PRELOG CHOW GROUPS OF SELF-PRODUCTS OF DEGENERATIONS OF CUBIC THREEFOLDS

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Abstract. It is unknown whether smooth cubic threefolds have an (integral Chow-theoretic) decomposition of the diagonal, or whether they are stably rational or not in general. As a first step towards making progress on these questions, we compute the (saturated numerical) prelog Chow group of the self-product of a certain degeneration of cubic threefolds.

1. Introduction

A large area within the study of the birational geometry of rationally connected varieties is concerned with varieties that are *close* to projective space as in Definition 1.1. Deciding which varieties are how close to projective space is surprisingly hard and is not known in many simple cases such as the ones of Conjecture 1.2.

Definition 1.1. Let V be a variety over $\mathbb{C}$. V is said to be

- *rational* if it is birational to a projective space $\mathbb{P}^n$,
- *stably rational* if $V \times \mathbb{P}^m$ is rational for some $m \geq 0$,
- *unirational* if there is a dominant rational morphism $\mathbb{P}^n \dashrightarrow V$ for some $n \geq 0$.

Note that (rational) $\Rightarrow$ (stably rational) $\Rightarrow$ (unirational). Outside of dimension 3 (cf. [B-CT-S-S85]), there is no known invariant that separates rational varieties from strictly stably rational ones. There is however a powerful method due to Voisin [Voi15] that can be employed to show that a rationally connected variety V is not stably rational (i.e. is stably irrational). It proceeds by showing that a suitable degeneration of V (after possibly a resolution) does not admit a decomposition of the diagonal. This method has found wide applicability such as in the work of Colliot-Thélène/Pirutka [CT-P16], Totaro [To16], Schreieder [Schrei19-1, Schrei19-2] and others. Let us just mention the result in [Schrei19-2] showing that for $n \geq 3$, a very general hypersurface of degree $d \geq \lceil \log_2 n \rceil + 2$ in $\mathbb{P}^{n+1}_k$ is not stably rational over the algebraic closure of k. Here k is an uncountable field of characteristic not 2.

How sharp the bounds are remains completely open, even in low dimensions. It is possible that they are constant:

Conjecture 1.2. A very general cubic in $\mathbb{P}^{n+1}$, for $n \geq 3$, is not stably rational.

Definition 1.3. Let V be a smooth n-dimensional rationally connected projective variety over $\mathbb{C}$. We say that V admits a *Chow-theoretic decomposition of the diagonal* if one can write

$$[\Delta_V] = [V \times \mathrm{pt}] + [Z] \text{ in the Chow ring } \mathrm{CH}_n(V \times V), \tag{1}$$

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where Δ_V is the diagonal, pt is a point of V and Z is a cycle supported on $D \times V$ for some divisor D in V. Less restrictively, V is said to admit a *cohomological decomposition of the diagonal* if (1) holds in cohomology with $\mathbb{Z}$-coefficients.

Since the existence of such a decomposition is a birational invariant, finding obstructions to existence is an approach to proving stable irrationality. Note that while powerful, this is also a rather subtle invariant. Indeed, if V is a unirational variety with unirational parametrization $\mathbb{P}^n \dashrightarrow V$ of degree N, then there is a decomposition

$$N\,[\Delta_V] = N\,[V \times \mathrm{pt}] + [Z] \text{ in } \mathrm{CH}_n(V \times V), \tag{2}$$

with Z supported on $D \times V$ for D a divisor in V. In fact, (2) holds more generally for rationally connected varieties.

Let now V be a smooth cubic threefold. Denote by $J(V)$ the intermediate Jacobian of V and let $\theta \in \mathrm{H}^2(\mathrm{J(V)}, \mathbb{Z})$ be the class of the theta divisor of $J(V)$. $(J(V), \theta)$ is a principally polarized abelian variety. Clemens-Griffiths in [CG72] prove that V is irrational using that $(J(V), \theta)$ is not a product of Jacobians of curves.

Voisin in [Voi13, Voi17] investigated the existence of a decomposition of the diagonal on smooth cubic threefolds:

Theorem 1.4 (Voisin, Theorem 1.7 in [Voi17])**.** *The smooth cubic threefold V admits a decomposition of the diagonal if and only if the class $\theta^4/4!$ on $J(V)$ is algebraic. On the moduli space of smooth cubic threefolds, this algebraicity is satisfied (at least) on a countable union of closed subvarieties of codimension* ≤ 3.

In Theorem 1.1 in [Voi17] Voisin also proves that V admits a Chow-theoretic decomposition of the diagonal if and only if it admits a cohomological decomposition of the diagonal.

In this article, we consider a degeneration of a very general cubic threefold into the union of a hyperplane and a quadric in $\mathbb{P}^4$. In our previous article [BBG19-1] we showed that a decomposition of the diagonal on the geometric generic fibre has a specialization to the saturated prelog Chow group of the central fibre of a strictly semistable modification of the product family. In this article we compute a natural quotient of this saturated prelog Chow group constructed using numerical equivalence. Our main result is

Theorem 6.15. $\mathrm{Num}_{\mathrm{sat,prelog},3}(Y) \simeq \mathbb{Z}^6$ and is generated by the classes in Theorem 6.14 and a half of their sum.

2. Recollection of prelog Chow rings and saturated prelog Chow groups

We work over the complex numbers $\mathbb{C}$ throughout. In this Section we give a summary of those concepts developed in [BBG19-1] that are used in this paper.

Let X be a simple normal crossing variety with irreducible components X_i and normalization $\nu\colon X^\nu \to X$. Let $X_{ij} = X_i \cap X_j$ and define the following ring:

$$R(X) := \Big\{ (\alpha_i) \in \bigoplus_i \mathrm{Num}^*(X_i) \,\Big|\, \forall\, i, j :\, \alpha_i \mid_{X_{ij}} = \alpha_j \mid_{X_{ij}} \text{ in } \mathrm{Num}^*(X_{ij}) \Big\}.$$

Here we simply write a restriction symbol to denote pull-backs to X_{ij}. We call the condition $\alpha_i \mid_{X_{ij}} = \alpha_j \mid_{X_{ij}}$ the *prelog condition*, and the above ring the *ring of compatible classes*.

Definition 2.1. Let X be a snc variety with at worst triple intersections. We say that X satisfies the *Friedman condition* if for every intersection X_{ij} we have

$$\mathscr{N}_{X_{ij}/X_i} \otimes \mathscr{N}_{X_{ij}/X_j} \otimes \mathscr{O}(T) = \mathscr{O}_{X_{ij}}.$$

Here T is the union of all triple intersections X_{ijk} that are contained in X_{ij}.

Definition 2.2. Let X be an snc variety that has at worst triple intersections and satisfies the Friedman condition. Then we define $\mathrm{Num}^*_{\mathrm{prelog}}(X)$ via the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & R(X) & \longrightarrow & \mathrm{Num}^*_{\mathrm{prelog}}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\bigoplus \mathrm{Num}^*(X_{ij}) & \xrightarrow{\delta} & \bigoplus \mathrm{Num}^*(X_i) & \longrightarrow & \mathrm{coker}(\delta) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & & & \\
\bigoplus \mathrm{Num}^*(X_{ijk}) & \xrightarrow{\delta'} & \bigoplus \mathrm{Num}^*(X_{ij}) & & & &
\end{array}$$

Here the maps $\rho, \rho', \delta, \delta'$ are defined as follows, using the convention $a < b < c$, $i < j < k$:

$$\big(\delta(z_{ij})\big)_a = \begin{cases} \iota_{\{ij\}>\{i\}*}(z_{ij}) & \text{if } a = i, \\ -\iota_{\{ij\}>\{j\}*}(z_{ij}) & \text{if } a = j, \\ 0 & \text{otherwise.} \end{cases}$$

$$\big(\rho(z_i)\big)_{ab} = \begin{cases} \iota^*_{\{ab\}>\{i\}}(z_i) & \text{if } i = a \\ -\iota^*_{\{ab\}>\{i\}}(z_i) & \text{if } i = b \\ 0 & \text{otherwise} \end{cases}$$

$$\big(\rho'(z_{ij})\big)_{abc} = \begin{cases} \iota^*_{\{abc\}>\{ij\}}(z_{ij}) & \text{if } (i,j) = (a,b) \\ -\iota^*_{\{abc\}>\{ij\}}(z_{ij}) & \text{if } (i,j) = (a,c) \\ \iota^*_{\{abc\}>\{ij\}}(z_{ij}) & \text{if } (i,j) = (b,c) \\ 0 & \text{otherwise} \end{cases}$$

$$\big(\delta'(z_{ijk})\big)_{ab} = \begin{cases} -\iota_{\{ijk\}>\{ab\}*}(z_{ijk}) & \text{if } (a,b) = (i,j) \\ \iota_{\{ijk\}>\{ab\}*}(z_{ijk}) & \text{if } (a,b) = (i,k) \\ -\iota_{\{ijk\}>\{ab\}*}(z_{ijk}) & \text{if } (a,b) = (j,k) \\ 0 & \text{otherwise} \end{cases}$$

Notice that being in the kernel of ρ amounts to the prelog condition. The fact that the lower left hand square commutes is proven in [BBG19-1, Prop. 2.8].

We define the *saturated numerical prelog Chow group* $\mathrm{Num}_{\mathrm{prelog,sat}}{}^*(X)$ as the saturation of $\mathrm{Num}^*_{\mathrm{prelog}}(X)$ in the lattice $\mathrm{coker}(\delta)/(torsion)$. Notice that this is the same definition as in [BBG19-1, Def. 4.5].

Proposition 2.3. *The map δ in the above diagram is an $R(X)$-module homomorphism, hence $\mathrm{Num}^*_{\mathrm{prelog}}(X)$ is naturally a quotient ring of $R(X)$ and coincides with the numerical prelog Chow ring defined in [BBG19-1, Def. 2.9], up to torsion.*

Proof. Let $Y_1, Y_2, Y_1 \cap Y_2 =: Y_{12}$ be smooth varieties and $\dim Y_1 = \dim Y_2 = \dim Y_{12} + 1$. Let furthermore

$$\iota_i \colon Y_{12} \to Y_i$$

be the inclusions. Then for classes $z_i \in \mathrm{Num}^*(Y_i)$ and $y \in \mathrm{Num}^*(Y_{12})$ we have

$$z_i.(\iota_{i*}(y)) = \iota_{i*}(\iota_i^*(z_i).y)$$

Consider now

$$\delta \colon \mathrm{Num}^*(Y_{12}) \to \mathrm{Num}^*(Y_1) \oplus \mathrm{Num}^*(Y_2).$$

with $\delta(y) = \big(\iota_{1*}(y), -\iota_{2*}(y)\big)$.

If (z_1, z_2) is a compatible class, i.e

$$\iota_1^*(z_1) = \iota_2^*(z_2) =: r$$

Then

$$\begin{aligned}(z_1, z_2).\delta(y) &= \big(\iota_{1*}(y), -\iota_{2*}(y)\big).(z_1, z_2)\\ &= \big(z_1.\iota_{1*}(y), -z_2.\iota_{2*}(y)\big)\\ &= \big(\iota_{1*}(\iota_1^*(z_1).y), -\iota_{2*}(\iota_2^*(z_2).y)\big)\\ &= \big(\iota_{1*}(r.y), -\iota_{2*}(r.y)\big)\\ &= \delta(r.y)\end{aligned}$$

This proves that the image of

$$\delta \colon \bigoplus \mathrm{Num}(Y_{ij}) \to \bigoplus \mathrm{Num}(Y_i).$$

is an R-Module. □

Given a strictly semistable degeneration $\pi \colon \mathscr{X} \to C$ (strictly semistable=total space smooth+ central fibre reduced simple normal crossing) over some curve with marked point t_0 and $X \simeq \mathscr{X}_{t_0}$, the specialization homomorphism induces a natural homomorphism

$$\sigma_{\mathscr{X}} \colon \mathrm{CH}_*(X_K) \to \mathrm{Num}^*_{\mathrm{prelog}}(X)$$

(where X_K is the generic fibre). This follows from [BBG19-1, Thm. 3.2] since $\mathrm{Num}^*_{\mathrm{prelog}}(X)$ is a natural quotient of $\mathrm{Chow}^*_{\mathrm{prelog}}(X)$ in that paper.

If we consider a cover $C' \to C$ of smooth curves branched at t_0, the specialization homomorphism $\sigma_{\mathscr{X}'}$ of the pull-back family $\mathscr{X}' = \mathscr{X} \times_C C' \to C'$ (where we fix a distinguished point t_0' in C' mapping to t_0) gives a homomorphism into ${\mathrm{Num}_{\mathrm{prelog,sat}}}^*(X)$ by [BBG19-1, Prop. 4.2].

3. Recollection of some formulas for Chow groups

We now recall a few formulas for Chow rings of projective bundles and blow ups needed in the sequel.

Proposition 3.1. *Let X be a smooth projective variety and $\mathscr{E}$ a vector bundle of rank $r+1$ on X. Let $\pi \colon \mathbb{P}(\mathscr{E}) \to X$ be the associated projective bundle, and let ζ be the first Chern class of the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ in $\mathrm{CH}^1(\mathbb{P}(\mathscr{E}))$. Then as rings*

$$\mathrm{CH}^*(\mathbb{P}(\mathscr{E})) \simeq \mathrm{CH}^*(X)[\zeta]/(\zeta^{r+1} + c_1(\mathscr{E})\zeta^r + \cdots + c_{r+1}(\mathscr{E}))$$

where $c_i(\mathscr{E})$ are the Chern classes of $\mathscr{E}$ and $\mathrm{CH}^*(X)$ is considered as a subring of $\mathrm{CH}^*(\mathbb{P}(\mathscr{E}))$ via the injective map $\pi^* \colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(\mathbb{P}(\mathscr{E}))$.

Proof. This is [E-H16, Thm. 9.6]. □

Proposition 3.2. *Let X be a smooth projective variety, $Z \subset X$ a smooth subvariety. Let $\pi\colon \mathrm{Bl}_Z X \to X$ be the blow up, $\mathscr{N}_{Z/X}$ the normal bundle of Z in X, $E = \mathbb{P}(\mathscr{N}_{Z/X}) \subset W$ the exceptional divisor, and i, j the natural inclusions as in the following diagram:*

$$\begin{array}{ccc} E & \xrightarrow{\ j\ } & \mathrm{Bl}_Z X \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\ i\ } & X \end{array}$$

Let ζ be the first Chern class of the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{N}_{Z/X})}(1)$ in $\mathrm{CH}^1(E)$. Then $\mathrm{CH}^*(\mathrm{Bl}_Z X)$ is generated by $\pi^* \mathrm{CH}^*(X)$ and $j_* \mathrm{CH}^*(E)$; more precisely, there is an exact sequence

$$0 \longrightarrow \mathrm{CH}^*(Z) \xrightarrow{\ \varphi\ } \mathrm{CH}^*(E) \oplus \mathrm{CH}^*(X) \xrightarrow{\ \psi\ } \mathrm{CH}^*(\mathrm{Bl}_Z X) \longrightarrow 0$$

where

$$\varphi(z) = (c_{m-1}(\mathscr{Q})\pi_E^*(z), -i_*(z)),$$
$$\psi(\gamma, \alpha) = j_*(\gamma) + \pi^*(\alpha).$$

and m is the codimension of Z in X, $\mathscr{Q}$ is the universal quotient bundle on $E \simeq \mathbb{P}(\mathscr{N}_{Z/X})$. Moreover,

$$c_{m-1}(\mathscr{Q}) = \zeta^{m-1} + c_1(\mathscr{N})\zeta^{m-2} + \cdots + c_{m-1}(\mathscr{N}).$$

There are the following rules for multiplication:

$$\begin{aligned} \pi^*(\alpha) \cdot \pi^*(\beta) &= \pi^*(\alpha\beta) && \text{for } \alpha, \beta \in \mathrm{CH}^*(X) \\ \pi^*(\alpha) \cdot j_*(\gamma) &= j_*(\pi_E^* i^* \alpha \cdot \gamma) && \text{for } \alpha \in \mathrm{CH}^*(X), \gamma \in \mathrm{CH}^*(E) \\ j_*\gamma \cdot j_*\delta &= -j_*(\gamma \cdot \delta \cdot \zeta) && \text{for } \gamma, \delta \in \mathrm{CH}^*(E) \end{aligned}$$

Proof. This is [E-H16, Prop. 13.12, Thm. 13.14] and [Ful98, Prop. 6.7]. □

Sometimes one can find a more economical set of generators for the Chow ring of a blowup:

Proposition 3.3. *In the situation of Proposition 3.2 the ring $\mathrm{CH}^*(X)$ is generated by $\pi^* \mathrm{CH}^*(X)$ and $j_*\pi_E^* \mathrm{CH}^*(Z)$.*

Proof. $j_* \mathrm{CH}^*(E)$ is generated as a $\mathbb{Z}$-module by elements of the form $j_*(\zeta^c \pi_E^* z)$. If $c \geq 1$ we have, by the last line of Proposition 3.2, the following relation:

$$(j_*E)^c \cdot j_*(\pi_E^* z) = (-1)^c j_*(\zeta^c \pi_E^* z).$$

We can therefore express all generators with $c \geq 1$ by such with $c = 0$. □

Proposition 3.4. *In the situation of Proposition 3.2 let I the ideal generated by the image of i^* in $\mathrm{CH}^*(Z)$ and*

$$\mathscr{Z} \subset \mathrm{CH}^*(Z)$$

a set of elements whose images generate $\mathrm{CH}^*(Z)/I$ as a $\mathbb{Z}$-module. Then $\mathrm{CH}^*(X)$ is generated as a ring by $\pi^* \mathrm{CH}^*(X)$ and elements $j_*\pi_E^* z$, $z \in \mathscr{Z}$.

Proof. By Proposition 3.3, we only need to show that elements of the form $j_*\pi_E^*(y)$ with y in the ideal I are in the subring generated by $\pi^* \operatorname{CH}^*(X)$ and elements $j_*\pi_E^* z$, $z \in \mathscr{Z}$. Every such element y is a sum of elements of the form $i^*(x) \cdot z$ with $z \in \operatorname{CH}^*(Z)$. By induction on the codimension of z and using that i^* is a ring homomorphism, one sees that y is also a sum of elements of the form $i^*(x) \cdot z'$ with $z' \in \mathscr{Z}$. But

$$j_*\Big(\pi_E^*\big(i^*(x) \cdot z'\big)\Big) = j_*\Big(\pi_E^*\big(i^*(x)\big) \cdot \pi_E^*\big(z'\big)\Big) = \pi^*(x) \cdot j_*\pi_E^*\big(z'\big).$$

□

We also need a few facts about Chow groups and rings of products. The first result says when a Künneth formula can be expected to hold.

Proposition 3.5. *Let X be a linear variety. By this we mean a variety in the class of varieties constructed by an inductive procedure starting with an affine space of any dimension, in such a way that the complement of a linear variety imbedded in affine space in any way is a linear variety, and a variety stratified into a finite disjoint union of linear varieties is a linear variety. Then for any variety Y*

$$\operatorname{CH}_*(X \times Y) \simeq \operatorname{CH}_*(X) \otimes_{\mathbb{Z}} \operatorname{CH}_*(Y).$$

Proof. This is [To14, Prop. 1]. □

The next concerns Chow groups, modulo numerical equivalence, for self-products of very general curves of genus ≥ 1.

Proposition 3.6. *Let C be a smooth projective curve that is very general in a linear system $|L|$ on a surface S with trivial Albanese variety $\operatorname{Alb}(S)$, hence a regular surface. Then $\operatorname{Jac}(C)$ is a simple abelian variety. For C of positive genus, the Neron-Severi group of $C \times C$, which is the Chow group of 1-cycles modulo numerical equivalence, is generated by the class of the diagonal $[\Delta_C]$ and by $[C \times \{p\}]$ and $[\{p\} \times C]$ where $p \in C$ is a point.*

Proof. The first assertion is in [CG92]. Now, as is well known, for smooth curves C, D

$$\operatorname{Pic}(C \times D) \simeq \operatorname{Pic}^0(C) \times \mathbb{Z} \times \operatorname{Pic}^0 D \times \mathbb{Z} \times \operatorname{Hom}(\operatorname{Jac} C, \operatorname{Jac} D)$$

(after tensoring by line bundles pulled back from the factors, we can think of a line bundle on the product as a family of degree 0 line bundles on D parametrised by C and trivialised along some fixed slice $\{p\} \times D$; these are classified by morphisms $C \to \operatorname{Jac}(D)$ since the Jacobian is isomorphic to the Pic^0 of D, but since the Jacobian is also the Albanese variety of a curve, such morphism are the same thing as morphisms $\operatorname{Jac}(C) \to \operatorname{Jac}(D)$ by the universal property of the Albanese). Hence

$$\operatorname{CH}^1_{\mathrm{num}}(C \times C) \simeq \mathbb{Z}^2 \times \operatorname{End}(\operatorname{Jac} C),$$

and $\operatorname{End}(\operatorname{Jac} C)$ is not a point for C of positive genus and generated by the diagonal correspondence since $\operatorname{Jac}(C)$ is simple, thus the second assertion follows. □

4. Numerical Chow rings via dual socle generators

From now on we work with Chow rings modulo numerical equivalence, and denote these by $\operatorname{Num}^*(X)$ for a smooth projective variety X. We would like to be able to write these numerical Chow rings, which are Artin rings, in more compact and computationally convenient form. For this we briefly recall some facts about zero-dimensional Gorenstein rings from [Ei04, Section 21.2], partly to set up notation.

Let k be a field (later we will work with a subring, too, for us $k = \mathbb{Q}$ and the subring will be $\mathbb{Z}$), and let

$$R = k[x_1, \dots, x_r], \quad R^* = k[x_1^{-1}, \dots, x_r^{-1}]$$

the polynomial rings in variables x_i and their inverses, respectively, both considered as subrings of $K = k(x_1, \dots, x_r)$. We make R^* into an R-module by decreeing that for monomials $m \in R$ and $n \in R^*$, $m \cdot n$ is to be the product $mn \in K$ if this lies in the subring R^*, and zero otherwise. Now [Ei04, Thm. 21.6] says that the ideals $I \subset (x_1, \dots, x_r)$ such that R/I is a local zero-dimensional Gorenstein ring are precisely the ideals of the form $I = \mathrm{Ann}_R(f)$ for some nonzero element $f \in R^*$. Here f is called the dual socle generator of R/I.

Notation 4.1. Let X be a smooth projective variety of dimension d and let $\mathrm{Num}^*(X)$ and $\mathrm{Num}^*(X)_{\mathbb{Q}}$ be its Chow ring of cycles modulo numerical equivalence with $\mathbb{Z}$ and $\mathbb{Q}$ coefficients, respectively. Let $x_1, \dots, x_r$ be variables corresponding to homogeneous generators $\alpha_1, \dots, \alpha_r$ of $\mathrm{Num}^*(X)$. We use multi-index notation and write

$$\begin{aligned} a =&(a_1, \dots, a_r) \in \mathbb{N}^r \\ |a| =& \sum_{i=1}^{r} \deg(x_i) a_i \\ \alpha^a =& \alpha_1^{a_1} \cdot \dots \cdot \alpha_r^{a_r} \in \mathrm{Num}^*(X) \\ x^{-a} =& x_1^{-a_1} \cdot \dots \cdot x_r^{-a_r} \in \mathbb{Z}[x_1^{-1}, \dots, x_r^{-1}]. \end{aligned}$$

Lemma 4.2. *With the previous notation, let*

$$f_X = \sum_{|a|=d} \deg(\alpha^a) \cdot x^{-a} \in \mathbb{Q}[x_1^{-1}, \dots, x_r^{-1}].$$

Then

$$\mathrm{Num}^*(X) = \mathbb{Z}[x_1, \dots, x_r]/\mathrm{Ann}(f_X), \quad \mathrm{Num}^*(X)_{\mathbb{Q}} = \mathbb{Q}[x_1, \dots, x_r]/\mathrm{Ann}(f_X).$$

In particular, $\mathrm{Num}^*(X)_{\mathbb{Q}}$ *is a Gorenstein local ring with dual socle generator* f_X*.*

Proof. Suppose a homogenoeus class α of degree δ that can be written as some polynomial $p(\alpha_1, \dots, \alpha_r)$ is numerically trivial. We need to prove that then the corresponding $p = p(x_1, \dots, x_r)$ is in $\mathrm{Ann}(f_X)$, and conversely, if p is in $\mathrm{Ann}(f_X)$, then $p(\alpha_1, \dots, \alpha_r)$ is numerically trivial. Let us start with the first part. We put $R = \mathbb{Z}[x_1, \dots, x_r]$, $R^* = \mathbb{Z}[x_1^{-1}, \dots, x_r - 1]$, which are graded rings. Since α is numerically trivial, in $\mathrm{Num}^*(X)$, $\alpha \cdot \beta = 0$ for every $\beta \in \mathrm{Num}^{d-\delta}(X)$. Now every polynomial q in $x_1, \dots, x_r$ such that $q(\alpha_1, \dots, \alpha_r) = \alpha \cdot \beta$ annihilates f_X: indeed, $q \cdot f_X = \deg(\alpha \cdot \beta)$ by definition of the pairing and f_X. Hence, $p \cdot R_{d-\delta}$ annihilates f_X. Then we have to have $p \cdot f_X = 0$: indeed, all of $R_{d-\delta}$ annihilates $p \cdot f_X$, and the pairing $R_{d-\delta} \times R^*_{-(d-\delta)} \to \mathbb{Z}$ is perfect (or, becomes perfect over $\mathbb{Q}$).

For the converse, suppose that p is in $\mathrm{Ann}(f_X)$, we have to show $p(\alpha_1, \dots, \alpha_r)$ is numerically trivial. This is basically the same line of argument: all of $p \cdot R_{d-\delta}$ annihilates f_X, which just translates into the statement that $\alpha \cdot \beta = 0$ for every $\beta \in \mathrm{Num}^{d-\delta}(X)$ has degree 0, hence is numerically trivial. □

5. The case of cubic threefolds

Consider a degeneration $\pi_{\mathscr{V}}\colon \mathscr{V} \to B$ of smooth cubic threefolds, over $B = \mathbb{A}^1$, into the union of a smooth quadric and a hyperplane in $\mathbb{P}^4$, given by an equation

$$\{lq - tf = 0\} \subset \mathbb{P}^4 \times \mathbb{A}^1$$

where $l, q, f \in \mathbb{C}[X_0, \dots, X_4]$ are homogeneous of degree $1, 2, 3$, respectively, and

a) q defines a nonsingular quadric Q;
b) f is general, in particular $f = 0$ defines a smooth cubic threefold V;
c) the hyperplane L defined by l in $\mathbb{P}^4$ intersects Q transversely in a smooth quadric surface $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$;
d) $S \cap V$ is a smooth divisor C of bidegree $(3, 3)$, which is a genus 4 canonical curve in $L \simeq \mathbb{P}^3$, general if f is general.

Notice that the total space $\mathscr{V}$ is singular in C.

We blow up the non-Cartier divisor L in the total space $\mathscr{V}$ and get a strictly semistable degeneration $\pi_{\mathscr{X}}\colon \mathscr{X} \to B$ with central fibre

$$X = L_C \cup Q.$$

Here, by slight abuse of notation, we denote by Q the irreducible component of X mapping isomorphically to Q in $\mathscr{V}$ under the natural morphism $\mathscr{X} \to \mathscr{V}$. L_C is the blowup of L in C with exceptional divisor E_C. L_C and Q intersect in a surface which is naturally isomorphic to S: in Q we have the previous copy of S, and in L_C the strict transform of S. Hence we denote this new surface by S as well.

Lemma 5.1. *Let H be the pullback of a hyperplane class in $\mathbb{P}^3$, $E = \mathbb{P}(\mathscr{N}_{C/\mathbb{P}^3})$ the class of the exceptional divisor in L_C and $F := \pi_E^*(P)$ with P a point on C (the class of a fiber of E). Then*

$$\operatorname{Num}^*(L_C) = \mathbb{Z}[H, E, F]/\operatorname{Ann}(f_{L_C})$$

with

$$f_{L_C} = H^{-3} - 6H^{-1}E^{-2} - 30E^{-3} - E^{-1}F^{-1}.$$

Proof. The Chow ring of $\mathbb{P}^3$ blown up in a smooth curve C (for rational equivalence) is calculated in [E-H16, Prop. 13.13]. From this it follows that H, E, F are ring generators of $\operatorname{Num}^*(L_C)$ and the intersection numbers in the dual socle generator as defined in Lemma 4.2 are the ones given above. □

Lemma 5.2. *Let S be the class of a hyperplane section of Q and let L be the class of a line in Q. Then*

$$\operatorname{Num}^*(Q) = \mathbb{Z}[S, L]/\operatorname{Ann}(f_Q)$$

with

$$f_Q = 2S^{-3} + S^{-1}L^{-1}.$$

Proof. Apply [Ful98, Example 1.9.1]. □

Lemma 5.3. *Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ as above. Let R_1, R_2 be the classes of the two rulings. Then*

$$\operatorname{Num}^*(S) = \mathbb{Z}[R_1, R_2]/\operatorname{Ann}(f_S)$$

with

$$f_S = R_1^{-1}R_2^{-1}.$$

Proof. This is obvious. □

Lemma 5.4. *Let* $\iota_{S,L_C}\colon S \to L_C$ *and* $\iota_{S,Q}\colon S \to Q$ *be the natural inclusions. Then*

$$\iota^*_{S,L_C}(H,E,F) = \big(R_1+R_2, 3(R_1+R_2), R_1R_2\big)$$
$$\iota^*_{S,Q}(S,L) = \big(R_1+R_2, R_1R_2\big).$$

and

$$(\iota_{S,L_C})_*(1,R_1,R_2,R_1R_2) = \big(2H-E, H^2-3F, H^2-3F, H^3\big)$$
$$(\iota_{S,Q})_*(1,R_1,R_2,R_1R_2) = \big(S,L,L,SL\big).$$

Proof. Only the third formula is not obvious. For the third formula, remark that the class of the strict transform of S in L_C is given by $2H-E$, and that the lines of the rulings on S are trisecants to the bidegree $(3,3)$ curve C, hence give classes H^2-3F as claimed. □

6. The product family

Consider the product family $\mathscr{X}\times_B\mathscr{X}\to B$. The total space is singular in a variety isomorphic to $S\times S$ contained in the central fibre as the locus where all four irreducible components $L_C\times L_C$, $L_C\times Q$, $Q\times L_C$, $Q\times Q$ intersect. We now blow up $L_C\times Q$ in the total space and obtain a strictly semistable degeneration $\pi_{\mathscr{Y}}\colon\mathscr{Y}\to B$ with components of the central fibre Y given by

$$\begin{aligned} Y_1 =& L_C\times L_C,\\ Y_2 =& \mathrm{Bl}_{S\times S}\left(L_C\times Q\right),\\ Y_3 =& \mathrm{Bl}_{S\times S}\left(Q\times L_C\right),\\ Y_4 =& Q\times Q. \end{aligned}$$

Compare Figure 1.

The mutual intersections $Y_{ij}=Y_i\cap Y_j$ of these components are

$$\begin{aligned} Y_{12} =& L_C\times S,\\ Y_{13} =& S\times L_C\\ Y_{23} =& \mathbb{P}\left(\mathscr{N}_{(S\times S)/(L_C\times Q)}\right)\simeq\mathbb{P}\left(\mathscr{N}_{(S\times S)/(Q\times L_C)}\right)\\ Y_{24} =& S\times Q\\ Y_{34} =& Q\times S. \end{aligned}$$

The inclusions $\iota_{\{i,j\},\{k\}}\colon Y_{ij}\to Y_k$, for $k=1,4$, are given componentwise by the inclusions described at the start of Section 5. The inclusions of Y_{23} into Y_2 and Y_3 are the inclusions of the exceptional divisors of the respective blow-ups. The inclusion of Y_{13} into Y_3 is obtained as follows: one has the inclusions

$$S\times S\subset S\times L_C\subset Q\times L_C.$$

Thus we see that, blowing up the $S\times S$ in $Q\times L_C$, the strict transform of $S\times L_C$ is isomorphic to $S\times L_C$. This isomorphism composed with the inclusion into Y_3 gives $\iota_{\{1,3\},\{3\}}$. Similarly for the remaining cases.

The triple intersections are

$$Y_{123}=Y_{234}=S\times S.$$

The inclusions $\iota_{\{1,2,3\},\{i,j\}}$ are clear unless $\{i,j\}=\{2,3\}$ in which case we deal with it in Proposition 6.10.

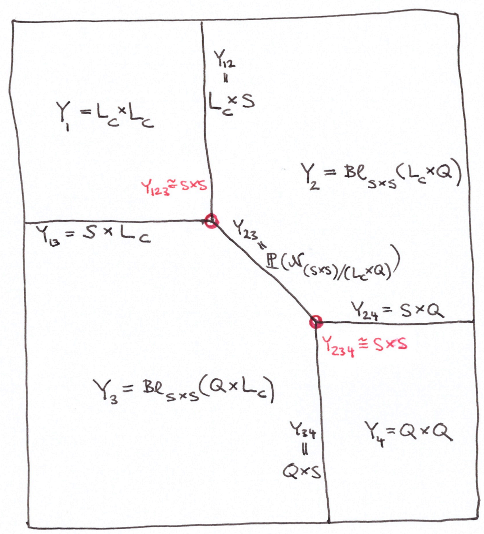

FIGURE 1.

We seek to compute the saturated prelog Chow group of the central fibre Y of $\mathscr{Y} \to B$.

6.1. **The numerical Chow ring of Y_1.** For this we begin with $\mathrm{Num}^*(Y_1)$. Notice that $Y_1 = L_C \times L_C$ is the blowup of $L_C \times \mathbb{P}^3$ in $L_C \times C$, and that furthermore $L_C \times C$ is the blowup of $\mathbb{P}^3 \times C$ in $C \times C$. This gives the following combined blow up diagram:

$$\begin{array}{ccccc}
 & & E_{L_C \times C} & \overset{j'}{\hookrightarrow} & L_C \times L_C \\
 & & \downarrow \pi'_E & & \downarrow \pi' \\
E_{C \times C} & \overset{j}{\hookrightarrow} & L_C \times C & \overset{i'}{\hookrightarrow} & L_C \times \mathbb{P}^3 \\
\downarrow \pi_E & & \downarrow \pi & & \\
C \times C & \overset{i}{\hookrightarrow} & \mathbb{P}^3 \times C & &
\end{array}$$

Lemma 6.1. *We have*

$$\mathrm{Num}^*(C \times C) = \mathbb{Z}[p, P, \Delta_C]/\mathrm{Ann}(f_{C \times C})$$

where p is the pullback of the class of a point on C via the first projection, P the same via the second projection, and Δ_C the class of the diagonal and

$$f_{C \times C} = p^{-1}P^{-1} + p^{-1}\Delta_C^{-1} + \Delta_C^{-1}P^{-1} - 6\Delta_C^{-2}.$$

Proof. We use Proposition 3.6 and the fact that C has topological Euler characteristic -6. $\square$

Proposition 6.2. *$\mathrm{Num}^*(L_C \times C)$ is generated as a ring by the elements*

$$\{h, e, f, \bar{P}, \overline{\Delta}_C\},$$

where h, e, f are the classes coming from L_C and $\bar{P}$ the point class coming from the second factor of $L_C \times C$. Furthermore $\overline{\Delta}_C = j_*\pi_E^*(\Delta_C)$ is the $\mathbb{P}^1$ bundle over Δ_C in $E_{C\times C}$.

Proof. Let $I \subset \mathrm{Num}^*(C \times C)$ the ideal generated by the image of i^*. This ideal contains $P = i^*(\bar{P})$. Since

$$\mathrm{Num}^*(C \times C)/(P)$$

is generated by $\{1, p, \Delta_C, p\Delta_C\}$ as a module and $p\Delta_C = P\Delta_C$ is also in I, we have that $\{1, p, \Delta_C\}$ generates $\mathrm{Num}^*(C \times C)/I$ as a $\mathbb{Z}$-module. Noticing that

$$j_*\pi_E^*(1) = e,$$
$$j_*\pi_E^*(p) = f,$$
$$j_*\pi_E^*(\Delta_C) = \overline{\Delta}_C$$

and that $\mathrm{Num}^*(\mathbb{P}^3 \times C)$ is generated by $\{h, \bar{P}\}$, Proposition 3.4 then gives the claim. □

Proposition 6.3. *We have that*

$$\mathrm{Num}^*(L_C \times L_C)$$

is generated as a ring by

$$\{h, e, f, H, E, F, D\},$$

where h, e, f, H, E, F are the classes as in Lemma 5.1 coming from the two factors, and $D := j'_*(\pi'_E)^* j_*\pi_E^*\Delta_C$ is the class is the class of the $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over the diagonal in $C \times C$.

Proof. Let $I' \subset \mathrm{Num}^*(L_C \times C)$ be the ideal generated by the image of $(i')^*$. This ideal contains e,f and h. Since

$$\mathrm{Num}^*(L_C \times C)/(e, f, h)$$

is generated by $\{1, \bar{P}, \overline{\Delta}_C, \bar{P}\overline{\Delta}_C\}$ as a $\mathbb{Z}$-Module, and

$$\bar{P}\overline{\Delta}_C = \bar{P}j_*\pi_E^*(\Delta_C) = j_*\pi_E^*(P\Delta_C) = j_*\pi_E^*(Pp) = \bar{P}j_*\pi_E^*(p) = \bar{P}f$$

is also in I, we have that $\{1, \bar{P}, \overline{\Delta}_C\}$ generates $\mathrm{Num}(L_C \times C)/I$ as a $\mathbb{Z}$-module. Noticing that

$$j'_*(\pi'_E)^*(1) = E$$
$$j'_*(\pi'_E)^*(\bar{P}) = F$$
$$j'_*(\pi'_E)^*(\overline{\Delta}_C) = j'_*(\pi'_E)^* j_*\pi_E^*(\Delta_C) = D$$

and that $\mathrm{Num}^*(L_C \times \mathbb{P}^3)$ is generated by $\{h, e, f, H\}$, the claim follows from Proposition 3.4. □

Proposition 6.4. *With the notation of the previous Proposition we have*

$$\mathrm{Num}^*(Y_1) = \mathrm{Num}^*(L_C \times L_C) = \mathbb{Z}[h, e, f, H, E, F, D]/\mathrm{Ann}(f_{L_C\times L_C})$$

where

$$\begin{aligned} f_{L_C\times L_C} =& f_{L_C} \cdot f'_{L_C} \\ &+ D(30e^{-2}E^{-1} + 30e^{-1}E^{-2} + 6e^{-1}E^{-1}h^{-1} + 6e^{-1}E^{-1}H^{-1} + E^{-1}f^{-1} + e^{-1}F^{-1}) - 6D^{-2}. \end{aligned}$$

Proof. We have to calculate the intersection numbers of all monomials of degree 6 in the generators. Here $\{H, E, h, e\}$ have degree 1, $\{F, f\}$ have degree 2 and D has degree 3. We can write every such polynomial as mMD^c where m is a monomial in the generators of $\mathrm{Num}^*(L_C)$ of the first factor (lower case letters) and M is a monomial in the generators of $\mathrm{Num}^*(L_C)$ of the second factor (upper case letters). We then have the following cases

$c = 0$: Here the monomial is mM and the intersection number of the product is the product of the intersection numbers on the first and second L_C respectively. These intersection numbers are calculated by the dual socle generators of the factors and we obtain the summand

$$f_{L_C} \cdot f'_{L_C}$$

of $f_{L_C \times L_C}$.

$c = 1$: We can calculate these intersection numbers on D. We recall that D is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle on C, where we identify C with the diagonal of $C \times C$. The intersection ring $\mathrm{Num}^*(D)$ is generated by the pullback of a point P on D, and the relative hyperplane class γ and Γ of the first and second factor. We have $\gamma^2 = -30\gamma P$ and $\Gamma^2 = -30\Gamma P$. Let i be the inclusion of D in $L_C \times L_C$. The pullbacks of the generators to D are

$$\begin{aligned} i^*(h) &= 6P & \qquad i^*(H) &= 6P \\ i^*(e) &= -\gamma & \qquad i^*(E) &= -\Gamma \\ i^*(f) &= -\gamma P & \qquad i^*(F) &= -\Gamma P \end{aligned}$$

The class of a point in D is $\gamma\Gamma P$ so the non zero intersection numbers are collected in the summand

$$D(30e^{-2}E^{-1} + 30e^{-1}E^{-2} + 6e^{-1}E^{-1}h^{-1} + 6e^{-1}E^{-1}H^{-1} + E^{-1}f^{-1} + e^{-1}F^{-1}).$$

$c = 2$: We compute

$$\begin{aligned} D^2 &= \big(j'_*(\pi'_E)^* j_* \pi_E^* \Delta_C\big)^2 \\ &= -j'_*\Big(\Gamma\big((\pi'_E)^* j_* \pi_E^* \Delta_C\big)^2\Big) \\ &= -j'_*\Big(\Gamma(\pi'_E)^* \big(j_* \pi_E^* \Delta_C\big)^2\Big) \\ &= j'_*\Big(\Gamma(\pi'_E)^* j_* \big(\gamma(\pi_E^* \Delta_C)^2\big)\Big) \\ &= j'_*\Big(\Gamma(\pi'_E)^* j_* \big(\gamma \pi_E^*(\Delta_C^2)\big)\Big) \end{aligned}$$

Since $\Delta_C^2 = -6$ this proves that also $D^2 = -6$.

□

6.2. **The numerical Chow ring of Y_2.** Next we turn to Y_2, the blow up of $L_C \times Q$ in $S \times S$, as in the blowup diagram

$$\begin{array}{ccc} N & \xrightarrow{\;j\;} & Y_2 \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi} \\ S \times S & \xrightarrow{\;i\;} & L_C \times Q. \end{array}$$

Lemma 6.5. *We have*

$$\mathrm{Num}^*(N) = \mathrm{Num}^*(S \times S)[\xi]/\big((\xi - r_1 - r_2)(\xi + R_1 + R_2)\big).$$

Here ξ is the relative hyperplane class of the projectivisation of $\mathscr{N}_{(S\times S)/(L_C\times Q)}$, which is naturally isomorphic to N.

Proof. We have

$$\mathscr{N}_{(S\times S)/(L_C\times Q)} \simeq \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(-1,-1,0,0) \oplus \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(0,0,1,1).$$

The assertion then follows from Proposition 3.1. □

Proposition 6.6. *We have*

$$\mathrm{Num}^*(Y_2) = \mathbb{Z}[h,e,f,S,L,N_x]/I$$

where x runs over a generating set of $\mathrm{Num}^*(N)$ as a $\mathbb{Z}$-module, and $N_x := j_*(x)$. Furthermore, I is the ideal of relations from Proposition 3.2 with $\zeta = \xi$.

Proof. This is clear because a Künneth formula holds for $L_C \times Q$ by Proposition 3.5. □

6.3. **The numerical Chow ring of Y_3.** Next we turn to Y_3, the blow up of $Q\times L_C$ in $S\times S$, as in the blowup diagram

$$\begin{array}{ccc} M & \xrightarrow{\ j\ } & Y_3 \\ \Big\downarrow{\scriptstyle \pi_M} & & \Big\downarrow{\scriptstyle \pi} \\ S\times S & \xrightarrow{\ i\ } & Q\times L_C. \end{array}$$

Lemma 6.7. *There is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\quad \mu \quad} & N \\ & {\scriptstyle \pi_M}\searrow \quad \swarrow{\scriptstyle \pi_N} & \\ & S\times S & \end{array}$$

where μ is an isomorphism mapping η to $\xi - r_1 - r_2 + R_1 + R_2$ where η is the relative hyperplane class of the projectivisation of $\mathscr{N}_{(S\times S)/(Q\times L_C)}$, which is naturally isomorphic to M.

Proof. We have

$$\mathscr{N}_{(S\times S)/(L_C\times Q)} \simeq \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(-1,-1,0,0) \oplus \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(0,0,1,1).$$

and

$$\mathscr{N}_{(S\times S)/(Q\times L_C)} \simeq \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(1,1,0,0) \oplus \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(0,0,-1,-1).$$

Hence

$$\mathscr{N}_{(S\times S)/(L_C\times Q)} \otimes \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(1,1,-1,-1) \simeq \mathscr{N}_{(S\times S)/(Q\times L_C)}.$$

Thus letting $\mathscr{E} = \mathscr{N}_{(S\times S)/(Q\times L_C)}$ and $\mathscr{L} = \mathscr{O}_{(\mathbb{P}^1\times\mathbb{P}^1)\times(\mathbb{P}^1\times\mathbb{P}^1)}(1,1,-1,-1)$ and observing that

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathscr{O}_{\mathbb{P}(\mathscr{E}\otimes\mathscr{L})}(1) \otimes \mathscr{L},$$

we get

$$\xi = \mu_*(\eta) + r_1 + r_2 - R_1 - R_2.$$

□

Proposition 6.8. *We have*

$$\operatorname{Num}^*(Y_3) = \mathbb{Z}[s, l, H, E, F, M_x]/I$$

where x runs over a generating set of $\operatorname{Num}^*(N)$ as a $\mathbb{Z}$-module, and $M_x := j_* \circ \mu_*(x)$. Furthermore, I is the ideal of relations from Proposition 3.2 with $\zeta = \xi - r_1 - r_2 + R_1 + R_2$.

Proof. This is clear because of Lemma 6.7 and because again a Künneth formula holds for $Q \times L_C$ by Proposition 3.5. □

6.4. **The numerical Chow ring of Y_4.** Since $Y_4 = Q \times Q$, this can be computed by the Künneth formula. $\operatorname{Num}^*(Y_4)$ is generated by s, l, S, L.

6.5. **The numerical Chow rings of Y_{ij}, Y_{ijk}.** The numerical Chow rings of double and triple intersections can be computed by the Künneth formula except for $\operatorname{Num}^*(Y_{23})$ which is $\operatorname{Num}^*(N)$.

6.6. **Computing pushforwards and pullbacks via $\iota_{\{i,j,k\},\{a,b\}} \colon Y_{ijk} \to Y_{ab}$.** These pushforwards and pullbacks are all easy to obtain using the Künneth formula except for $\iota_{\{1,2,3\},\{2,3\}}$ and $\iota_{\{2,3,4\},\{2,3\}}$.

Lemma 6.9. *Let $\mathscr{E}$ be a vector bundle on a smooth projective variety X. Consider the diagram*

$$\begin{array}{c} \mathbb{P}(\mathscr{E}) \\ \pi \big\downarrow \big) \sigma \\ X \end{array}$$

where σ is a section and $\Sigma = \sigma(X)$. Then

$$\sigma_*(x) = \pi^*(x) \cdot \Sigma$$

for $x \in \operatorname{Num}^*(X)$. For $y \in \operatorname{Num}^*(\mathbb{P}(\mathscr{E}))$ we have

$$\sigma^*(y) = \pi_*(y \cdot \Sigma).$$

If $\mathscr{E}$ is rank two and $y = \zeta \cdot c + d$ with $c, d \in \pi^* \operatorname{Num}^*(X)$ and ζ the relative hyperplane class, then

$$\pi_*(y) = c.$$

Proof. This is clear. □

Proposition 6.10. *We have*

$$[\iota_{\{1,2,3\},\{2,3\}}(S \times S)] = \xi + R_1 + R_2,$$
$$[\iota_{\{2,3,4\},\{2,3\}}(S \times S)] = \xi - r_1 - r_2$$

as classes in $\operatorname{Num}^*(N)$.

Using Lemma 6.9, this gives a complete description of pushforwards and pullbacks via $\iota_{\{1,2,3\},\{2,3\}}$ and $\iota_{\{2,3,4\},\{2,3\}}$.

Proof. The subvarieties $\iota_{\{1,2,3\},\{2,3\}}(S \times S)$ resp. $\iota_{\{2,3,4\},\{2,3\}}(S \times S)$ of $Y_{23} = N$ consist of all those normal directions to $S \times S$ in $L_C \times Q$ that are contained in $Y_{12} = L_C \times S$ resp. $Y_{24} = S \times Q$. Recall that N is the projectivisation of

$$\mathscr{N}_{(S\times S)/(L_C \times Q)} \simeq \mathscr{O}_{(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1)}(-1, -1, 0, 0) \oplus \mathscr{O}_{(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1)}(0, 0, 1, 1).$$

and the (projectivisation of the) first summand here corresponds to normal directions contained in $Y_{12} = L_C \times S$ and the second to those contained in $Y_{24} = S \times Q$. The assertion follows because these are just the two natural sections $(1 : 0)$ and $(0 : 1)$ in this projective bundle. □

6.7. Computing pushforwards and pullbacks via $\iota_{\{i,j\},\{a\}} \colon Y_{ij} \to Y_a$.

Lemma 6.11. *Consider smooth varieties $Z \subset Y \subset X$ with Z a divisor in Y and Y a divisor in X. Blowing up Z in X gives the following diagram*

$$\begin{array}{ccccc} E & \xrightarrow{j_1} & Y \cup E & \xrightarrow{j_2} & \mathrm{Bl}_Z(X) \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi_{Y\cup E}} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{i_1} & Y & \xrightarrow{i_2} & X \end{array}$$

We denote by $\pi_Y := \pi_{Y\cup E}|_Y$ the restriction of π to the strict transform of Y. It is an isomorphism since Z is a divisor in Y and Y is smooth. Also we set $j := j_2 \circ j_1$.

Consider the map

$$j_Y \colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*\big(\mathrm{Bl}_Z(X)\big)$$

with $j_Y := (j_2)_* \circ \pi_Y^*$. Then for $y \in \mathrm{CH}^*(Y)$ we have

$$j_Y(y) = \pi^*(i_2)_*(y) - j_*\pi_E^*(i_1)^*(y).$$

Proof. Let Γ be a subscheme of Y determining a cycle representing the class y; by the Moving Lemma [E-H16, Appendix A, Lemma A.1 (a)], we can assume that Γ intersects the codimension one subvariety Z of Y generically transversely and all components of Γ are reduced. Now to conclude we only have to show that

(1) $\pi^{-1}(\Gamma)$ (the scheme-theoretic pullback) has an underlying cycle whose class is precisely $\pi^*(i_2)_*(y)$;
(2) $\pi_Y^{-1}(\Gamma)$ (scheme-theoretic pullback) is a component of $\pi^{-1}(\Gamma)$ with residual scheme giving a cycle representing $j_*\pi_E^*(i_1)^*(y)$.

The residual scheme is nothing but $\pi_E^{-1}(\Gamma\cap Z)$, where by our choice $\Gamma \cap Z$ represents the intersection $y \cdot [Z]$. Hence (2) is clear, and (1) follows from [E-H16, Thm. 1.23, (a)], applied to each irreducible component of Γ: if $f \colon V \to W$ is a morphism of smooth projective varieties and A is a subvariety of W representing a cycle class γ, then the scheme-theoretic pullback $f^{-1}(A)$ represents the class $f^*(\gamma)$ provided A is *generically transverse to* f, meaning: (a) A is dimensionally transverse to f, i.e. all components of $f^{-1}(A)$ have the same codimension in V as A in W, (b) $f^{-1}(A)$ is generically reduced. $\square$

With this Lemma we can compute pushforwards and pullbacks for:

a) $\iota_{\{1,2\},\{2\}} \colon Y_{12} \to Y_2$.
b) $\iota_{\{1,3\},\{3\}} \colon Y_{13} \to Y_3$.
c) $\iota_{\{2,4\},\{2\}} \colon Y_{24} \to Y_2$.
d) $\iota_{\{3,4\},\{3\}} \colon Y_{34} \to Y_3$.

The inclusions $\iota_{\{2,3\},\{2\}} \colon Y_{23} \to Y_2$ and $\iota_{\{2,3\},\{3\}} \colon Y_{23} \to Y_3$ are just the inclusions of the exceptional divisor into the blowup.

The inclusions $\iota_{\{2,4\},\{4\}} \colon Y_{24} \to Y_4$ and $\iota_{\{3,4\},\{4\}} \colon Y_{34} \to Y_4$ can be handled using the Künneth formula.

Lastly, we have to consider $\iota_{\{1,2\},\{1\}} \colon Y_{12} \to Y_1$ and $\iota_{\{1,3\},\{1\}} \colon Y_{13} \to Y_1$. Here everything follows using the Künneth formula except the pullback of D:

Lemma 6.12. *We have*

$$\iota^*_{\{1,2\},\{1\}}(D) = eR_1R_2 + 3f(R_1 + R_2)$$
$$\iota^*_{\{1,3\},\{1\}}(D) = r_1r_2E + 3(r_1 + r_2)F.$$

Proof. We only need to prove one of these formulas because the other follows by symmetry. We have the two inclusion

$$j := \iota_{\{1,2,3\},\{1,2\}} \colon S \times S \to L_C \times S, \quad i := \iota_{\{1,2\},1} \colon L_C \times S \to L_C \times L_C.$$

Now the pullback $(i \circ j)^*(D)$ is the diagonal $\Delta_C \subset C \times C \subset S \times S$. Hence, intersecting with a basis of the divisors in $S \times S$, we find

$$(i \circ j)^*(D) = 3(r_1 + r_2)R_1R_2 + 3r_1r_2(R_1 + R_2).$$

Therefore, $i^*(D)$ must be of the form

$$\alpha h R_1 R_2 + \beta e R_1 R_2 + \gamma h^2(R_1 + R_2) + \delta f (R_1 + R_2)$$

with $\alpha, \beta, \gamma, \delta$ integers. Now we have the equation

$$D \cdot i_*(L_C \times S) = i_*(i^*(D)).$$

Using Proposition 6.4 and the fact that we can compute i_* using the Künneth formula, we obtain linear equations for the unknowns $\alpha, \beta, \gamma, \delta$. The only solution is the one in the statement. This is checked in [BBG-M2]. ☐

Theorem 6.13. *We have*

$$\mathrm{Num}^3_{\mathrm{prelog}}(Y) = \mathbb{Z}^6$$

modulo torsion.

Proof. For this computation we use Definition 2.2:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & R(Y) & \longrightarrow & \mathrm{Num}^3_{\mathrm{prelog}}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\bigoplus \mathrm{Num}^2(Y_{ij}) & \xrightarrow{\delta} & \bigoplus \mathrm{Num}^3(Y_i) & \longrightarrow & \operatorname{coker}\delta & \longrightarrow & 0 \\
\downarrow \rho' & & \downarrow \rho & & & & \\
\bigoplus \mathrm{Num}^2(Y_{ijk}) & \xrightarrow{\delta'} & \bigoplus \mathrm{Num}^3(Y_{ij}) & & & &
\end{array}$$

with explicitly given maps ρ, ρ' in terms of the push forwards ι_* and δ, δ' in terms of the pullbacks ι^* calculated above.

For the convenience of the reader we give a slow walk through the necessary computations. A Macaulay2 script doing the same work is available at [BBG-M2].

Using Propositions 6.4, 6.6, 6.8 and Subsections 6.4, 6.5 we can calculate the intersection rings of Y_i, Y_{ij} and Y_{ijk} in degree 3:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & R(Y) & \longrightarrow & \mathrm{Num}^3_{\mathrm{prelog}}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\bigoplus \mathbb{Z}^{32} & \xrightarrow{\delta} & \bigoplus \mathbb{Z}^{39} & \longrightarrow & \operatorname{coker} \delta & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & & & \\
\bigoplus \mathbb{Z}^{11} & \xrightarrow{\delta'} & \bigoplus \mathbb{Z}^{32} & & & &
\end{array}$$

A good check that we got everything right is that indeed $\delta\rho = \rho'\delta'$ (this is the Friedman condition).

Now one can check that δ has rank 22 in every characteristic except 2 where it has rank 21. The same is true for ρ.

This reduces the diagram to

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}^{17} & \longrightarrow & \mathrm{Num}^3_{\mathrm{prelog}}(Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\
\bigoplus \mathbb{Z}^{32} & \xrightarrow{\delta} & \bigoplus \mathbb{Z}^{39} & \xrightarrow{\gamma} & \mathbb{Z}^{17} \oplus \mathbb{Z}/2 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & & & \\
\bigoplus \mathbb{Z}^{11} & \xrightarrow{\delta'} & \bigoplus \mathbb{Z}^{32} & & & &
\end{array}$$

The degree 3 part of the numerical prelog ring $\mathrm{Num}^3_{\mathrm{prelog}}(Y)$, modulo torsion, is therefore the image of an explicitly given 17×17 matrix M. We calculate that this matrix M has rank 6. Therefore

$$\mathrm{Num}^3_{\mathrm{prelog}}(Y) = \mathbb{Z}^6$$

modulo torsion. □

We now identify explicit effective generators of $\mathrm{Num}^3_{\mathrm{prelog}}(Y)$ modulo torsion.

Theorem 6.14. *The following cycles satisfy the prelog condition and are mapped to a $\mathbb{Z}$-basis of* $\mathrm{Num}^3_{\mathrm{prelog}}(Y)$*, modulo torsion:*

$$\begin{aligned}
Z_{03} &= (h^3, h^3, 0, 0)\\
Z_{30} &= (H^3, 0, H^3, 0)\\
Z_{12} &= \big((h^2-2f)H, (h^2-2f)S, 0, 0\big)\\
Z_{21} &= \big(h(H^2-2F), 0, s(H^2-2F), 0\big)\\
Z_\Delta &= \big(h^3 + h^2H + hH^2 + H^3 - D,\\
&\qquad N_{r_1r_2} + N_{R_1r_2} + N_{r_1R_2} + N_{R_1R_2},\\
&\qquad M_{r_1r_2} + M_{R_1r_2} + M_{r_1R_2} + M_{R_1R_2},\\
&\qquad sl + sL + Sl + SL\big)\\
Z_D &= (D - eF - fE, 0, 0, 0)
\end{aligned}$$

The cycles Z_{03}, Z_{30}, Z_{12}, Z_{21}, and Z_Δ are the specializations of (point) $\times V$, $V \times$ (point), (line)$\times$(hyperplane section), (hyperplane section)$\times$(line), and the diagonal, respectively.

Proof. In [BBG-M2] we check that the cycles in the statement satisfy the prelog condition and are a $\mathbb{Z}$-basis: we do this in [BBG-M2] by showing that they are linearly independent and writing the images under $\gamma \circ \sigma$ of all standard generators of $\mathbb{Z}^{17}$ as $\mathbb{Z}$-linear combinations of the elements above (modulo torsion).

The statement that these arise as specializations as claimed is clear for the first four cycles. That Z_Δ is the specialization of the diagonal can be seen as follows. The class

$$sl + sL + Sl + SL$$

is the class of the diagonal on $Y_4 = Q \times Q$ whereas

$$h^3 + h^2H + hH^2 + H^3 - D$$

is the class of the diagonal on $L_C \times L_C$: indeed, $h^3 + h^2H + hH^2 + H^3$ is the class of $\Delta_{\mathbb{P}^3}$ on $\mathbb{P}^3 \times \mathbb{P}^3$, and $L_C \times L_C$ is obtained from $\mathbb{P}^3 \times \mathbb{P}^3$ by first blowing up $C \times \mathbb{P}^3$ and then $L_C \times C$. The diagonal intersects $C \times \mathbb{P}^3$ in $\Delta_C \subset C \times C$, and hence as schemes

$$\sigma^{-1}(\Delta_{\mathbb{P}^3}) = \Delta_{L_C} \cup D$$

where $\sigma\colon L_C \times L_C \to \mathbb{P}^3 \times \mathbb{P}^3$ is the composition of the two blow-ups. Since D is three-dimensional, we have $\sigma^*([\Delta_{\mathbb{P}^3}]) = [\Delta_{L_C}] + D$ as cycle classes. To justify the components of Z_Δ on Y_2 and Y_3, note that the diagonal in the product family $\mathscr{X} \times_B \mathscr{X} \to B$ intersects $S \times S \subset X \times X$ in Δ_S. Note also that $S \times S$ is precisely the locus in which the total space $\mathscr{X} \times_B \mathscr{X}$ is singular. The class of Δ_S in $S \times S$ is

$$r_1r_2 + R_1r_2 + r_1R_2 + R_1R_2$$

and pulling this back to Y_{23} and pushing forward to Y_2 and Y_3 we obtain the middle two entries in Z_Δ. □

We now calculate the saturated numerical prelog Chow group in degree 3.

Theorem 6.15. $\mathrm{Num}_{\mathrm{sat,prelog},3}(Y) \simeq \mathbb{Z}^6$ *and is generated by the classes in Theorem 6.14 and a half of their sum.*

Proof. This is calculated in [BBG-M2]: writing the generators given in Theorem 6.14 in the standard basis of $\mathbb{Z}^{17}$ gives a 17×6 matrix N. The gcd of its maximal minors is equal to 2. Therefore, N has full rank in every characteristic except 2. Moreover, in characteristic 2, N has rank 5. The kernel of N in characteristic 2 is generated by the sum of the 6 generators of Theorem 6.14. □

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