = \sum_{k|\beta} \frac{1}{k^3} n_{\beta/k}.
\]
The Aspinwall-Morrison formula was proven for rigid rational curves in [53, 69]. Note that for S del Pezzo, (3.1) is equivalent to Definition 1.5 as explained in Section 3.3 of [11]. Inverting this formula yields that

\[ n_\beta = \sum_{k | \beta} \frac{1}{k^3} \mu(k) \mathcal{GW}_{\beta/k}(X), \]

where \( \mu \) is the Möbius function. Under the assumption that \( E \) is nef, combining with Theorem 3.2, we obtain that

(3.2) \((-1)^{w-1} w n_\beta = \sum_{k | \beta} \frac{1}{k^3} \mu(k) (-1)^{w-1} (-1)^{w-1} \frac{1}{w/k} N_{\beta/k}(S, E).\]

Noting that \( w + w/k \) and \( (k - 1)w/k \) have the same parity, we rewrite (3.2) as follows.

(3.3) \((-1)^{w-1} w n_\beta = \sum_{k | \beta} (-1)^{(k-1)w/k} k^2 \mu(k) N_{\beta/k}(S, E).\]

By the decomposition (2.1), this turns into

(3.4) \((-1)^{w-1} w n_\beta = \sum_{k | \beta} (-1)^{(k-1)w/k} k^2 \mu(k) \sum_{P \in E(\beta/k)} \mathcal{N}_P^{P}(S, E).\]

This suggests that we define the log BPS numbers at \( P \) as

(3.5) \( m_P^P := \sum_{\{k | \beta : P \in E(\beta/k)\}} (-1)^{(k-1)w/k} k^2 \mu(k) N_{\beta/k}(S, E).\]

Note that \( \mathcal{N}_P^{P}(S, E) = 0 \) if \( P \notin E(\beta') \). Setting \( m_{P'}^P = 0 \) if \( P \notin E(\beta') \) and inverting equation (3.4) yields the following definition.

**Definition 3.3** (See Definition 1.1). Let \((S, E)\) be a log Calabi-Yau surface with respect to \( \beta \in H_2(S, \mathbb{Z}) \) and assume that \( S \) is regular and \( E \) elliptic. We define \( m_P^P \), the log BPS number at \( P \), via

\[ \mathcal{N}_P^{P}(S, E) = \sum_{k | \beta} \frac{(-1)^{(k-1)w/k}}{k^2} m_P^P. \]

Note that in the above sum, the number of terms varies with the arithmetic properties of \( P \), as is illustrated by the examples of \( \S 6.1 \). Note also that [65, Remark 4.11] contains an equivalent description of this formula for \( \mathbb{P}^2 \). By the above calculation,

(3.6) \[ \sum_{P \in E(\beta)} m_P^P = (-1)^{w-1} w n_\beta. \]

**Conjecture 3.4** (See Conjecture 1.3). Let \((S, E)\) be a log Calabi-Yau surface with respect to \( \beta \in H_2(S, \mathbb{Z}) \) and assume that \( S \) is regular and \( E \) elliptic. For all \( P, P' \in E(\beta) \),

\[ m_P^P = m_{P'}^{P'} \]

If in addition \( E \) is nef, this is equivalent to the assertion that for all \( P \in E(\beta) \),

\[ n_\beta = (-1)^{w-1} w m_{P}^P. \]
In §4.2, we elaborate on Conjecture 3.4 in the case of some special points \( P \in E(\beta) \) which we call \( \beta \)-primitive (Definition 4.6). For such \( P \), \( m^P_\beta \) is a weighted count of rational curves, see Proposition 4.20. In that setting, the conjecture hence gives an enumerative interpretation of \( n_\beta \).

4. Primitive points of contact

For this section and next, we assume that \( S \) is a del Pezzo surface and consider certain points \( P \in E(\beta) \) such that \( \overline{M^P_\beta}(S,E) \) is zero-dimensional.

4.1. Preliminaries on del Pezzo surfaces. In this section, we collect basic facts about curve classes on del Pezzo surfaces. Let \( S \) be a del Pezzo surface. Denote by \( S_r \) the blowup of \( \mathbb{P}^2 \) along \( r \) general points. Then \( S \) is either \( S_r \) for \( 0 \leq r \leq 8 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \). We will mainly consider the case \( S = S_r \) and will make remarks for for \( \mathbb{P}^1 \times \mathbb{P}^1 \) separately whenever needed. The results of this paper hold for \( \mathbb{P}^1 \times \mathbb{P}^1 \) as well.

Definition 4.1. A class \( \beta \in H_2(S,\mathbb{Z}) \) is a curve class if it can be represented by a nonempty subscheme of dimension one. We often consider \( \beta \) as a divisor on \( S \).

Recall that \( p_a(\beta) := \frac{1}{2} \beta(\beta + K_S) + 1 \) is the arithmetic genus of \( \beta \). Since del Pezzo surfaces are rational, by Poincaré duality, \( \text{Pic}(S) \simeq H_2(S,\mathbb{Z}) \). So when we write \( |\mathcal{O}_S(\beta)| \) or simply \( |\beta| \), we mean the complete linear system \( |L| \) for the unique \( L \in \text{Pic}(S) \) such that \( c_1(L) = \beta \).

For \( S_r \), let \( h \) be the pullback of \( \mathcal{O}_{\mathbb{P}^2}(1) \) and let \( e_i \) for \( 1 \leq i \leq r \) be the exceptional divisors. The Picard group \( \text{Pic}(S_r) \) is generated by \( h \) and the \( e_i \)'s. The anticanonical divisor is \( -K_{S_r} = 3h - \sum_{i=1}^r e_i \). For \( \mathbb{P}^1 \times \mathbb{P}^1 \), we denote by \( h_1 \) and \( h_2 \) the pullback of \( \mathcal{O}_{\mathbb{P}^1}(1) \) from each factor. The anticanonical divisor is \( -K_{\mathbb{P}^1 \times \mathbb{P}^1} = 2h_1 + 2h_2 \).

Definition 4.2. A line class on \( S \) is a class \( l \in \text{Pic}(S) \) such that \( l^2 = -1 \) and \( (-K_S).l = 1 \).

It is well-known that each line class contains a unique irreducible line and there are only finitely many lines on \( S \).

Example 4.3. By numerical calculation, we list all line classes up to permutation of the \( e_i \)'s:

\[
e_i, (1;1^2), (2;1^5), (3;21^6), (4;2^31^5), (5;2^61^2), (6;3^2)\]

Here, we used the notation \( (d; a_1, \cdots, a_r) \) for the divisor \( dh - \sum a_i e_i \). The superscripts indicate the number of repetitions.

Definition 4.4. In the case of \( S_r \), a point \( P \in E \) is said to be a flex point if there is a curve \( C \) of class \( h \) such that \( C \) meets \( E \) only at \( P \) (of tangency \( h.E = 3 \)).

Lemma 4.5 (See [14]). Let \( \beta \in H_2(S,\mathbb{Z}) \) be a curve class containing a reduced irreducible curve and with \( p_a(\beta) \geq 1 \). Assume that \( \beta \) is not \(-K_{S_r}, -K_{S_b} \) or \(-2K_{S_b} \) (neither of which are very ample). Then \( \beta \) is very ample if and only if there are no line classes \( l \) such that \( \beta.l = 0 \).
Proof. We first note that \( \beta \) is nef and big. In fact, let \( C \) be a reduced irreducible member of \( \beta \). Since \( S \) is del Pezzo, \( K_S.C < 0 \) holds, so \( C^2 > 0 \) follows from \( p_a(C) \geq 1 \) and adjunction. Since \( C \) is irreducible, it is nef.

Now we apply the criterion of [14]. When \( S = \mathbb{P}^2 \), the assertion holds. When \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), the criterion is that \( \beta.h_i \geq 1 \) for \( i = 1, 2 \). By our assumptions, this is satisfied.

For \( S = S_1 \), the criterion is that \( \beta.e_1 \geq 1 \) and that \( \beta.(h - e_1) \geq 1 \). The first condition follows from the assumption that there are no line classes \( l \) with \( \beta.l = 0 \) (and nefness). If we assume that the second condition does not hold, then by nefness of \( \beta \) we have \( \beta.(h - e_1) = 0 \). If we write \( \beta = dh - ae_1 \), this translates into \( d = a \). This contradicts the bigness of \( \beta \). Finally, when \( S = S_r \) for \( r \geq 2 \), the criterion is that \( \beta.l \geq 1 \) for any line class \( l \), which is satisfied by our assumptions. \( \square \)

4.2. \( \beta \)-primitive points.

Definition 4.6. A point \( P \in E \) is said to be \( \beta \)-primitive if \( \beta|_E \sim wP \), but there is no decomposition into non-zero pseudo-effective classes \( \beta = \beta' + \beta'' \), with \( \beta'|_E \sim w'P \), where \( w' = \beta'.(-K_S) > 0 \).

This definition guarantees that if \( P \) is \( \beta \)-primitive, any curve of class \( \beta \) meeting \( E \) only at \( P \) is irreducible and no multiple covers appear in the moduli space \( M(P) \). Note that for del Pezzo surfaces, the effective cone is generated by finitely many classes, and thus there are only finitely many ways of decomposing \( \beta \) as above.

Lemma 4.7. Assume that \( S = \mathbb{P}^2 \) and let \( d \geq 1 \). Then the following are equivalent.

1. \( P \) is \( dh \)-primitive.
2. \( P \) is of order 3d for a choice (not necessarily all) of 0 \in E a flex point.
3. For a fixed flex point 0 \in E,

\[
\begin{align*}
P \text{ is of order } 3d, & \quad \text{if } 3 \mid d. \\
P \text{ is of order } d \text{ or } 3d, & \quad \text{if } 3 \nmid d.
\end{align*}
\]

Proof. Choose 0 \in E to be any flex point. Let \( \text{ord}(P) \) be the order of \( P \) for the resulting group law, so that \( \text{ord}(P)|3d \). From the definition, we see that \( P \) is \( \beta \)-primitive if and only if \( \text{ord}(P)/3d_1 \) for all \( d_1 \) such that \( 1 \leq d_1 < d \). This is equivalent to (3). Condition (2) then is obtained by noting that all other flex points are of order 3 with respect to 0. \( \square \)

Lemma 4.8. Let \( \beta \in H_2(S, \mathbb{Z}) \) be a curve class. Assume that \( (S,E) \) is general and let \( P \in E(\beta) \). Then \( P \) is not \( \beta \)-primitive if and only if there is a pseudo-effective class \( \beta \) and an integer \( k > 1 \) such that \( \beta = k\beta \) and \( \beta|_E \sim \frac{w}{k}P \).

Proof. We are looking at deformation classes of \( (S,E) \). Since the Picard group does not change, we consider a class \( \gamma \) as being a class in each deformation.

Consider a decomposition of \( \beta \) into pseudo-effective classes \( \beta = \beta' + \beta'' \) with \( \beta'|_E \sim w'P \). Assume first that \( \beta = k\beta' \) for \( k \in \mathbb{Q} \). Note that we can check equalities with \( \mathbb{Q} \)-coefficients as
Pic(S) is torsion-free. Choose \(a, b \in \mathbb{Z}\) to satisfy \(aw + bw' = \gcd(w, w')\) and set \(\tilde{\beta} = a\beta + b\beta'\). Then \(\overline{\beta}E \sim \gcd(w, w')P\). Since \(w = kw', akw' + bw' = \gcd(w, w')\) and \(\overline{\beta} = (ak + b)\beta' = \frac{w}{\gcd(w, w')}\beta'\). It follows that \(\frac{w}{\gcd(w, w')}\beta' = \frac{w}{w'}\beta' = k\beta' = \beta\). Since \(\frac{w}{\gcd(w, w')}\) is an integer greater than 1, we are done with the case \(\beta = k\beta'\).

Assume now that \(\beta \neq k\beta'\) for any \(k \in \mathbb{Q}\). This case excludes \(\mathbb{P}^2\). We will show that this can only happen on a proper closed subset of the moduli of the \((S, E)\). Let \(S = S_r\) be the blowup of \(\mathbb{P}^2\) in \(r\) general points \(P_1, \ldots, P_r\) for \(1 \leq r \leq 8\). Then \(E \subset S_r\) is the strict transform of a cubic in \(\mathbb{P}^2\) through \(P_1, \ldots, P_r\). We identify points on \(E\) with those on this cubic. Recall that \(e_i\) is the exceptional divisor over \(P_i\). Choose also a flex point \(P_0\). Let

\[
\beta \sim dh - \sum a_i e_i \quad \text{and} \quad \beta' \sim d'h - \sum a'_i e_i
\]

with \(d, d', a_i, a'_i \in \mathbb{Z}\). Note that there are only a finite number of possibilities for \(\beta'\). Indeed, the set

\[
\{ \beta' \in H_2(S, \mathbb{R}) \text{ pseudo-effective and } (E, \beta') \leq (E, \beta) \}
\]

is compact. We find that

\[
3d'P_0 - \sum a'_i P_i \sim w'P \quad \text{on } E,
\]

as well as

\[
3dP_0 - \sum a_i P_i \sim wP \quad \text{on } E,
\]

so that

\[
(4.1) \quad 3(d'w - dw')P_0 + \sum (w'a_i - wa'_i)P_i \sim 0.
\]

Assume that the \(w'a_i - wa'_i\) are not all zero and view \((4.1)\) as an equation in the \(P_i\) on \(E\). As such, it defines a proper closed subset of the set of tuples of blow up loci \(\{(P_i)\}\). Outside of this set,

\[
(4.2) \quad a_i = \frac{w}{w'}a'_i.
\]

By calculating the degree of the divisor \((4.1)\), we see that \(d'w - dw' = 0\), from which it follows that \(\beta = \frac{d}{d'}\beta'\) using \((4.2)\). Thus \(\beta \neq k\beta'\) is only possible on a closed proper subset of the parameter space.

In the case of \(\mathbb{P}^1 \times \mathbb{P}^1\), we blow up a point on \(E\) away from \(E(\beta)\) and deduce the result from the \(S_2\) case. \(\Box\)

We introduce some more notation. Let \(\beta\) be a curve class, let \((S, E)\) be general with respect to \(\beta\) and let \(P \in E(\beta)\). It follows from Lemma 4.8 that there is a unique \(k(\beta, P) \in \mathbb{N}\) such that \(P\) is \(\beta/k(\beta, P)\)-primitive. If moreover \(\beta/k(\beta, P)\) is a primitive curve class, i.e. is a primitive vector in the cone of effective classes, we say that \(P\) is a \(\beta\)-zero point of \(E\).

For example, flex points are \(dh\)-zero points for all \(d \geq 1\). Indeed, in this case \(k(dh, P) = d\) since \(h\) is primitive (but no multiple of it is). For \(S_r\) and a class \(\beta = dh - \sum a_i e_i\) with \(\gcd(d, a_1, \ldots, a_r) = 1\), each point of \(E(\beta)\) is both \(\beta\)-zero and \(\beta\)-primitive.
Lemma 4.9. Let $\beta$ be a curve class, let $(S, E)$ be general with respect to $\beta$ and let $P \in E(\beta)$. Then $P$ is $\beta$-primitive if and only if $P$ is of order $w$ for a choice (not necessarily all) of $\beta$-zero point $0 \in E$.

Proof. This is a natural extension of the proof of Lemma 4.7. □

Proposition 4.10. For each curve class $\beta$ and a general pair $(S, E)$, there is a $\beta$-primitive point $P \in E(\beta)$.

Proof. This follows from Lemma 4.9 and the fact that $\mathbb{Z}/w \times \mathbb{Z}/w$ has elements of order $w$. □

4.3. Log BPS numbers at primitive points.

Conjecture 4.11 (Special case of Conjecture 3.4). For a curve class $\beta$, for a general pair $(S, E)$ and for $\beta$-primitive $P, P' \in E(\beta)$, $m_P^\beta = m_{P'}^\beta$.

The number $m_P^\beta = N_P^\beta(S, E)$ is invariant under log smooth deformation, cf. [52]. Deforming the triple $S \supset E \ni P$ keeping $E$ smooth and $P$ $\beta$-primitive is such a log smooth deformation. Hence a sufficient condition for Conjecture 4.11 to hold is that in the moduli space of such $S \supset E \ni P$ there is a unique connected component.

For $P, P' \in E(\beta)$ that are $\beta$-primitive for $\beta' = \beta/k$, $k \geq 1$, a similar argument is expected to show that $N_P^\beta(S, E) = N_{P'}^{\beta'}(S, E)$.

In fact, Conjecture 4.11 is true for $S = \mathbb{P}^2$.

Lemma 4.12. Let $S = \mathbb{P}^2$ and fix $d \geq 1$. Then the universal locus $U_d = \{(P, E) \mid E$ is a smooth plane cubic and $P$ is a $d$-primitive point of $E\}$ of $d$-primitive points is irreducible.

Proof. We adapt an argument from [37]. It suffices to show that for any smooth cubic $E$ and distinct $d$-primitive points $P, P' \in E$, we can find a path in $U_d$ connecting $(P, E)$ and $(P', E)$.

To that end, choose a flex $P_0$ of $E$ distinct from $P$ and $P'$. As before, the map $Q \mapsto Q - P_0$ identifies $E(d)$ with the set of $3d$-torsion points of $\text{Pic}^0(E)$. The $d$-primitivity condition on a point $q \in E$ becomes

\begin{equation}
Q - P_0 \text{ is not } 3d'\text{-torsion for any } d' < d,
\end{equation}

independent of the choice of flex $P_0$. In particular, $P - P_0$ and $P' - P_0$ satisfy (4.3). Furthermore, by Lemma 4.7, we know that the orders of $P - P_0$ and $P' - P_0$ are exactly $3d$ if $3|d$ and either $d$ or $3d$ if $3 \not{|} d$.

Up to change of coordinates in $\mathbb{P}^2$, we can identify $E$ with the image of $E_{\omega_1, \omega_2} := \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ under the map

\begin{equation}
\iota : E_{\omega_1, \omega_2} \to \mathbb{P}^2, \quad \iota(z) = (\psi(z), \psi'(z), 1)
\end{equation}
for some independent periods $\omega_1$ and $\omega_2$, with $\iota(0) = P_0$. In \cite{4}, $\varphi$ is the Weierstrass function on $\mathbb{C}$, periodic with respect to the lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. We put $\omega = (\omega_1, \omega_2)$ for convenience. For some $m, n, m', n' \in ((1/3d)\mathbb{Z})/\mathbb{Z}$, let $\tilde{P} = m\omega_1 + n\omega_2$ and $\tilde{P}' = m'\omega_1 + n'\omega_2$ be the $3d$-torsion points of $E_\omega$ corresponding to $P$ and $P'$ respectively via $\iota$.

We claim that the flex $P_0$ can be chosen so that $P - P_0$ and $P' - P_0$ have the same order. There is nothing to show unless $3 \nmid d$, $P - P_0$ has order $3d$, and $P' - P_0$ has order $d$, up to interchange of $P$ and $P'$. Replacing $P_0$ by a different flex is tantamount to adding a nontrivial $3$-torsion point $t$ to both $P - P_0$ and $P' - P_0$. We observe that for some choice of flex, both points have order $3d$. Otherwise for all nontrivial $3$-torsion points $t$, either $d(P - P_0 + t) = 0$ or $d(P' - P_0 + t) = 0$, which is immediately seen to be impossible, recalling that $3 \nmid d$.

Having guaranteed that $P - P_0$ and $P' - P_0$ have the same order by a judicious choice of flex, then since $\text{SL}(2, \mathbb{Z})$ acts transitively on the set of torsion points of $E_\omega \simeq (\mathbb{R}/\mathbb{Z})^2$ of any fixed order, we can find $A \in \text{SL}(2, \mathbb{Z})$ so that $(m', n') = (m, n)A$. We then choose a path $A(t), 0 \leq t \leq 1$ in $\text{SL}(2, \mathbb{R})$ satisfying $A(0) = I$ and $A(1) = A$. Then the required path in $U_d$ can be taken to be

$$t \mapsto \left( \iota \left( (m, n) A(t)^t \omega \right), \iota \left( E_{\omega, A(t)} \right) \right).$$

As discussed above, Lemma \ref{lemma12} establishes our Conjecture for $S = \mathbb{P}^2$.

**Proposition 4.13.** Conjecture \ref{conjecture4.1} is true for $S = \mathbb{P}^2$.

As the results of our calculations in this section are independent of $P$, we sometimes omit $P$ and write $m_\beta = m_\beta^P$.

**Proposition 4.14.** Let $\beta \in H_2(S, \mathbb{Z})$ be a curve class that contains a reduced irreducible curve and choose $P \in E(\beta)$. Then there is a short exact sequence

$$0 \to H^0(O_S(\beta - E)) \to H^0(O_S(\beta)) \xrightarrow{\text{res}} H^0(O_E(wP)) \to 0,$$

where $\text{res}$ is the restriction map. Moreover, $h^0(O_S(\beta - E)) = p_a(\beta)$, $h^0(O_S(\beta)) = \chi(O_S(\beta)) = p_a(\beta) + w$ and $h^1(O_S(\beta)) = h^2(O_S(\beta)) = 0$.

**Proof.** Consider the short exact sequence

$$0 \to O_S(-E) \to O_S \to O_E \to 0.$$

Since $\beta|_E \sim wP$, tensoring with $O_S(\beta)$ yields

$$0 \to O_S(\beta - E) \to O_S(\beta) \to O_E(wP) \to 0.$$

Consider the maps induced on sections

$$0 \to H^0(O_S(\beta - E)) \to H^0(O_S(\beta)) \to H^0(O_E(wP)),$$
where the last map is the restriction map. Denote by $\mathcal{O}_\beta$ the structure sheaf $\mathcal{O}_C$ of an irreducible curve of class $\beta$. From the short exact sequence

$$0 \to \mathcal{O}_S(-\beta) \to \mathcal{O}_S \to \mathcal{O}_\beta \to 0$$

and Serre duality, we obtain $H^1(\mathcal{O}_S(\beta-E)) = H^1(\mathcal{O}_S(-\beta)) = 0$. It follows that the restriction map in $[4.7]$ is surjective. Moreover, $H^2(\mathcal{O}_S(\beta-E)) = H^0(\mathcal{O}_S(-\beta)) = 0$ by Serre duality. Thus it follows from Riemann-Roch and the adjunction formula that

$$h^0(\mathcal{O}_S(\beta-E)) = \chi(\mathcal{O}_S(\beta-E)) = \frac{1}{2}(\beta - E).\beta + 1 = p_a(\beta).$$

Finally, from the long exact cohomology sequence associated to $[4.6]$, we find that $h^1(\mathcal{O}_S(\beta)) = h^2(\mathcal{O}_S(\beta)) = 0$ and therefore $h^0(\mathcal{O}_S(\beta)) = \chi(\mathcal{O}_S(\beta))$. □

**Definition 4.15.** Set

$$|\mathcal{O}_S(\beta, P)| := \{C \in |\beta| : C \cap E \supseteq wP \text{ as subschemes of } E\}.$$

Up to scalar multiple, there is only one section in $H^0(\mathcal{O}_E(wP))$ that vanishes to order $w$ at $P$. Denote the corresponding one-dimensional subspace of $H^0(\mathcal{O}_E(wP))$ by $L_P$. By Proposition 4.14

$$|\mathcal{O}_S(\beta, P)| = \mathbb{P}\left(\text{res}^{-1}(L_P)\right)$$

is of dimension $p_a(\beta)$.

**Remark 4.16.** If $P$ is $\beta$-primitive, we identify $M_{\beta,P}$ with the set of rational curves in $|\mathcal{O}_S(\beta, P)|$ with only one branch at $P$. The set $M_{\beta,P}$ is finite by Proposition 2.12.

**Proposition 4.17** (See also Theorem 3.10 of [11]). Let $\beta \in H_2(S, \mathbb{Z})$ be a curve class and $P$ a $\beta$-primitive point. Consider the blow down $\pi : S \to S'$ of $S$ along some line classes $l$ such that $\beta.l = 0$. Then $\pi(P)$ is $\pi_*\beta$-primitive on $\pi(E)$, and each curve $C \in M_{\beta,P}$ is isomorphic to exactly one curve $C' \in M_{\pi_*\beta,\pi(P)}$ and vice-versa. Furthermore, for each such $C$, a neighborhood of $C$ in $S$ is isomorphic to a neighborhood of $C'$ in $S'$ and $m_{\beta}^{P} = m_{\pi_*\beta}^{\pi(P)}$.

**Proof.** It is straightforward to see that $\pi(P)$ is $\pi_*\beta$-primitive.

Blowing down a line $l$ does not change members of $|\mathcal{O}_S(\beta, P)|$ except for those containing $E$, since they are irreducible and $\beta.l = 0$. In particular, it does not change the geometry in a neighborhood of elements of $M_{\beta,P}$ and $M_{\pi_*\beta,\pi(P)}$. Hence the moduli spaces are isomorphic and $m_{\beta}^{P} = m_{\pi_*\beta}^{\pi(P)}$. □

**Corollary 4.18.** Let $\beta \in H_2(S, \mathbb{Z})$ be a curve class containing a reduced irreducible member and such that $p_a(\beta) \geq 1$. Each distinct two lines $l_1$ and $l_2$ with $\beta.l_1 = \beta.l_2 = 0$ are mutually disjoint. Let $\eta$ be the number of disjoint lines $l$ with $\beta.l = 0$ and let $\pi : S \to S'$ be the del Pezzo surface obtained by blowing down these lines. Assume that $\beta \neq -K_S$, $-K_S$ or $-2K_S$ (which are ample). Then $\pi_*\beta$ is very ample and $m_{\beta}^{P} = m_{\pi_*\beta}^{\pi(P)}$. □
Then:

\[ \beta \] smooth divisor and let \( \beta \) of class \( e \)

Proposition 4.20 (See Proposition 1.7) (analogously [11, Theorem 1.1] for the local BPS invariants).

Remark 4.19. By Proposition 4.17, the log BPS number \( \beta \) should depend on \( e(S) - \eta \) and not simply on \( e(S) \). This is exactly what we observe in the prediction of Theorem 1.9 (analogously [11, Theorem 1.1] for the local BPS invariants).

Proposition 4.20 (See Proposition 1.7). Let \((S, E)\) be a regular log Calabi-Yau surface with smooth divisor and let \( \beta \in H_2(S, \mathbb{Z}) \) be a curve class. Let \( C \) be an (irreducible) rational curve of class \( \beta \) maximally tangent to \( E \) at \( P \), and denote the normalization map by \( n : \mathbb{P}^1 \to C \). Then:

1. The map \( n \) gives an isolated point of \( \overline{M}_{\beta}(S, E) \), and contributes a positive integer to \( N_{\beta}(S, E) \).
2. If \( C \) is immersed outside \( P \), \([n]\) contributes 1 to \( N_{\beta}(S, E) \).

Assume that \( S \) is del Pezzo.

3. If \( n \) is smooth at \( P \), then it contributes \( e(\text{Pic}^0(C)) \) to \( N_{\beta}(S, E) \).
4. If \( C \) is smooth at \( P \) and has arithmetic genus 1, then \([n]\) contributes \( e(C) \) to \( N_{\beta}^P(S, E) \).
5. Let \( P \in E(\beta) \) be \( \beta \)-primitive. Assume that all \( C \in M_{\beta,P} \) are smooth at \( P \). Then

\[
\beta^P = \sum_{C \in M_{\beta,P}} e(\text{Pic}^0(C)).
\]

Proof. (1) By Proposition 2.17, \( n \) lifts to a unique stable log map. By [35, Example 7.1] or more generally Proposition 2.11, in a neighborhood of \([n]\), the moduli of stable log maps is isomorphic to a certain locally closed substack of the moduli of 1-marked ordinary stable maps parametrizing maps \( f \) with \( f^*E = w.(\text{marked point}) \). It follows from Proposition 2.12 that \([n]\) is isolated. Since \( n \) has no nontrivial automorphisms, this isolated component is a scheme, and the contribution of \([n]\) is given by its length.

(2) Let \( x \in \mathbb{P}^1 \) be such that \( n^{-1}(E) = \{x\} \). The deformation of \([n]\) in \( \overline{M}_{\beta}(S, E) \) is governed by the complex \( T^* := [T_{\mathbb{P}^1}(-\log x) \cdot \sum \cdot n^*T_S(-\log E)] \) in degrees \(-1\) and \(0\). In particular, the tangent space is given by \( R^0\Gamma(\mathbb{P}^1, T^*) \).

If \( n \) is immersible outside \( x \), \( dn \) is an injective bundle map, and by calculating the degree, we see that \( T^* \) is quasi-isomorphic to \( O_{\mathbb{P}^1}(-1) \). So \([n]\) is infinitesimally rigid.

(3) Let \( M_{[n]} \) be the connected component of \( \overline{M}_{\beta}(S, E) \) at \([n]\) and denote by \( m(C) \) the length of \( M_{[n]} \). The multiplicity of \([n]\) as a stable map to \( C \), i.e., the length of the scheme \( M_{0,0}(C, [C]) \), was calculated in [15], see also [61], to be \( e(\text{Pic}^0(C)) \).

As we noted above, the moduli of log structure is trivial and \( M_{[n]} \) can be considered as a subscheme of \( M_{0,0}(S, [C]) \) by [35, Example 7.1] or Proposition 2.11. In order to show that \( m(C) \) is the stable map multiplicity of \([n]\), we need to show that deformations of \([n]\) in \( M_{[n]} \) factor (scheme-theoretically) through \( C \). Note that since \( C \) is assumed to be smooth at \( P \),
in the notation of [35, Example 7.1], this guarantees that \( u_x = \mu_x \) at \( P \) remains true in a deformation.\(^3\)

Consider the open sub-linear system

\[
|O_S(\beta, P)|^\circ := \{ C \in |\beta| : C \cap E = wP \} \subseteq |O_S(\beta, P)|
\]

and denote by \( \text{Pic}^0(\beta, P) \) the relative compactified Jacobian of the universal curve over \( |O_S(\beta, P)|^\circ \). The main result of [12] states that \( \text{Pic}^0(\beta, P) \) is smooth at each point of the fiber \( \text{Pic}^0(C) \). Using this result, the statement follows from [15, Section F, proof of Theorem 2].

(4) In this case, we can recover (3) directly. Outside the member of \( \beta \) with a singularity at \( P \) (see §5.2), the curve family can be identified with the associated elliptic fibration, and with \( \text{Pic}^0(\beta, P) \) at reduced irreducible fibers. The total space of the elliptic fibration is given by successive blow-ups at \( P \), so it is a nonsingular surface.

(5) follows from (3). \( \square \)

The calculation of \( e(\text{Pic}^0(C)) \) reduces to the case of unibranched curves with only 1 singularity.

**Proposition 4.21.** (See e.g. [61, Formula (5)]) Let \( \tilde{C} \) be the minimal unibranch partial normalization of \( C \). Let \( \Delta \) be the set of singular points of \( \tilde{C} \). For \( y \in \Delta \), let \( C_y \) be a rational curve smooth away from a unique singularity analytically isomorphic to \( y \). Then

\[
e(\text{Pic}^0(C)) = \prod_{y \in \Delta} e(\text{Pic}^0(C_y)).
\]

It is shown in [61, §6] that in an appropriate sense and up to sign the contribution of an irreducible rational curve \( C \) to \( n_\beta \) is given by \( e(\text{Pic}^0(C)) \). In light of Conjecture 1.6, Proposition 4.20 states that in a normalized way, \( C \) contributes the same to the local and log BPS numbers.

5. Calculations

In this section, we compute \( m_\beta^P \) for all del Pezzo surfaces, classes \( \beta \) of arithmetic genus up to 2 and \( \beta \)-primitive points \( P \). First note that if there is no irreducible reduced curve \( C \) of degree \( \beta \), then \( m_\beta^P = 0 \) for \( P \)-\( \beta \)-primitive. In particular, if \( p_a(\beta) < 0 \), then \( m_\beta^P = 0 \).

So henceforth we assume that \( \beta \) contains an irreducible reduced curve.

5.1. Arithmetic genus 0. Assume that \( p_a(\beta) = 0 \) and let \( P \) be a \( \beta \)-primitive point. By Proposition 4.14, \( H^0(O_S(\beta)) \simeq H^0(O_E(wP)) \). There is a (unique up to scaling) non-zero section \( s \in H^0(O_E(wP)) \) that vanishes at \( P \) of order \( w \). By \( \beta \)-primitivity, the corresponding curve is necessarily reduced and irreducible, thus it is isomorphic to \( \mathbb{P}^1 \). Hence \( m_\beta = 1 \).

\(^3\)On the converse, say \( C \) has a cusp at \( P \). Then \( [n] \) has a first order deformation as a stable map, resolving the cusp. Under this deformation the pullback of \( E \) is no longer a multiple of \( \mu_x \) (e.g. if \( \mu_x = 2 \), resp. 3, and \( x \) is defined by \( t = 0 \), then the pullback of \( E \) is defined by \( t^2 + \epsilon \), resp. \( t^3 + \epsilon t \), for an infinitesimal parameter \( \epsilon \)).
5.2. Arithmetic genus 1. Assume now that \( p_\eta(\beta) = 1 \) and let \( P \in E(\beta) \) be a \( \beta \)-primitive point. Let \( \eta \) be the number of disjoint lines \( l \) with \( \beta \cdot l = 0 \). We blow down all such lines in \( S \), yielding a del Pezzo surface \( \pi : S \to S' \) with divisor \( E' \) and a curve class \( \beta' := \pi_* \beta \). Then \( e(S') = e(S) - \eta \) and by Corollary 4.18, \( \beta \) is ample and \( m_\beta^P = m_\beta^{P'} \) for \( P' := \pi(P) \). By \cite{11}, Lemma 4.3 or Proposition 2.18 (3), \( \beta' = |E'| \).

By Proposition 4.14, \( \Lambda := |O_{S'}(E', P')| \) is a linear pencil on \( S' \). Note that any member of \( \Lambda \setminus \{E'\} \) has at worst a node or a cusp, since it is irreducible by \( \beta \)-primivity and is of arithmetic genus 1; \( E' \) itself is of course nonsingular.

Assume first that \( w > 1 \). There is one curve \( D \) in the pencil \( \Lambda \) that is nodal or cuspidal at \( P' \). Indeed, by restricting to an open affine neighborhood \( U \) of \( P' \), assume that the members of \( \Lambda \) are parametrized by \( sf|_U + tg|_U \) for \( [s : t] \in \mathbb{P}^1 \) where \( f \) and \( g \) are distinct sections of \( \Lambda \). Taking the differential at \( P' \), we obtain a linear family of elements of \( m_{P'}/m_{P'}^2 \), which vanishes for a single value of \( [s : t] \). The corresponding curve \( D \) is nodal or cuspidal at \( P' \).

In the former case, one branch meets \( E' \) with multiplicity 1, and the other branch meets \( E' \) with multiplicity \( w - 1 \).

Next, we blow up \( w \) times along the inverse image of \( P' \) in the strict transforms of \( E' \). Assume \( D \) is nodal at \( P' \). The first blowup will separate the two branches of \( D \) and the total transform of \( D \) becomes a cycle of two \( \mathbb{P}^1 \)'s. Each successive blowup but the last will introduce another \( \mathbb{P}^1 \) in the cycle. After the first \( w - 1 \) blowups, the preimage \( \tilde{D} \) of \( D \) consists of a cycle of \( w \) \( \mathbb{P}^1 \)'s with topological Euler characteristic \( \omega \). The last blowup then separates all of the curves in our pencil and we obtain a family \( \mathcal{C} \to \mathbb{P}^1 \), where the last exceptional divisor maps isomorphically to the base \( \mathbb{P}^1 \).

We apply the same procedure for \( D \) cuspidal at \( P' \). In this case \( \tilde{D} \) is a chain of \( \mathbb{P}^1 \)'s and the Euler characteristic of \( \tilde{D} \) is \( w + 1 \).

In both cases, \( \tilde{D} \) is a fiber of \( \mathcal{C} \to \mathbb{P}^1 \). The other fibers are members of \( \Lambda \) other than \( D \). So, we can calculate \( m_\beta \) by computing the topological Euler characteristic of \( \mathcal{C} \). For a smooth curve \( C \) of the pencil, \( e(C) = 0 \). If \( C \) is nodal, then \( e(C) = 1 \) and if \( C \) is cuspidal, \( e(C) = 2 \).

In case \( D \) is nodal at \( P' \),

\[
e(C) = \# \{ \text{nodal fibers} \} + 2 \cdot \# \{ \text{cuspidal fibers} \} + w.
\]

In case \( D \) is cuspidal at \( P' \),

\[
e(C) = \# \{ \text{nodal fibers} \} + 2 \cdot \# \{ \text{cuspidal fibers} \} + w + 1.
\]

On the other hand,

\[
e(C) = e(S') + \# \{ \text{blow ups} \} = e(S') + w
\]

By Proposition 4.20, if \( D \) is nodal at \( P' \),

\[
m_\beta = \# \{ \text{nodal fibers} \} + 2 \cdot \# \{ \text{cuspidal fibers} \} = e(S') = e(S) - \eta.
\]

(5.1)
where $D \not\in M_{\beta', P'}$ has been used in the first equality in (5.1). If $D$ is cuspidal at $P'$, then $D$ is also in $M_{\beta', P'}$ with multiplicity 1. Therefore

$$m_{\beta} = 1 + \# \{\text{nodal fibers}\} + 2 \cdot \# \{\text{cuspidal fibers}\} = e(S') = e(S) - \eta$$

as well.

In case that $w = 1$, then it follows that $\beta = -K_{S_8}$. Since $w = 1$, there is no curve in the pencil $\Lambda$ that is nodal at $P$. To obtain the universal family, we only need to blow up once and obtain that

$$e(S_8) + 1 = e(C) = \# \{\text{nodal fibers}\} + 2 \cdot \# \{\text{cuspidal fibers}\},$$

so that $m_{\beta} = e(S_8) + 1 = 12$. Note that $\eta = 0$ in this case, since $\beta.\ell = 1$ for any line.

**Remark 5.1.** In light of the previous argument, note that we expect that for general $(S, E)$, curves with a cusp at the marked point are avoided.

**Remark 5.2.** In the case of $\mathbb{P}^2$, we can rule out cuspidal degree 3 curves meeting $E$ in a 3h-primitive point, as was noted in the proof of [64, Proposition 1.5]. Indeed, suppose that there is a degree 3 cuspidal curve $C$ meeting $E$ in a 9-torsion point $P$, where we take the group law on $E$ obtained by choosing a flex point as 0. Then $C$ also has a group law, and is isomorphic to $\mathbb{G}_a$ away from its cusp. The zero of $C$ is a flex point. Moreover, $E$ induces the equation $9P = 0$ on $C$. Since $C$ has no torsion, $P = 0$ is a flex point, which is impossible since $C$ and $E$ have 9-tuple intersection at $P$ and since $P$ is not a flex point for $E$.

**5.3. Approach for arithmetic genus 2 and higher.** Assume that $\beta$ is a curve class with $p_a(\beta) = 2$ containing a reduced irreducible member, let $P \in E(\beta)$ be a $\beta$-primitive point and let $\eta$ be the number of disjoint lines $l$ with $\beta.l = 0$. By Corollary 4.18, after blowing down these lines, we may assume that $\beta$ is very ample, except when $\beta = -2K_{S_8}$. See Remark 1.10 for the case $\beta = -2K_{S_8}$.

There are no genus 2 classes for $\mathbb{P}^2$. In [64], a method was developed for general $E$ for calculating counts of rational curves in blow ups of $\mathbb{P}^2$ with relative conditions of maximal tangency along a smooth effective anticanonical divisor. All log BPS numbers $m_{\beta}$ for $\beta$ as above with $d \leq 4$ and for $d \leq 6$ with some additional assumptions are calculated in [64]. This covers all genus 2 classes of $S_r$ as long as $r \leq 7$. Examining the numbers yields that

$$m_{\beta}^P = \left(\frac{e(S_r) - \eta}{2}\right) + 5,$$

as predicted by Theorem 1.9. Note that although the calculation in [64] is not in terms of the binomial coefficients, the counts do agree.

The method of [64] uses a 1-parameter family of a smooth rational projective surfaces $X$, a smooth anticanonical divisor $E$ and a divisor class $D$ corresponding to $\beta$. Let $X_t$ denote a fiber of $X$. Let $P$ be a section corresponding to one of the finitely many points such that $D_t|_{E_t} \sim wP_t$, where $w = D_t.E_t$. Then the moduli space $M(X, E, P, D)$ is constructed (cf. Definition 3.3 in [64]) with the following properties. According to Lemma 3.4 of [64], it has
an open subspace $M_0(\mathcal{X}, \mathcal{E}, \mathcal{P}, D)$ which parametrizes morphisms from $\mathbb{P}^1$ to $X_t$ that meet $E_t$ with maximal tangency at $P_t$.

The interesting part is the boundary. Namely, $M(\mathcal{X}, \mathcal{E}, \mathcal{P}, D)$ is constructed in such a way that allows only degenerations into a rational curve with two components. This is simple enough and yields relations between a surface and its blowups in Theorem 3.8 of [64]. Using this theorem and many calculations, [64] calculates the log BPS numbers in many cases. In particular, the log BPS numbers are computed for $S_r$ and classes of arithmetic genus up to two, for $\mathbb{P}^2$ and degrees up to 6, and for degrees up to 8 with some additional hypotheses, are computed. Conjecture 1.6 is verified for these cases.

It remains to consider $\mathbb{P}^1 \times \mathbb{P}^1$. Up to permuting the generators $h_1$ and $h_2$, the only genus 2 class is $2h_1 + 3h_2$. Blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ in a general point, $2h_1 + 3h_2$ pulls back to a (non-very ample) class $\beta'$ on $S_2$ and the exceptional divisor for the blow up is the unique line class not meeting $\beta'$. By Proposition 4.17, $m_{\beta'} = m_{\beta}$. By the calculations of [64],

$$m_{\beta'} = \left(\frac{e(S_2) - 1}{2}\right) + 5 = \left(\frac{e(\mathbb{P}^1 \times \mathbb{P}^1)}{2}\right) + 5$$

as expected.

**Remark 5.3.** Note that in all calculations that we encountered, $m_{\beta}$ was given in terms of polynomials of topological numbers associated to $S$.

Combining the calculations of this section and the ones of [64] with Theorem 1.9 (Theorem 1.1 in [11]) proves Theorem 1.8.

6. Multiple covers and loop quiver DT invariants

Let $(S, E)$ be a log Calabi-Yau surface pair. Let $C$ be an irreducible nodal rational curve in $S$ that meets $E$ with maximal tangency at $P$. Then the corresponding immersion $[\mathbb{P}^1 \to C] \in \overline{M}_{[C]}^P(S, E)$ is infinitesimally rigid, cf. Proposition 2.12. Here, $[C]$ is the class of $C$, as usual.

Let $l \in \mathbb{N}$. We claim that the contributions of multiple covers over $C$ to $m_{\lfloor l \lfloor C \rfloor}$ is given by a certain quiver DT invariant. Let $\text{Contr}(l,C)$ be the contribution of the component of $\overline{M}_{[C]}^P(S, E)$ corresponding to maps $f : D \to C$ with $f_*D = lC$ as cycles. The contribution $\text{Contr}(l,C)$ is defined and calculated in [32, Section 6] to be

$$\text{Contr}(l,C) = \frac{1}{l!} \left( \frac{l(C.E - 1)}{l - 1} \right).$$

**Definition 6.1.** Let $l \in \mathbb{N}$. Define the contribution $\text{Contr}^{\text{BPS}}(l,C)$ of $C$ to $m_{\lfloor l \lfloor [C] \rfloor}^P$ to be

$$\text{Contr}^{\text{BPS}}(l,C) := \sum_{\{k \mid \Phi \in E(\lfloor l \rfloor [C]/k)\}} \frac{(-1)^{(k-1)(l(C.E)/k)}}{k^2} \mu(k) \text{Contr}(l/k,C).$$

We will express these contributions in terms of generalized Donaldson-Thomas (DT) invariants of loop quivers. Motivated by the framework of Kontsevich-Soibelman in [44], Reineke
in [60] and [59] calculated these invariants. We state Reineke’s calculation in Theorem 6.3 below.

Fix $m \geq 1$ and consider the $m$-loop quiver, consisting of one vertex and $m$ loops. The associated framed $m$-loop quiver $L_m$ contains an additional vertex and an arrow directed towards the original vertex. The quiver $L_m$ is depicted below.

Let $n \geq 0$ and denote by

$$U \subset \mathbb{C}^n \oplus M_n(\mathbb{C})^\oplus m$$

the open subset of stable representations of $L_m$ of dimension vector $(1, n)$. Then $GL_n(\mathbb{C})$ acts on $U$ via

$$g \cdot (v, \phi_i) = (gv, g\phi_i g^{-1}).$$

The geometric quotient for this action is the noncommutative Hilbert scheme $\text{Hilb}^{(m)}_n$. Consider the generating function

$$F^{(m)}(t) := \sum_{n \geq 0} \chi(\text{Hilb}^{(m)}_n) t^n \in \mathbb{Z}[[t]].$$

Since $F(0) = 1$, $F(t)$ admits a product expansion.

**Definition 6.2** (Definition 3.1 in [60], following [44]). Define the generalized Donaldson-Thomas invariants $DT^{(m)}_n \in \mathbb{Q}$ of $L_m$ through the product expansion (Kontsevich-Soibelman wall-crossing formula):

$$F^{(m)}((-1)^{m-1}t) = \prod_{n \geq 1} (1 - t^n)^{-(1)^{m-1}n} \mu_{DT^{(m)}_n}.$$ 

For the following theorem, note that the formula as stated in [60] has a typo.

**Theorem 6.3** (Theorem 3.2 in [60], see also Lemma 12 of [22] and [17, 18]). We have $DT^{(m)}_n \in \mathbb{N}$ and

$$DT^{(m)}_n = \frac{1}{n \phi} \sum_{d|n} \mu\left(\frac{n}{d}\right) (-1)^{d-1} \left(\frac{dm - 1}{d - 1}\right).$$

Coming back to contributions of multiple covers to the $m^P$, we have the following unexpected result.

**Proposition 6.4.** Assume that the normalization map $[\mathbb{P}^1 \to C] \in \overline{M}_{g}(S, E)$ is infinitesimally rigid. Then

$$\text{Contr}^{\text{BPS}}(l, C) = DT^{(C, E-1)}_l.$$
Proof. The argument follows the same lines as the proof of the main result in [22] and we therefore do not reproduce it here. \[\Box\]

Remark 6.5. Let \(\beta\) be a line class or a conic class and let \(k \geq 2\) be an integer.

For \(P \in E(\beta)\), we have \(m_{k\beta}^P = 0\) from the previous proposition.

We also have \(m_{k\beta}^P = 0\) for \(P \in E(k\beta) \setminus E(\beta)\). In fact, \(\overline{\mathcal{M}}^P_{l\beta}(S,E) = \emptyset\) holds for any \(l \geq 1\).

If \(\beta\) is a line class consisting of a \((-1)\)-curve, then \(l\beta = \{lC\}\), so our assertion is clear. If \(\beta\) is a conic class corresponding to a fibration \(\pi : S \to \mathbb{P}^1\) whose generic fiber is \(\mathbb{P}^1\), then \(\pi|_C\) is ramified at each point of \(E(\beta)\) and is 2-to-1 at other points. For any \(l\), an element of \(l\beta\) is a sum of \(l\) fibers of \(\pi\), so if it meets \(E\) at only 1 point, this point must belong to \(E(\beta)\).

Thus we have verified Conjectures 1.2 and 1.3 for the class \(k\beta\).

6.1. Calculations. We prove that Conjecture 3.4 holds for \(S = \mathbb{P}^2\) and degrees up to 4. In order to do so, we collect some calculations from [64, 65]. Choose as zero element \(0 \in E\) a flex point. For a given degree \(dh \in H_2(\mathbb{P}^2, \mathbb{Z})\), recall from Lemma 2.14 that

\[
E(dh) \simeq \mathbb{Z}/3d \times \mathbb{Z}/3d.
\]

Let \(k|d\) and set (cf. Lemma 1.7)

\[
P(k) := \{P \in E(dh) \mid P\text{ is of order } k \text{ or } 3k \text{ if } 3|k, \text{ of order } 3k \text{ if } 3|k\}.
\]

Note that the implicit dependence on \(d\) has been suppressed from the notation for \(P(k)\). For \(k|d\) and \(P \in P(k)\), set

\[
\overline{\mathcal{M}}^P_{dh}(\mathbb{P}^2, E) := \overline{\mathcal{M}}^P_{dh}(\mathbb{P}^2, E).
\]

Implicit in the definition is an assumption that for \(E\) general and \(P, P' \in P(k)\), there is a 1-to-1 correspondence between \(\overline{\mathcal{M}}^P_{dh}(\mathbb{P}^2, E)\) and \(\overline{\mathcal{M}}^{P'}_{dh}(\mathbb{P}^2, E)\) preserving the type of singularity. For the examples of this section, this follows from the calculations of [64, 65].

For \(d = 1, 2\), cf. [64, 65],

\[
\begin{array}{|c|c|}
\hline
\overline{\mathcal{M}}^{P(1)}_h(\mathbb{P}^2, E) = \{\text{flex line}\} & m^{P(1)}_h = N^{P(1)}_h(\mathbb{P}^2, E) = 1 \\
\overline{\mathcal{M}}^{P(2)}_{2h}(\mathbb{P}^2, E) = \{\text{conic}\} & m^{P(2)}_{2h} = N^{P(2)}_{2h}(\mathbb{P}^2, E) = 1 \\
\overline{\mathcal{M}}^{P(1)}_{2h}(\mathbb{P}^2, E) = \{\text{double covers over flex line}\} & N^{P(1)}_{2h}(\mathbb{P}^2, E) = \text{Contr}(2, \text{line}) = \frac{3}{4} \\
\hline
\end{array}
\]

Moreover, by (3.5)

\[
m^{P(1)}_{2h} = N^{P(1)}_{2h}(\mathbb{P}^2, E) + \frac{1}{4} N^{P(1)}_h(\mathbb{P}^2, E) = 1,
\]

as expected. We could alternatively have computed by Proposition 6.4

\[
m^{P(2)}_{2h} = \text{DT}_1(5) = 1,
\]

\[
m^{P(1)}_{2h} = \text{DT}_2(2) = 1.
\]

The calculations for \(d = 3\) and a flex point \(P(1)\) depend on \(E\). Denote by \(E^{\text{gen}}\) an elliptic curve such that \(j(E^{\text{gen}}) \neq 0\) and let \(E^0\) be an elliptic curve with \(j(E^0) = 0\), such as the
curve $Y^2Z - X^3 - Z^3$. Then there is a cuspidal cubic meeting $E^0$ in a flex point and in fact this is the only case where this happens. Such a cuspidal cubic contributes 2 to the log GW invariant. Then [64, 65] gives

\[
\begin{array}{|c|c|}
\hline
\bar{M}^{P(3)}_{3h}(\mathbb{P}^2, E^{gen}) &= \{3 \text{ nodal cubics}\} \\
m^{P(3)}_{3h} &= N^{P(3)}_{3h}(\mathbb{P}^2, E^{gen}) = 3 \\
\bar{M}^{P(1)}_{3h}(\mathbb{P}^2, E^{gen}) &= \{2 \text{ nodal cubics}\} \cup \{\text{triple covers over flex line}\} \\
N^{P(1)}_{3h}(\mathbb{P}^2, E^{gen}) &= 2 + \text{Contr}(3,\text{line}) = 2 + \frac{10}{9} \\
\bar{M}^{P(1)}_{3h}(\mathbb{P}^2, E^{0}) &= \{1 \text{ cuspidal cubic}\} \cup \{\text{triple covers over flex line}\} \\
N^{P(1)}_{3h}(\mathbb{P}^2, E^{0}) &= 2 + \text{Contr}(3,\text{line}) = 2 + \frac{10}{9} \\
\hline
\end{array}
\]

For both $E = E^{gen}, E^{0}$, we obtain

\[
m^{P(1)}_{3h} = N^{P(1)}_{3h}(\mathbb{P}^2, E) - \frac{1}{9} N^{P(1)}_{h}(\mathbb{P}^2, E) = 3,
\]

as predicted. For $E^{gen}$, the relevant log maps are infinitesimally rigid and we also obtain

\[
m^{P(1)}_{3h} = 2 \cdot DT_1^{(8)} + DT_3^{(2)} = 2 + 1 = 3.
\]

To treat $d = 4$, we use two results from [65]. We state them for the setup at hand.

**Theorem 6.6** (Theorem 1.4 in [65]). Let $C_1$ and $C_2$ be two (irreducible) immersed rational curves of classes $\beta_1$ and $\beta_2$ with $\beta = \beta_1 + \beta_2$ such that

- $C_1$ and $C_2$ meet $E$ only at $P \in E(\beta_1 + \beta_2)$ in maximal tangency.
- $(C_1,C_2)_P = \text{min } \{(C_1,E),(C_2,E)\}$.

Consider $[f] \in \mathbb{M}^P_{\beta}(S,E)$ the unique (isolated) point with image $C_1 \cup C_2$. Then the contribution of $[f]$ to $N^P_{\beta}(S,E)$ is $\text{min } \{(C_1,E),(C_2,E)\}$.

**Proposition 6.7** (Proposition 4.4 in [65]). Assume that $E$ is general. Then:

1. $\bar{M}^{P(4)}_{4h}(\mathbb{P}^2, E) = \{16 \text{ nodal quadrics}\}$.
2. $\bar{M}^{P(2)}_{4h}(\mathbb{P}^2, E) = \{14 \text{ nodal quadrics}\} \cup \{\text{double covers over conic}\}$.
3. The components of $\bar{M}^{P(3)}_{4h}(\mathbb{P}^2, E)$ are
   - $\{8 \text{ nodal quadrics}\}$,
   - $2 \text{ isolated maps with reducible image } C + L$, where $C = \text{nodal cubic}$ and $L = \text{flex line}$,
   - $\{4\text{-fold covers over flex line}\}$.

Combining these results yields

- $N^{P(4)}_{4h}(\mathbb{P}^2, E) = 16$,
- $N^{P(2)}_{4h}(\mathbb{P}^2, E) = 14 + \text{Contr}(2,\text{conic}) = 14 + \frac{9}{4}$,
- $N^{P(1)}_{4h}(\mathbb{P}^2, E) = 8 + 2 \cdot \text{min } \{9,3\} + \text{Contr}(4,\text{line}) = 14 + \frac{35}{16}$.

At the log BPS level:

- $m^{P(4)}_{4h} = 16$,
- $m^{P(2)}_{4h} = N^{P(2)}_{4h}(\mathbb{P}^2, E) - \frac{1}{4} N^{P(2)}_{2h}(\mathbb{P}^2, E) = 14 + \frac{9}{4} - \frac{1}{4} = 16$, 

\[ m_{4h}^{P(1)} = \mathcal{N}_{4h}^{P(1)}(\mathbb{P}^2, E) - \frac{1}{4} \mathcal{N}_{2h}^{P(1)}(\mathbb{P}^2, E) + 0 \cdot \mathcal{N}_h^{P(1)}(\mathbb{P}^2, E) = 14 + \frac{36}{16} - \frac{3}{16} = 16, \]

as conjectured. For \( E \) general and \( P(2) \), we can also calculate \( m_{4h}^{P(2)} \) directly from the quadrics meeting \( E \) only at \( P(2) \):

\[ m_{4h}^{P(2)} = 14 \cdot \text{DT}^{(11)}_1 + \text{DT}^{(5)}_2 = 14 + 2 = 16. \]

This completes the proof of Theorem 1.4, namely that Conjecture 3.4 holds for \( P^2 \) up to degree 4.

**Open Question 6.8.** For each irreducible component \( \overline{M} \subset \overline{M}^P_{\beta}(S, E) \), is its contribution to \( \mathcal{N}_P^\beta(S, E) \) given by a DT invariant of some quiver? The next case to understand is the situation of Theorem 6.6.

**References**


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