PRELOG CHOW RINGS AND DEGENERATIONS

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Abstract. For a simple normal crossing variety X, we introduce the concepts of prelog Chow ring, saturated prelog Chow group, as well as their counterparts for numerical equivalence. Thinking of X as the central fibre in a (strictly) semistable degeneration, these objects can intuitively be thought of as consisting of cycle classes on X for which some initial obstruction to arise as specializations of cycle classes on the generic fibre is absent. After proving basic properties for prelog Chow rings and groups, we explain how they can be used in an envisaged further development of the degeneration method by Voisin et al. to prove stable irrationality of very general fibres of certain families of varieties; this extension would allow for much more singular degenerations, such as toric degenerations as occur in the Gross-Siebert programme, to be usable. We illustrate that by looking at the "baby example" of degenerations of elliptic curves, and we compute the saturated prelog Chow group of degenerations of cubic surfaces.

1. Introduction

Using ideas from logarithmic geometry we develop in this article a conceptual framework that extends the scope of Voisin's degeneration method [Voi15] (as developed further by Colliot-Thélène/Pirutka [CT-P16], Totaro [To16], Schreieder [Schrei17, Schrei18] et al.) to more singular degenerations, for example toric degenerations used in the Gross-Siebert programme [GS06, GS10, Gross11, GS11, GS11a, GS16, GS19]. Our main construction concerns prelog Chow rings (and variations thereof) associated to central fibres of strictly semistable degenerations and we show that the specialization morphism takes values in these rings (Theorem 3.2). Related ideas have been pursued in [BGS94, BGS95, CCGGK12].

Whereas we will give more details in Section 5, the main idea on how to apply Voisin's method to strictly semistable degenerations can informally be described as follows: suppose $\mathscr{X} \to \Delta$ is a degeneration of projective varieties $\mathscr{X}_t$ over a small disk Δ centered at 0 in $\mathbb{C}$ with coordinate t. Suppose $\mathscr{X}_t$ is smooth for t nonzero. Let $\mathscr{X}^* \to \Delta^*$ be the induced family over the puctured disk obtained by removing the central fibre $\mathscr{X}_0$. We would like to prove that a very general fibre $\mathscr{X}_t$ is not retract rational (or weaker, not stably rational). Arguing by contradiction and assuming to the contrary that $\mathscr{X}_t$ is stably rational, we obtain (cf. [Voi15, Proof of Thm. 1.1]) that, after replacing t by t^k for some positive integer k and shrinking Δ, there is a relative decomposition of the diagonal on $\mathscr{X}^* \times_{\Delta^*} \mathscr{X}^* \to \Delta^*$: there exists a section $\sigma\colon \Delta^* \to \mathscr{X}^*$ and a relative cycle $\mathscr{Z}^* \subset \mathscr{X}^* \times_{\Delta^*} \mathscr{X}^*$ together with a relative divisor $\mathscr{D}^* \subset \mathscr{X}^*$ such that, for all $t \in \Delta^*$

$$\Delta_{\mathscr{X}_t} = \mathscr{X}_t \times \sigma(t) + \mathscr{Z}_t^* \text{ in } \mathrm{CH}^*(\mathscr{X}_t \times \mathscr{X}_t)$$

and $\mathscr{Z}^* \subset \mathscr{D}^* \times_{\Delta*} \mathscr{X}^*$. Closing everything up in $\mathscr{X} \times_\Delta \mathscr{X}$, and intersecting with $\mathscr{X}_0 \times \mathscr{X}_0$, we can specialize this to obtain a decomposition of the diagonal on $\mathscr{X}_0 \times \mathscr{X}_0$.

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If the singularities of $\mathscr{X}_0$ are sufficiently mild (e.g. only nodes, but more general classes of singularities are admissible), then a resolution $\tilde{\mathscr{X}}_0$ inherits a decomposition of the diagonal. Now we can derive a contradiction to our initial assumption that $\mathscr{X}_t$ is stably rational for t very general since one can obstruct the existence of decompositions of the diagonal on nonsingular projective varieties by, for example, nonzero Brauer classes, and other unramified invariants. However, we can also try to bypass the necessity that $\mathscr{X}_0$ have mild singularities if we are willing to obstruct *directly the existence of a decomposition of the diagonal on $\mathscr{X}_0 \times \mathscr{X}_0$ that arises as a "limit" of decompositions of the diagonal on the fibres of $\mathscr{X}^* \times_{\Delta^*} \mathscr{X}^* \to \Delta^*$.* Here the qualifying relative clause is important: if the degeneration $\mathscr{X}_0$ is sufficiently drastic, e.g. $\mathscr{X}_0$ could be simple normal crossing with toric components, then decompositions of the diagonal may well exist, but still possibly none that arise as a limit in the way described above.

The desire to single out decompositions of the diagonal for $\mathscr{X}_0$ that would stand a chance to arise via such a limiting procedure naturally leads to the idea of endowing $\mathscr{X}_0$ with its natural log structure $\mathscr{X}_0^\dagger$ (cf. [Gross11, Chapter 3.2], [GOR15]) obtained by restricting the divisorial log structure for $(\mathscr{X}, \mathscr{X}_0)$ to $\mathscr{X}_0$, and develop some kind of log Chow theory on $\mathscr{X}_0^\dagger$ in which cycles carry some extra decoration that encodes information about the way they can arise as limits. Although a promising theory that may eventually lead to a realization of this hope is currently being developed by Barrott [Bar18], it is still at this point unclear to us if and how it can be used for our purposes, so we take a more pedestrian approach in this article, always keeping the envisaged geometric applications in view.

While we believe that the correct framework is ultimately likely to be the general theory of log structures in the sense of Deligne, Faltings, Fontaine, Illusie, Kato et al. as exposed in [Og18], our point of departure in Section 2 is the following simple observation: given a strictly semistable degeneration, cycle classes in $\mathrm{CH}_*(\mathscr{X}_0)$ that arise as specializations in fact come from cycle classes on the normalization of $\mathscr{X}_0$ satisfying an obvious coherence/compatibility condition that we call the prelog condition following Nishinou and Siebert [Ni15, NiSi16] (in the case of curves). Despite its simplicity, the idea is very effective in applications and it needs a little work to cast it in the appropriate algebraic structures, which we do in Section 2 in the form of the prelog Chow ring of a simple normal crossing scheme and the numerical prelog Chow ring, which is easier to handle computationally. We also discuss a relation between the prelog condition and the Friedman condition.

In Section 3, we treat specialization homomorphisms of strictly semistable degenerations into prelog Chow rings, and in Section 4 we deal with the problems that arise by the necessity to perform a ramified base change $t \mapsto t^k$ in Voisin's method outlined above. This leads to the concepts of saturated prelog Chow groups and their numerical counterparts.

In Section 5 we finally outline how (saturated) prelog Chow groups can be used for rationality problems and describe a souped-up version of the degeneration technique. In Section 7, we consider the very simple case of a degeneration of a family of elliptic curves, realized as plane cubics, into a triangle of lines. We show that one can still read off from the saturated prelog Chow group of the central fibre that a smooth elliptic curve does not have universally trivial Chow group of zero cycles. This is, if trivial, at least morally reassuring we are on the right track.

In Section 6 we compute the saturated prelog Chow group of a degeneration of cubic surfaces. As expected we explicitely recover the 27 lines as prelog cycles in the central fibre.

We deal with the more complicated and lengthier example of cubic threefolds in a forthcoming article [BBG19]. We would also like to think that the concepts in this article, such as prelog Chow rings and groups, may be viewed as receiving natural forgetful maps from some log Chow groups, still to be properly defined, whose elements are certain cycle classes together with log structures on, or equivalence classes of certain types of log morphisms into, $\mathscr{X}_0^\dagger$. We believe that the type of geometric problems arising from rationality questions considered here may serve as good concrete guiding questions to test the concepts of any emerging log Chow theory against.

In the final stages of the preparation of this manuscript, the preprint [NiOt19] came to our attention. This also seeks to (and successfully does) use wider classes of degenerations to prove stable irrationality results. It is, however, quite different in its details: it uses the motivic obstruction to stable rationality introduced by Nicaise-Shinder, and, it seems to us, cannot make use of degenerations all of whose components are smooth rational. In particular, there seems to be no way to make it work for cubic hypersurfaces. We, on the contrary, eventually want to obstruct the existence of decompositions of the diagonal on the singular central fibre that have a chance to deform in the family, and this, if successful, could have even wider applicability than the method in [NiOt19].

2. Prelog Chow rings of simple normal crossing schemes

We work over the complex numbers $\mathbb{C}$ throughout.

Let $X = \bigcup_{i\in I} X_i$ be a simple normal crossing (snc) scheme; here I is some finite set, and all irreducible components X_i are smooth varieties. Moreover, for a nonempty subset $J \subset I$, we denote by X_J the intersection $\bigcap_{j\in J} X_j \subset X$. In this way, each X_J is a smooth variety (possibly not connected). The irreducible components of X_J then form the $(|J|-1)$-dimensional cells in the dual intersection complex of X. It is a regular cell complex in general, and simplicial if and only if all X_J are irreducible. For nonempty subsets $J_1 \subset J_2$ of I, we denote by

$$\iota_{J_2>J_1}\colon X_{J_2} \hookrightarrow X_{J_1}$$

and by

$$\iota_J\colon X_J \hookrightarrow X$$

the inclusions. Let

$$\nu\colon X^\nu = \bigsqcup_{i\in I} X_i \to X$$

be the normalization.

Definition 2.1. Denote by

$$R(X) = R \subset \mathrm{CH}^*(X^\nu) = \bigoplus_{i\in I} \mathrm{CH}^*(X_i)$$

the following subring of the Chow ring of the normalization, which we call the *ring of compatible classes*: elements in R are tuples of classes $(\alpha_i)_{i\in I}$ with the property that for any two element subset $\{j,k\} \subset I$

$$\iota^*_{\{j,k\}>\{j\}}(\alpha_j) = \iota^*_{\{j,k\}>\{k\}}(\alpha_k).$$

We call this property the *prelog condition*. Furthermore, denote by

$$M = M(X) = \mathrm{CH}_*(X)$$

the Chow group of X.

Notice that there is in general no well-defined intersection product on M. We will show, however, that one can turn M into an R-module.

Definition 2.2. Let $\alpha = (\alpha_i)_{i\in I}$ be an element in R and let Z be a prime cycle (=irreducible subvariety) on X. Let $J \subset I$ be the largest subset such that $Z \subset X_J$ and let $j \in J$ be arbitrary. Then we define

$$\langle \alpha, Z\rangle := \iota_{J,*}\left(\iota^*_{J>\{j\}}(\alpha_j).[Z]\right).$$

This is independent of the choice of j since if j' is another element of J, the class $\iota^*_{J>\{j'\}}(\alpha_{j'})$ is the same as $\iota^*_{J>\{j\}}(\alpha_j)$ since by the definition of R we have that $\iota^*_{\{j,j'\}>\{j\}}(\alpha_j) = \iota^*_{\{j,j'\}>\{j'\}}(\alpha_{j'})$, so the definition is well-posed. If Z is an arbitrary cycle on X, we define $\langle\alpha, Z\rangle$ by linearity.

Proposition 2.3. *If Z_1 and Z_2 are rationally equivalent cycles on X, then*

$$\langle\alpha, Z_1\rangle = \langle\alpha, Z_2\rangle.$$

In particular, the pairing descends to rational equivalence on X and makes $M = \mathrm{CH}_*(X)$ into an R-module. The push forward map induced by the normalization

$$\nu_* \mid_{R(X)} \colon R = R(X) \to M$$

is an R-module homomorphism. Indeed, $\nu_*\colon \bigoplus_i \mathrm{CH}_*(X_i) \to \mathrm{CH}_*(X)$ is an R-module homomorphism.

Proof. The main point is the following consequence of the projection formula that gives a way to calculate the pairing $\langle\alpha, Z\rangle$ in a very flexible way: if Z is a prime cycle contained in X_J as in Definition 2.2, then we can write

$$\begin{aligned}\langle\alpha, Z\rangle = \iota_{J,*}\left(\iota^*_{J>\{j\}}(\alpha_j).[Z]\right) &= \iota_{\{j\},*}\iota_{J>\{j\},*}\left(\iota^*_{J>\{j\}}(\alpha_j).[Z]\right)\\ &= \iota_{\{j\},*}\left(\alpha_j.\iota_{J>\{j\},*}[Z]\right).\end{aligned}$$

Moreover, as already remarked in Definition 2.2, here $j \in J$ is arbitrary and the result independent of it. Hence, for an arbitrary cycle Z on X and $\alpha \in R$, we can compute $\langle\alpha, Z\rangle$ as follows: first we write, in whichever way we like,

$$Z = \sum_{i\in I} Z_i$$

where Z_i is a cycle supported on X_i, then form the intersection products $\alpha_i.Z_i$ on X_i, push these forward to X and sum to get the cycle $\langle\alpha, Z\rangle$. In particular, this makes it clear that if Z_1 and Z_2 are rationally equivalent on X, then $\langle\alpha, Z_1\rangle = \langle\alpha, Z_2\rangle$ as elements of M. Indeed, cycles of dimension d rationally equivalent to zero are sums of cycles T on X that arise as follows: take an irreducible subvariety $Y \subset X$ of dimension $d+1$, its normalization $\nu_Y\colon Y^\nu \to Y$, and let T be $\nu_{Y,*}$ of the divisor of zeros and poles of a rational function on Y^ν. Now Y is necessarily contained entirely within one of the irreducible components of X, X_i say, and hence T is rationally equivalent to zero on X_i. Hence $\langle\alpha, T\rangle = 0$ since we have the flexibility to compute this entirely on X_i.

To show that $\nu_*\colon \bigoplus_i \mathrm{CH}_*(X_i) \to M$ is an R-module homomorphism, we take two elements $\alpha = (\alpha_i)_{i\in I} \in R, \beta = (\beta_i)_{i\in I} \in \bigoplus_i \mathrm{CH}_*(X_i)$ and represent β_i by a cycle Z_i on X_i. All we have to show then is that

$$\nu_*(\alpha\beta) = \langle \alpha, Z\rangle, \quad Z := \sum_{i\in I} \iota_{\{i\},*}(Z_i).$$

But this directly follows from the definition of ν_* and the way we can compute the pairing. □

Definition 2.4. We call the quotient $R(X)/(\ker \nu_* \mid_{R(X)})$ the pre-log Chow ring of X, and denote it by $\mathrm{CH}^*_{\mathrm{prelog}}(X)$. It is indeed naturally a ring since $\ker \nu_* \mid_{R(X)}$ is an ideal in $R(X)$.

Proposition 2.5. *There is an exact sequence*

$$\bigoplus \mathrm{CH}_*(X_{ij}) \xrightarrow{\delta} \bigoplus \mathrm{CH}_*(X_i) \xrightarrow{\nu_*} \mathrm{CH}_*(X) \longrightarrow 0$$

where for $z_{ij} \in \mathrm{CH}_*(X_{ij})$ with $i < j$

$$\big(\delta(z_{ij})\big)_a = \begin{cases} \iota_{\{ij\}>\{i\}*}(z_{ij}) & \text{if } a = i, \\ -\iota_{\{ij\}>\{j\}*}(z_{ij}) & \text{if } a = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We recall that by [Ful98, Ex. 1.8.1] if

$$\begin{tikzcd} Y' \arrow[r, "j"] \arrow[d, "q"] & Z' \arrow[d, "p"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

is a fibre square, with i a closed embedding, p proper, such that p induces an isomorphism of $Z' - Y'$ onto $Z - Y$, then: there is an exact sequence

$$\mathrm{CH}_k Y' \xrightarrow{a} \mathrm{CH}_k Y \oplus \mathrm{CH}_k Z' \xrightarrow{b} \mathrm{CH}_k Z \longrightarrow 0$$

where $a(\alpha) = (q_*\alpha,\ -j_*\alpha)$, $b(\alpha, \beta) = i_*\alpha + p_*\beta$. Apply this inductively with Y one component of an snc scheme and Z' all remaining components. □

Definition 2.6. Let X be a snc scheme with at worst triple intersections. We say that X satisfies the *Friedman condition* if for every intersection $X_{ij} = X_i \cap X_j$ we have

$$\mathscr{N}_{X_{ij}/X_i} \otimes \mathscr{N}_{X_{ij}/X_j} \otimes \mathscr{O}(T) = \mathscr{O}_{X_{ij}}.$$

Here T is the union of all triple intersections X_{ijk} that are contained in X_{ij}.

Remark 2.7. By [Fried83, Def. 1.9 and Cor. 1.12] any X that is smoothable with smooth total space has trivial infinitesimal normal bundle and in particular satisfies the Friedman condition.

The following Proposition describes a relation between the prelog condition, Friedman condition and Fulton's description of the kernel of ν_*.

Proposition 2.8. *Let X be an snc scheme that has at worst triple intersections and satisfies the Friedman condition. Then the following diagram commutes*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & R(X) \arrow[r] \arrow[d] & \mathrm{CH}^*_{\mathrm{prelog}}(X) \arrow[r] \arrow[d] & 0 \\
\bigoplus \mathrm{CH}^*(X_{ij}) \arrow[r, "\delta"] \arrow[d, "\rho'"] & \bigoplus \mathrm{CH}^*(X_i) \arrow[r, "\nu_*"] \arrow[d, "\rho"] & \mathrm{CH}_*(X) \arrow[r] & 0 \\
\bigoplus \mathrm{CH}^*(X_{ijk}) \arrow[r, "\delta'"] & \bigoplus \mathrm{CH}^*(X_{ij}) & &
\end{tikzcd}$$

Here the maps ρ, ρ', δ' are defined as follows, using the convention $a < b < c$, $i < j < k$:

$$\big(\rho(z_i)\big)_{ab} = \begin{cases} \iota^*_{\{ab\}>\{i\}}(z_i) & \text{if } i = a \\ -\iota^*_{\{ab\}>\{i\}}(z_i) & \text{if } i = b \\ 0 & \text{otherwise} \end{cases}$$

$$\big(\rho'(z_{ij})\big)_{abc} = \begin{cases} \iota^*_{\{abc\}>\{ij\}}(z_{ij}) & \text{if } (i,j) = (a,b) \\ -\iota^*_{\{abc\}>\{ij\}}(z_{ij}) & \text{if } (i,j) = (a,c) \\ \iota^*_{\{abc\}>\{ij\}}(z_{ij}) & \text{if } (i,j) = (b,c) \\ 0 & \text{otherwise} \end{cases}$$

$$\big(\delta'(z_{ijk})\big)_{ab} = \begin{cases} -\iota_{\{ijk\}>\{ab\}*}(z_{ijk}) & \text{if } (a,b) = (i,j) \\ \iota_{\{ijk\}>\{ab\}*}(z_{ijk}) & \text{if } (a,b) = (i,k) \\ -\iota_{\{ijk\}>\{ab\}*}(z_{ijk}) & \text{if } (a,b) = (j,k) \\ 0 & \text{otherwise} \end{cases}$$

Notice that being in the kernel of ρ amounts to the prelog condition.

Proof. It remains to be seen that the lower square commutes. For this we want to prove

$$\big((\delta' \circ \rho')(z_{ij})\big)_{ab} = \big((\rho \circ \delta)(z_{ij})\big)_{ab}.$$

There are three cases:

Case 1: $|\{i,j\} \cap \{a,b\}| = 0$. In this case both sides of the equation are 0.

Case 2: $|\{i,j\} \cap \{a,b\}| = 1$. In this case we can assume $\{i,j\} \cup \{a,b\} = \{i,j,k\}$. Depending on the relative size of the indices involved there are a number of cases to consider. We check only the case $a = j, b = k$, the others are similar. The left hand side then is

$$\big((\delta' \circ \rho')(z_{ij})\big)_{jk} = \big(\delta'(\iota^*_{\{ijk\}>\{ij\}}(z_{ij}) \pm \dots)\big)_{jk} = -\iota_{\{ijk\}>\{jk\}*}\iota^*_{\{ijk\}>\{ij\}}(z_{ij}).$$

Similarly the right hand side is

$$\begin{aligned} \big((\rho \circ \delta)(z_{ij})\big)_{jk} &= \Big(\rho\big(\iota_{\{ij\}>\{i\}*}(z_{ij}) - \iota_{\{ij\}>\{j\}*}(z_{ij})\big)\Big)_{jk} \\ &= 0 - \Big(\rho\big(\iota_{\{ij\}>\{j\}*}(z_{ij})\big)\Big)_{jk} \\ &= -\iota^*_{\{jk\}>\{j\}}\iota_{\{ij\}>\{j\}*}(z_{ij}). \end{aligned}$$

So our claim is just the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{CH}^*(X_{ij}) \arrow[r, "\iota_*"] \arrow[d, "\iota^*"] & \mathrm{CH}^*(X_j) \arrow[d, "\iota^*"] \\
\mathrm{CH}^*(X_{ijk}) \arrow[r, "\iota_*"] & \mathrm{CH}^*(X_{jk}).
\end{tikzcd}$$

Case 3: $\{i,j\} = \{a,b\}$. Here we get on the left hand side

$$\begin{aligned}
\big((\delta' \circ \rho')(z_{ij})\big)_{ij} &= \left(\delta'\left(\sum_k \pm\iota^*_{\{ijk\}>\{ij\}}(z_{ij})\right)\right)_{ij} \\
&= -\sum_k \iota_{\{ijk\}>\{ij\}*}(z_{ij})\iota^*_{\{ijk\}>\{ij\}}(z_{ij})
\end{aligned}$$

since by our sign convention ρ' and δ' always induce opposite signs in this situation. On the right hand side we use the Friedman relation:

$$\begin{aligned}
\big((\rho\circ\delta)(z_{ij})\big)_{ij} &= \Big(\rho\big(\iota_{\{ij\}>\{i\}*}(z_{ij}) - \iota_{\{ij\}>\{j\}*}(z_{ij})\big)\Big)_{ij} \\
&= \iota^*_{\{ij\}>\{i\}}\iota_{\{ij\}>\{i\}*}(z_{ij}) - \big(-\iota^*_{\{ij\}>\{j\}}\iota_{\{ij\}>\{j\}*}(z_{ij})\big) \\
&= N_{X_{ij}/X_i}\cdot z_{ij} + N_{X_{ij}/X_j}\cdot z_{ij} \\
&= \left(-\sum_k \iota_{\{ijk\}>\{ij\}*}(X_{ijk})\right)\cdot z_{ij} \\
&= -\sum_k \iota_{\{ijk\}>\{ij\}*}\iota^*_{\{ijk\}>\{ij\}}(z_{ij}).
\end{aligned}$$

□

One drawback of $\mathrm{CH}^*_{\mathrm{prelog}}(X)$ is that it is rather hard to compute with in examples, for instance it can be very far from being finitely generated. Instead, we would like to have an object constructed using numerical equivalence that receives at the very least an arrow from $\mathrm{CH}^*_{\mathrm{prelog}}(X)$.

Definition 2.9. Define the subring $R_{\mathrm{num}}(X)$ of $\bigoplus_i \mathrm{Num}^*(X_i)$, where $\mathrm{Num}^*(X_i)$ are cycles on X_i modulo numerical equivalence, as consisting of those tuples (α_i) such that α_i and α_j restrict to the same element in $\mathrm{Num}^*(X_i \cap X_j)$. Let $I_{\mathrm{num}}(X)$ be the extension of the ideal $\ker(\nu_* \mid_{R(X)}) \subset R(X)$ into $R_{\mathrm{num}}(X)$. Then we call

$$\mathrm{Num}^*_{\mathrm{prelog}}(X) = R_{\mathrm{num}}(X)/I_{\mathrm{num}}(X)$$

the *numerical prelog Chow ring*.

Proposition 2.10. *The natural projection map*

$$\varpi\colon R(X) \to R_{\mathrm{num}}(X)$$

maps classes in the kernel of $\nu_* \mid_{R(X)}$ into $I_{\mathrm{num}}(X)$. Hence we obtain an induced homomorphism

$$\bar\varpi\colon \mathrm{CH}^*_{\mathrm{prelog}}(X) \to \mathrm{Num}^*_{\mathrm{prelog}}(X).$$

Proof. This is clear by construction. □

3. Specialization homomorphisms into prelog Chow rings

We start by recalling some facts about specialization homomorphisms, following the paper [Ful75]. Let

$$\pi\colon \mathscr{X} \to C$$

be a flat morphism from a variety $\mathscr{X}$ to a nonsingular curve C. We fix a distinguished point $t_0 \in C$, and call X the scheme-theoretic fibre $\mathscr{X}_{t_0} = \pi^{-1}(t_0)$. Let $i\colon X \to \mathscr{X}$ be the inclusion. As in [Ful75, §4.1, p. 161] one can then define a "Gysin homomorphism"

$$i^*\colon \mathrm{CH}_k(\mathscr{X}) \to \mathrm{CH}_{k-1}(X)$$

by defining the map $i^*\colon Z_k(\mathscr{X}) \to Z_{k-1}X$ and checking that this descends to rational equivalence; on the level of cycles, if an irreducible subvariety V of $\mathscr{X}$ satisfies $V \subset X$, one defines $i^*(V) = 0$, and otherwise as $i^*(V) = V_{t_0}$, where V_{t_0} is the cycle associated to the zero scheme on V of a regular function defining X inside $\mathscr{X}$ in a neighborhood of X (notice that X is a principal Cartier divisor in $\mathscr{X}$ after possibly shrinking C). In the latter case, the class of V_{t_0}, well-defined as an element in $\mathrm{CH}_{k-1}(|X| \cap V)$, is then the intersection $X \cdot V$ of V with the Cartier divisor X; see also [Ful98, Chapter 2.3 ff.] for further information on this construction.

From now on we will want to work more locally on the base, hence assume that $C = \operatorname{Spec} R$ is a *curve trait*, by which we mean that R is a discrete valuation ring that is the local ring of a point on a nonsingular curve, or a completion of such a ring. By [Ful75, §4.4], all what was said above remains valid in this set-up. Let $\mathbb{C} = R/\mathfrak{m}$ be the residue field of R, and K the quotient field. We then have the special fibre $X_{\mathbb{C}} = X$ and generic fibre X_K with inclusions $i\colon X \to \mathscr{X}$ and $j\colon X_K \to \mathscr{X}$.

By [Ful75, Prop. in §1.9], there is an exact sequence

$$\mathrm{CH}_{p+1} X \xrightarrow{i_*} \mathrm{CH}_{p+1} \mathscr{X} \xrightarrow{j^*} \mathrm{CH}_p X_K \longrightarrow 0$$

and $i^* i_* = 0$, one gets that there is a unique map $\sigma = \sigma_{\mathscr{X}}$ making

$$\begin{tikzcd} & \mathrm{CH}_p X_K \arrow[dd, "\sigma"] \\ \mathrm{CH}_{p+1} \mathscr{X} \arrow[ur, "j^*"] \arrow[dr, "i^*"'] & \\ & \mathrm{CH}_p X \end{tikzcd}$$

This σ is called the *specialization homomorphism.*

Definition 3.1. The flat morphism $\pi\colon \mathscr{X} \to C$ with C a curve trait, or a nonsingular curve, with marked point $t_0 \in C$, is called a strictly semistable degeneration if $X = \pi^{-1}(t_0)$ is reduced and simple normal crossing and $\mathscr{X}$ is a regular scheme.

The advantage of $\mathscr{X}$ being regular is that then all the components X_i of X are Cartier divisors in $\mathscr{X}$, and we can form intersections with each of them separately.

Theorem 3.2. *If $\pi\colon \mathscr{X} \to C$ is a strictly semistable degeneration, σ takes values in $\mathrm{CH}^*_{\mathrm{prelog}}(X)$.*

Proof. The technical heart of the proof is that by [Ful98, Chapter 2], given a k-cycle α and Cartier divisor D on some algebraic scheme, one can construct an intersection class $D \cdot \alpha \in \mathrm{CH}_{k-1}(|D| \cap |\alpha|)$, satisfying various natural properties. If $\alpha = [Y]$ is the class of a subvariety Y then if $Y \not\subset |D|$, D restricts to a well-defined Cartier divisor on D whose associated Weil divisor is defined to be $D \cdot \alpha$; if $Y \subset |D|$, then one defines $D \cdot \alpha$ as the linear equivalence class of any Weil divisor associated to the restriction of the line bundle $\mathscr{O}(D)$ to Y.

Let now V be an irreducible subvariety of $\mathscr{X}$ not contained in X. We have $X = \bigcup_i X_i$, and all components X_i are Cartier. By [Ful98, Prop. 2.3 b)], one has in $\mathrm{CH}_*(|X| \cap V)$

$$X \cdot V = (\sum_i X_i) \cdot V = \sum_i X_i \cdot V.$$

Hence defining $\alpha_i = X_i \cdot V$ (viewed as classes in $\mathrm{CH}_*(X_i)$), we will have proved the Proposition once we show that (α_i) is in the ring $R(X)$ of compatible classes. This follows from the important commutativity property of the pairing between Cartier divisors and cycles on any algebraic scheme: if D, D' are Cartier divisors and β a cycle, then

$$D \cdot (D' \cdot \beta) = D' \cdot (D \cdot \beta)$$

as classes in $\mathrm{CH}_*(|D| \cap |D'| \cap |\beta|)$ by [Ful98, Cor. 2.4.2 of Thm. 2.4]. We now only have to unravel that this does indeed come down to the property we seek to prove: indeed,

$$X_j \cdot \alpha_i = X_i \cdot \alpha_j$$

and it follows from the definitions that we can compute $X_j \cdot \alpha_i$ as follows: X_j restricts to a well-defined Cartier divisor on the subvariety X_i, which is nothing but the Cartier divisor associated to $X_i \cap X_j$; intersecting that Cartier divisor with the cycle α_i on X_i gives $X_j \cdot \alpha_i$; analogously, for $X_i \cdot \alpha_j$ with the roles of i and j interchanged. Hence (α_i) is in $R(X)$. $\square$

Remark 3.3. The assumption that the total space $\mathscr{X}$ be nonsingular in Theorem 3.2 is essential, and it is useful to keep the following example in mind: take a degeneration of a family of plane conics into two lines, and then consider the product family of this with itself. The central fibre of the product family is a union of four irreducible components all of which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ glued together along lines of the rulings as in Figure 1 on the left-hand side. The cycle indicated in green is the specialization of the diagonal, and the red cycles 1, 2, 3 are all rationally equivalent, but the intersections with the green cycle are *not* rationally equivalent. Moreover, the central fiber in ① is not snc, but in fact, one can nevertheless define the prelog Chow ring as in Section 2: one only needs the intersections of every two irreducible components to be smooth for this, and all results of that Section, in particular, Proposition 2.3 hold with identical proofs if in addition the intersection of every subset of the irreducible components is still smooth, which is the case here. So what is wrong? The point is that the specialization of the diagonal is *not* in $\mathrm{CH}^*_{\mathrm{prelog}}(X)$ here. For example, the green cycle in ① on the upper left hand $\mathbb{P}^1 \times \mathbb{P}^1$ intersects the mutual intersection of the two upper $\mathbb{P}^1 \times \mathbb{P}^1$'s in a point, but there is no cycle on the upper right-hand $\mathbb{P}^1 \times \mathbb{P}^1$ to match this to satisfy the pre-log condition.

However, suppose we blow up one of the four components in the total space (this component is a non-Cartier divisor in the total space), thus desingularizing the total space and getting a new central fibre as in ②. Then the specialization of the diagonal will look like the green cycle and *is* in $\mathrm{CH}^*_{\mathrm{prelog}}(X)$. The red cycle in 1 here is equivalent to the red cycle in 2 and, if we try to move it across to the upper left-hand irreducible component, we see that the cycle 1 is equivalent to the dashed cycle 3, with multiplicities assigned to the irreducible components of the cycle as indicated. The intersection with the green cycle remains constant in accordance with Proposition 2.3: first it is 0, then it is $+1-1=0$ again.

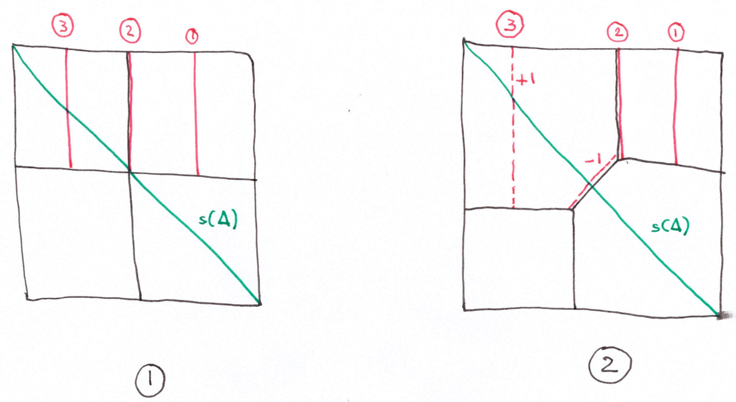

FIGURE 1.

4. Ramified base change and saturated prelog Chow groups

We keep working in the set-up of the previous Section, and consider a strictly semistable degeneration $\pi\colon \mathscr{X} \to C$. Suppose that $\beta\colon C' \to C$ is some cover of smooth curves or curve traits, in general ramified at the distinguished point $t_0 \in C$. Suppose t_0' is a distinguished point in C' mapping to t_0 under β. We consider the fibre product diagram

$$\begin{array}{ccc} \mathscr{X}' = \mathscr{X} \times_C C' & \xrightarrow{\beta'} & \mathscr{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ C' & \xrightarrow{\beta} & C \end{array}$$

Then $\mathscr{X}'$ will in general be singular. However, we can still prove that the specialization homomorphism $\sigma_{\mathscr{X}'}$ will take values, modulo torsion, in a group that is very similar to the prelog Chow ring of $X = \mathscr{X}_{t_0}$.

Definition 4.1. Let X be a simple normal crossing variety with normalization $\nu\colon X^\nu \to X$ as in Section 2. Then we define the ring $R(X)^{\mathbb{Q}}$ of *rational compatible classes* as the subring of $\mathrm{CH}^*(X^\nu)\otimes_{\mathbb{Z}}\mathbb{Q}$ consisting of tuples (α_i), $\alpha_i \in \mathrm{CH}^*(X_i)\otimes_{\mathbb{Z}}\mathbb{Q}$, such that α_i and α_j pull back to the same class in $\mathrm{CH}^*(X_i\cap X_j)\otimes_{\mathbb{Z}}\mathbb{Q}$. This may be different from $R(X)\otimes_{\mathbb{Z}}\mathbb{Q}$, but there is a natural map $R(X)\otimes_{\mathbb{Z}}\mathbb{Q} \to R(X)^{\mathbb{Q}}$.

Secondly, we define

$$\mathrm{Chow}_{\mathrm{prelog,sat},*}(X) := \mathrm{im}\left(\nu_*\colon R(X)^{\mathbb{Q}} \to \mathrm{CH}_*(X)\otimes_{\mathbb{Z}}\mathbb{Q}\right) \cap \left(\mathrm{CH}_*(X)/(\mathrm{torsion})\right)$$

and call it the *saturated prelog Chow group* of X.

Proposition 4.2. *With the notation introduced above, the specialization homomorphism $\sigma_{\mathscr{X}'}$ associated to $\pi'\colon \mathscr{X}' \to C'$ takes values in the group $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$ after we mod out torsion from $\mathrm{CH}_*(X)$.*

Proof. The punchline of the argument is very similar to that used in the proof of Theorem 3.2. The irreducible components X_i of X, viewed as the fibre of the family $\pi'\colon \mathscr{X}' \to C'$ over t_0', are $\mathbb{Q}$-Cartier, thus there is an integer N such that each $D_i := NX_i$ is Cartier. This is so because the X_i are Cartier in $\mathscr{X}$, and local equations of X_i in $\mathscr{X}$ pull back, under β', to local equations of X_i, with some multiplicity, inside $\mathscr{X}'$. Hence it makes sense, given an irreducible subvariety V of $\mathscr{X}'$ not contained in X, to form the intersection products $\alpha_i = D_i \cdot V$ and view them as classes in $\mathrm{CH}_*(X_i)$. In $\mathrm{CH}_*(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$, we can then define the classes $\gamma_i := (1/N)\alpha_i$. Clearly, we have again, as in the proof of Theorem 3.2,

$$NX \cdot V = (\sum_i D_i) \cdot V = \sum_i D_i \cdot V.$$

Thus $\sigma_{\mathscr{X}'}(V)$ and $\nu_*(\gamma_i)$ define the same class in $\mathrm{CH}_*(X)/(\text{torsion})$. Hence it remains to show that (γ_i) is in $R(X)^{\mathbb{Q}}$. We use once more the commutativity property for intersections of cycles with Cartier divisors:

$$D_i \cdot (D_j \cdot V) = D_j \cdot (D_i \cdot V)$$

holds in $\mathrm{CH}_*(|D_i| \cap |D_j| \cap |V|)$ by [Ful98, Cor. 2.4.2 of Thm. 2.4]. We need to convince ourselves that this indeed implies that (γ_i) is in $R(X)^{\mathbb{Q}}$. Immediately from the definition, we see that we can compute $D_i \cdot (D_j \cdot V)$ as follows: consider α_j as a class on X_j; D_i then restricts to a well-defined Cartier divisor on X_i, namely the Cartier divisor associated to $X_i \cap X_j$ (without multiplicity). Then $D_i \cdot (D_j \cdot V)$ is just the intersection of $X_i \cap X_j$ and α_j, taken on X_j. The same for $D_j \cdot (D_i \cdot V)$ with i and j interchanged. In other words, $N\gamma_i$ and $N\gamma_j$ pull-back to the same class in $\mathrm{CH}_*(X_i \cap X_j)$, hence the assertion. $\square$

Remark 4.3. Keeping the set-up of Proposition 4.2, let $\varrho\colon \mathscr{X}'' \to \mathscr{X}'$ be a proper birational morphism of schemes over C' such that $\mathscr{X}''$ is smooth and ϱ is an isomorphism outside the fibres over $t_0' \in C'$. We have a diagram

$$\begin{tikzcd} \mathscr{X}'' \arrow[r, "\varrho"] \arrow[dr, "\pi''"'] & \mathscr{X}' \arrow[d, "\pi'"] \\ & C' \end{tikzcd}$$

Then there is a commutative diagram for the specialization maps

$$\begin{tikzcd} \mathrm{CH}_*(\mathscr{X}''_\eta) \arrow[r, "(\varrho_L)_*"] \arrow[d, "\sigma_{\mathscr{X}''}"] & \mathrm{CH}_*(\mathscr{X}'_\eta) \arrow[d, "\sigma_{\mathscr{X}'}"] \\ \mathrm{CH}_*(X'') \arrow[r, "(\varrho|_{X''})_*"] & \mathrm{CH}_*(X') \end{tikzcd}$$

by part (1) of the Proposition on page 165 of [Ful75]. Here X' and X'' are the distinguished fibres of π' and π'', and $\mathscr{X}''_\eta$ and $\mathscr{X}'_\eta$ the generic fibres. In particular, $(\varrho\ |_{X''})_*$ maps the image of $\sigma_{\mathscr{X}''}$ inside $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X'')$ to the image of $\sigma_{\mathscr{X}'}$ inside $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X')$. It is not at all clear to us, though, if $(\varrho\ |_{X''})_*$ maps the entire subgroup $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X'')$ of $\mathrm{CH}_*(X'')/(\text{torsion})$ into $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X')$, although we observed this to be the case in some examples we considered. We leave it as an open question.

Lemma 4.4. *For a simple normal crossing variety, consider the exact sequence*

$$0 \longrightarrow K \longrightarrow \bigoplus_i \mathrm{CH}_*(X_i) \xrightarrow{\nu_*} \mathrm{CH}_*(X) \longrightarrow 0$$

where we recall that $K = \ker(\nu_*)$ can be computed using the diagram in Proposition 2.8. Then clearly the kernel $\bar{K}$ of the induced map of free $\mathbb{Z}$-modules

$$(\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion}) \xrightarrow{\nu_*} \mathrm{CH}_*(X)/(\mathrm{torsion}) \longrightarrow 0$$

is the saturation of the image of K inside $(\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion})$. We define the subring $\bar{R}(X)$ of $(\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion})$ as consisting of those tuples (α_i) such that α_i and α_j pull back to the same class in $\mathrm{CH}_*(X_i \cap X_j)/(\mathrm{torsion})$. The lattice $\bar{R}(X)$ might properly contain the image of $R(X)$ in $(\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion})$, but has the property that $\bar{R}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = R^{\mathbb{Q}}(X)$.

Then $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$ is naturally isomorphic to

$$(\bar{R}(X) + \bar{K})_{\mathrm{sat}}/\bar{K}$$

via ν_*, where the subscript "sat" denotes saturation in $(\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion})$.

Proof. Only the last assertion is possibly non-obvious: via ν_*, $(\bar{R}(X)+\bar{K})_{\mathrm{sat}}$ clearly maps into $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$, with kernel given by $\bar{K}$; we only need to verify that it maps surjectively. Thus let $x \in \mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$. It has a preimage $x' \in R^{\mathbb{Q}}(X)$ under ν_*, and a preimage $x'' \in (\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion})$ by definition. Then $k = x' - x''$ is in $\bar{K} \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, $x'' = x' - k$ is a class in $(\bigoplus_i \mathrm{CH}_*(X_i))/(\mathrm{torsion})$ a multiple of which is in $\bar{R}(X) + \bar{K}$. Hence x'' is a preimage of x under ν_* of x in $(\bar{R}(X) + \bar{K})_{\mathrm{sat}}$. □

As with the prelog Chow ring $\mathrm{CH}^*_{\mathrm{prelog}}(X)$, a problem with the saturated prelog Chow group is that it may in general be very non-finitely generated and hard to compute with. For that reason, we would like to work with some other group, constructed using numerical equivalence, and receiving at least a natural homomorphism from $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$.

Definition 4.5. Let K_{num} be the image of K in $\bigoplus_i \mathrm{Num}^*(X_i)$, and let $(R_{\mathrm{num}} + K_{\mathrm{num}})_{\mathrm{sat}}$ be the saturation in $\bigoplus_i \mathrm{Num}^*(X_i)$. Let $K_{\mathrm{num,sat}}$ be the saturation of K_{num} in $\bigoplus_i \mathrm{Num}^*(X_i)$. Then we call

$$\mathrm{Num}_{\mathrm{prelog,sat},*}(X) := (R_{\mathrm{num}} + K_{\mathrm{num}})_{\mathrm{sat}}/K_{\mathrm{num,sat}}$$

the *saturated numerical prelog Chow group* of X.

Proposition 4.6. *The natural projection*

$$q\colon (\bar{R}(X) + \bar{K})_{\mathrm{sat}} \to (R_{\mathrm{num}} + K_{\mathrm{num}})_{\mathrm{sat}}$$

maps $\bar{K}$ into $K_{\mathrm{num,sat}}$, hence induces a homomorphism of modules

$$\bar{q}\colon \mathrm{Chow}_{\mathrm{prelog,sat},*}(X) \to \mathrm{Num}_{\mathrm{prelog,sat},*}(X).$$

Proof. Clear by construction. □

5. Specializations of decompositions of the diagonal into (saturated) prelog Chow groups

In this Section we explain how we use the concept of prelog Chow groups and rings in a programme taking its point of departure from the degeneration method due to Voisin-Colliot-Thélène-Pirutka-Totaro-Schreieder et al.; the goal is to develop that method further so that it becomes possible to study stable rationality of very general fibres in families of varieties by looking at rather singular degenerations, for example toric degenerations as in the Gross-Siebert programme.

We recall some facts concerning the degeneration method in a way that will be suitable to develop our particular view point on it. We will follow [Voi15] and [CT-P16].

Definition 5.1. Let V be a smooth projective variety of dimension d (over any field K). We say that V has a *decomposition of the diagonal* if one can write

$$[\Delta_V] = [V \times p] + [Z] \quad \text{in } \mathrm{CH}_d(V \times V)$$

where p is a zero-cycle of degree 1 on V and $Z \subset V \times V$ is a cycle that is contained in $D \times V$ for some codimension 1 subvariety $D \subset V$.

This is equivalent to V being universally CH_0-trivial (meaning the degree homomorphism $\deg\colon \mathrm{CH}_0(V_{K'}) \to \mathbb{Z}$ is an isomorphism for any overfield $K' \supset K$) by [CT-P16, Prop. 1.4]. If it holds, it holds with p replaced by any other zero-cycle of degree 1, in particular for p a K-rational point if V has any. If V is smooth and projective and stably rational over K (or more generally retract rational), then V has a decomposition of the diagonal by [CT-P16, Lemm. 1.5].

We now consider a degeneration

$$\pi_{\mathscr{V}}\colon \mathscr{V} \to C$$

over a curve C with distinguished point t_0 and special fibre $V = \mathscr{V}_{t_0}$. We usually assume a general fibre of $\pi_{\mathscr{V}}$ to be rationally connected and smooth (in particular, all points will be rationally equivalent on it), since otherwise the question is not interesting. Let K be the function field of C. Suppose that a very general fibre of $\pi_{\mathscr{V}}$ is even stably rational, then this is also true for the geometric generic fibre $\mathscr{V}_{\bar{K}}$. Indeed, for b outside of a countable union of proper subvarieties of C we have a diagram

$$\begin{tikzcd} \mathscr{V}_{\bar{K}} \arrow[r, "\simeq", "j"'] \arrow[d] & \mathscr{V}_b \arrow[d] \\ \operatorname{Spec} K \arrow[r, "\simeq", "i"'] & \operatorname{Spec} \mathbb{C} \end{tikzcd}$$

for some isomorphism i and an isomorphism j of schemes. In particular, $\mathscr{V}_{\bar{K}}$ then has a decomposition of the diagonal and is universally CH_0-trivial.

Now suppose that we want to prove that a very general fibre of $\pi_{\mathscr{V}}$ is not stably rational. We would then assume the contrary, arguing by contradiction, and the classical degeneration method [CT-P16, Thm. 1.14] would then proceed as follows: *assume that the special fibre V is sufficiently mildly singular, in particular integral with a CH_0-universally trivial resolution of singularities $f\colon \tilde{V} \to V$ (meaning $f_*\colon \mathrm{CH}_0(\tilde{V}_{K'}) \to \mathrm{CH}_0(V_{K'})$ is an isomorphism for all overfields $K' \supset K$).* Then the fact that $\mathscr{V}_{\bar{K}}$ is universally CH_0-trivial would imply that $\tilde{V}$ is universally CH_0-trivial as well. $\tilde{V}$ being smooth, we can now use various obstructions, such as nonzero Brauer classes, to show that $\tilde{V}$ is in fact not universally CH_0-trivial and get a contradiction.

However, if we have that $\mathscr{V}_{\bar{K}}$ has a decomposition of the diagonal, then for some finite cover of smooth curves (or curve traits) $C' \to C$ with function field $\mathbb{C}(C') = L$, we have that $\mathscr{V}_L$ has a decomposition of the diagonal. Assume also that $\mathscr{V} \to C$ is strictly semistable (in applications, we may start with a more general degeneration and have to make some modifications before arriving at the strictly semistable set-up). Suppose then that in

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\varrho"] \arrow[dr, "\pi_{\mathscr{X}}"'] & \mathscr{V} \times_C \mathscr{V} \arrow[d] \\ & C \end{tikzcd}$$

the map ϱ is a birational morphism that is an isomorphism outside the central fibres, and such that $\mathscr{X}$ is again strictly semistable (this can be achieved according to [Har01, Prop 2.1] by some succession of blow-ups of components of the central fibre of $\mathscr{V} \times_C \mathscr{V} \to C$ that are not Cartier in the total space). Then we can consider the base change

$$\begin{tikzcd} \mathscr{X}' = \mathscr{X} \times_C C' \arrow[r, "\beta'"] \arrow[d, "\pi'"] & \mathscr{X} \arrow[d, "\pi"] \\ C' \arrow[r, "\beta"] & C \end{tikzcd}$$

and by Proposition 4.2, the specialization morphism $\sigma_{\mathscr{X}'}$ from $\mathrm{CH}_*(\mathscr{V}_L \times_L \mathscr{V}_L)$ into $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$ exists. In particular, we can apply $\sigma_{\mathscr{X}'}$ to an equation

$$[\Delta_{\mathscr{V}_L}] = [\mathscr{V}_L \times p] + [Z]$$

as in Definition 5.1, and see if we can derive a contradiction from assuming the resulting equality in $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$, or even more coarsely in $\mathrm{Num}_{\mathrm{prelog,sat},*}(X)$.

If we use the resolution procedure by Hartl in [Har01, Prop 2.1] to construct $\mathscr{X}$, every irreducible component $V_i \times V_j$ of the central fibre of $\mathscr{V} \times_C \mathscr{V} \to C$ will be birational, via ϱ, to an irreducible component X_{ij} of the central fibre X of $\mathscr{X} \to C$.

Definition 5.2. (a) Notation as above. We say that X has *a prelog decomposition of the diagonal in* $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$ if there is a class

$$[Z] \in \mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$$

that can be represented by a cycle Z that does not dominate any component of V when mapping it via

$$X \xrightarrow{\varrho|_X} V \times V \xrightarrow{\mathrm{pr}_1} V$$

and satisfies

$$\sigma_{\mathscr{X}'}\big([\Delta_{\mathscr{V}_L}]\big) - \sigma_{\mathscr{X}'}\big([\mathscr{V}_L \times p]\big) = [Z]$$

in $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$. If we replace $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$ by $\mathrm{Num}_{\mathrm{prelog,sat},*}(X)$ everywhere above, we say that X *has a prelog decomposition of the diagonal in* $\mathrm{Num}_{\mathrm{prelog,sat},*}(X)$.

(b) Suppose that $C = \mathrm{Spec}\,\mathbb{C}[[t]]$ above, $C' = \mathrm{Spec}\,\mathbb{C}[[s]]$, and β is given by $t = s^k$ for some positive integer k. In that case, [Har01, Prop. 2.2] tells us that there is a diagram

$$\begin{tikzcd} \mathscr{X}_k'' \arrow[r, "\varrho'"] \arrow[dr] & \mathscr{X}' \arrow[d] \\ & C' \end{tikzcd}$$

with $\mathscr{X}_k'' \to C'$ strictly semistable and ϱ' a succession of blow-ups in components of the central fibre that are non-Cartier. Let X_k'' be the central fibre of $\mathscr{X}_k'' \to C'$. Then suppose we have an equation in $\mathrm{CH}^*_{\mathrm{prelog}}(X_k'')$:

$$[\sigma_{\mathscr{X}_k''}(\Delta_{\mathscr{V}_L})] - [\sigma_{\mathscr{X}_k''}(\mathscr{V}_L \times p)] = [Z]$$

where Z is a cycle on X_k'' that does not dominate any component of V via

$$X_k'' \xrightarrow{\varrho'|_{X_k''}} X \xrightarrow{\varrho|_X} V \times V \xrightarrow{\mathrm{pr}_1} V.$$

In this case we say that X_k'' *has a prelog decomposition of the diagonal in* $\mathrm{CH}^*_{\mathrm{prelog}}(X_k'')$ *for the integer* k.

Proposition 5.3. *Notation as above.*

(1) *If $\mathscr{V}_{\bar{K}}$ is stably rational, then X has a prelog decomposition of the diagonal in* $\mathrm{Chow}_{\mathrm{prelog,sat},*}(X)$ *and* $\mathrm{Num}_{\mathrm{prelog,sat},*}(X)$.
(2) *If $\mathscr{V}_{\bar{K}}$ is stably rational, then X_k'' has a prelog decomposition of the diagonal in* $\mathrm{CH}^*_{\mathrm{prelog}}(X_k'')$ *for some integer k.*

Proof. We only need to notice the following: let $\mathscr{W} \to C'$ be one of $\mathscr{X}' \to C'$ or $\mathscr{X}'' \to C'$. Then we have a diagram:

$$\begin{tikzcd} \mathscr{W} \arrow[r] \arrow[dr] & \mathscr{V} \times_{C'} \mathscr{V} \arrow[d] \\ & C' \end{tikzcd}$$

If a cycle $\mathscr{Z}_0$ on $\mathscr{V} \times_{C'} \mathscr{V} \backslash (V \times V)$ is contained in $\mathscr{D}_0 \times_{C'} (\mathscr{V} \backslash V)$ for some relative divisor $\mathscr{D}_0 \in (\mathscr{V} \backslash V)$, then the closure of $\mathscr{Z}_0$ in $\mathscr{W}$ will map to the closure of $\mathscr{Z}_0$ in $\mathscr{V} \times_{C'} \mathscr{V}$ which is contained in $\mathscr{D} \times_{C'} \mathscr{V}$ where $\mathscr{D}$ is the closure of $\mathscr{D}_0$ in $\mathscr{V}$. Then $\mathscr{D}$ is also a relative divisor in $\mathscr{V}$. ☐

6. Saturated prelog Chow groups of degenerations of cubic surfaces

In this section we consider a degeneration of a smooth cubic surface S into three planes. More precisely let $f = 0$ be an equation of S and let

$$\begin{tikzcd} \mathscr{Y} \arrow[r] \arrow[d] & \mathbb{P}^3 \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r, equal] & \mathbb{A}^1 \end{tikzcd}$$

be the degeneration defined by

$$xyz - tf = 0.$$

Here x, y, z, w are coordinates of $\mathbb{P}^3$ and t is the coordinate of $\mathbb{A}^1$. The central fiber Y of $\mathscr{Y}$ consists of three coordinate planes. The singularities of $\mathscr{Y}$ lie on the intersection of two of these planes with $f = 0$. We assume f to be generic enough,

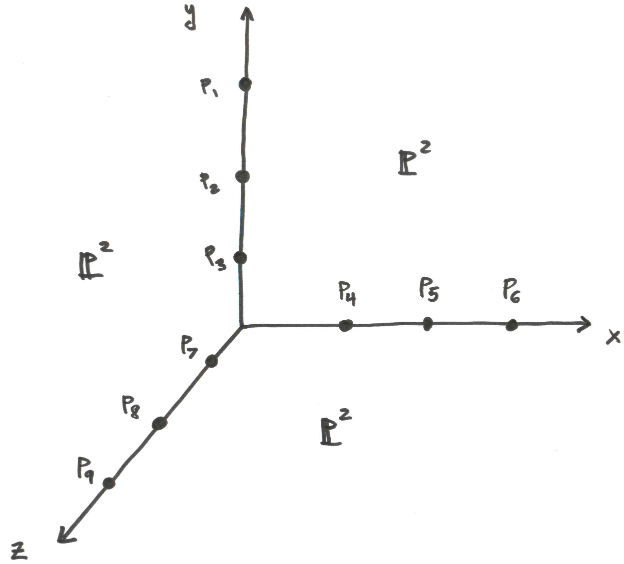

FIGURE 2.

such that these are three distinct points on each line, none of them in the origin. Figure 2 depicts this situation.

To desingularize $\mathscr{Y}$ we first blow up the $z = t = 0$ plane in $\mathscr{Y}$. In the central fiber this amounts to blowing up $P_1, \ldots, P_6$ in this plane. Next we blow up the strict transform of the $y = t = 0$ plane which amounts to blowing up P_7, P_8, P_9 in that plane. The third plane is left unchanged. We thus obtain a strictly semistable degeneration

$$\pi\colon \mathscr{X} \to \mathbb{A}^1$$

whose central fiber $X = X_1 \cup X_2 \cup X_3$ is skeched in Figure 3, where we have denoted the exceptional divisor over P_i by E_i.

Proposition 6.1. *In this situation*

$$\mathrm{CH}^1_{\mathrm{prelog}}(X) = \mathrm{CH}^1_{\mathrm{prelog,sat}}(X) = \mathbb{Z}^7.$$

which nicely coincides with $\mathrm{CH}^1(S)$.

Proof. For the convenience of the reader we give a slow walk through the necessary computations. A Macaulay2 script doing the same work is available at [BBG-M2].

We calculate the codimension 1 part of $\mathrm{CH}^*_{\mathrm{prelog}}(X)$. Here we have

$$\sum_{i=1}^{3} \mathrm{CH}_1(X_i) = \langle H_1, E_1, \ldots, E_6, H_2, E_7, \ldots, E_9, H_3 \rangle = \mathbb{Z}^{12}$$

The intersections of two components is always a $\mathbb{P}^1$ whose Chow group we can identify with $\mathbb{Z}$ via the degree map.

$$\sum_{1 \le i < j \le 3} \mathrm{CH}_1(X_{ij}) = \mathbb{Z}^3.$$

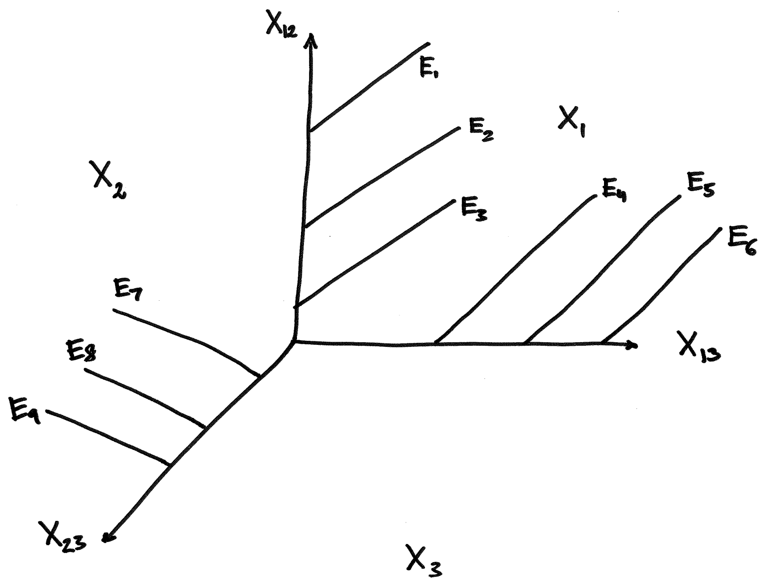

FIGURE 3.

Finally $X_{1,2,3}$ is a point with $\mathrm{CH}^0(X_{1,2,3}) = \mathbb{Z}$. The diagram 2.8 is therefore

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & R(X) \arrow[r] \arrow[d] & \mathrm{CH}^1_{\mathrm{prelog}}(X) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}^3 \arrow[r, "\delta"] \arrow[d, "\rho'"] & \mathbb{Z}^{12} \arrow[r, "\nu_*"] \arrow[d, "\rho"] & \mathrm{CH}_1(X) \arrow[r] & 0 \\
\mathbb{Z} \arrow[r, "\delta'"] & \mathbb{Z}^3 & &
\end{tikzcd}$$

Now notice that the image of $X_{1,2}$ in X_1 has the class

$$[\iota^*_{\{1,2\}>\{1\}}(X_{1,2})] = H_1 - E_1 - E_2 - E_3,$$

while its image in X_2 has the class

$$[\iota^*_{\{1,2\}>\{2\}}(X_{1,2})] = H_2.$$

Similarly

$$\begin{aligned}
[\iota^*_{\{1,3\}>\{1\}}(X_{1,3})] &= H_1 - E_4 - E_5 - E_6,\\
[\iota^*_{\{1,3\}>\{3\}}(X_{1,3})] &= H_3,\\
[\iota^*_{\{2,3\}>\{2\}}(X_{2,3})] &= H_2 - E_7 - E_8 - E_9,\\
[\iota^*_{\{2,3\}>\{3\}}(X_{2,3})] &= H_3.
\end{aligned}$$

With this we obtain

$$\delta = \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

Here for example the first line says that $X_{1,2}$ is mapped to $(H_1 - E_1 - E_2 - E_3) - H_2$. We also get

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}^t.$$

Here the first column says that H_1 intersects $X_{1,2}$ and $X_{1,3}$ but not $X_{2,3}$. Notice also that the sign convention says that the restriction of a divisor on X_1 is positive on both $X_{1,2}$ and $X_{1,3}$ because 1 is the smaller index in both cases. A divisor in X_2 is restricted with negative sign to $X_{1,2}$ but with positive sign to $X_{2,3}$. A divisor on X_3 is restricted with negative sign to both $X_{1,3}$ an $X_{2,3}$. Finally we get

$$\delta' = \begin{pmatrix} -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^t.$$

A good check that we got everything right is that indeed $\delta\rho = \rho'\delta'$ (this is the Friedman condition).

Now one can check that δ is injective and that the image of δ is saturated in $\mathbb{Z}^{12}$. This can be done for example by checking that the gcd of the 3×3 minors is 1. Similarly on can check that ρ is surjective.

This shows that the diagram reduces to

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathbb{Z}^9 \arrow[r] \arrow[d, "\varphi"] & \mathrm{CH}^1_{\mathrm{prelog}}(X) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}^3 \arrow[r, "\delta"] \arrow[d, "\rho'"] & \mathbb{Z}^{12} \arrow[r, "\nu_*"] \arrow[d, "\rho"] & \mathbb{Z}^9 \arrow[r] & 0 \\
\mathbb{Z} \arrow[r, "\delta'"] & \mathbb{Z}^3 & &
\end{tikzcd}$$

We can compute a representative matrix for φ by calculating generators for the kernel of ρ. Since the image of δ is saturated in $\mathbb{Z}^{12}$ a representative matrix for ν_* is obtained by

$$\ker(\delta^t)^t.$$

We can thus calculate $\varphi\nu_*$. One can check that this matrix has rank 7 and the image is saturated in $\mathbb{Z}^9$. Therefore

$$\mathrm{CH}^1_{\mathrm{prelog}}(X) = \mathrm{CH}^1_{\mathrm{prelog,sat}}(X) = \mathbb{Z}^7.$$

□

We now recall the classical log-geometric count of the 27 lines on a cubic surface. Assume that we have a degeneration

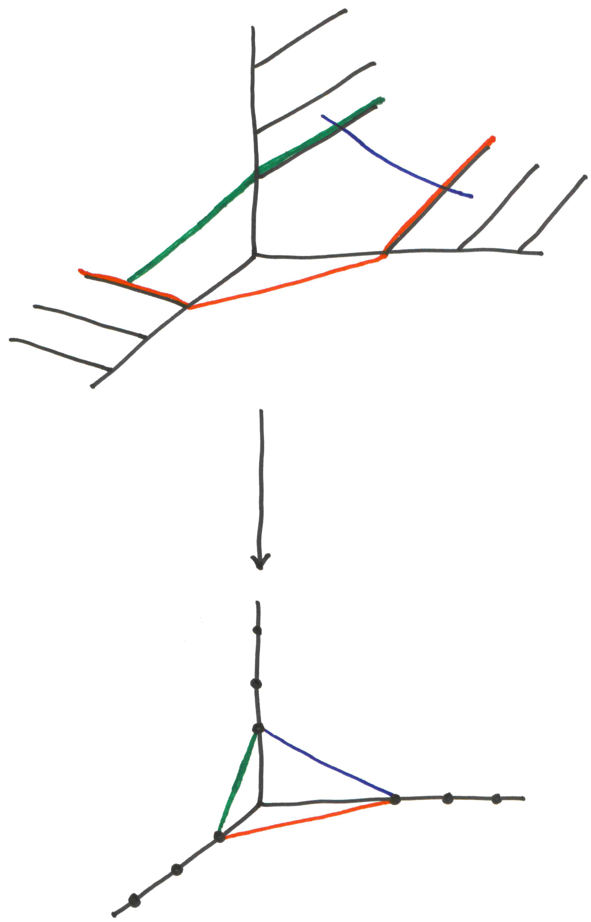

FIGURE 4.

$$\begin{tikzcd}
\mathscr{L} \arrow[r] \arrow[dr] & \mathscr{X} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[dl] \\
& \mathbb{A}^1 &
\end{tikzcd}$$

with the generic fiber of $\mathscr{L}$ a line in a cubic surface. The special fiber $L \subset Y$ is then a line in one of the coordinate planes. Its preimage $L \subset X$ is a prelog cycle. Because of the prelog condition L cannot intersect the coordinate lines outside of the singular points. Therefore we have on each of the 3 planes 3×3 possibilities to connect 2 singularities on the 2 adjacent coordinate lines. So in total we have $3 \cdot 3 \cdot 3 = 27$ possible "lines" on Y. Using log geometry one can then also prove that each of these lines indeed comes from a line on S and that each line in Y occurs only once as a limit.

Here we exhibit the 27 prelog cycles on X that map to the above 27 lines on Y:

Proposition 6.2. *The following 27 cycles on X are prelog cycles (see Figure 4):*

$$\begin{array}{rl} (H_1 - E_i - E_j, 0, 0) & i \in \{1,2,3\}, j \in \{4,5,6\} \\ (E_i, H_2 - E_j, 0) & i \in \{1,2,3\}, j \in \{7,8,9\} \\ (E_i, E_j, H_3) & i \in \{4,5,6\}, j \in \{7,8,9\} \end{array}$$

These are the preimages in X of lines connecting two singularities of $\mathscr{Y}$ in Y. One can choose 7 such cycles that generate $\mathrm{CH}^1_{prelog}(X)$.

Proof. In the first case $H_1 - E_i - E_j$ intersects neither $H_1 - E_1 - E_2 - E_3$, the images of $X_{1,2}$ in X_1, nor $H_1 - E_4 - E_5 - E_6$, the image of $X_{1,3}$ in X_1. Therefore there need not be cycles on X_2 or X_3 matching with it on $X_{1,2}$ or $X_{1,3}$.

In the second case E_j intersects the image of $X_{1,2}$ but not the image of $X_{1,3}$ on X_1. At the same time $H_2 - E_j$ intersects $X_{1,2}$ but not $X_{2,3}$ on X_2. Therefore there need not be a cycle on X_3 matching with them on the intersection. A similar reasoning proves the third case.

For a given set of 7 lines one can check the assertion by calculating their images in $\mathrm{CH}^1(X) = \mathbb{Z}^9$ and check whether they form a saturated sublattice of rank 7. A computer program can easily find a set of 7 cycles that works. One such is exhibited in [BBG-M2]. $\square$

7. Degenerations of elliptic curves and their prelog Chow rings

Certainly a smooth elliptic curve E is not stably rational and does not have a decomposition of the diagonal. To see the latter, suppose you could write, for rational or even just homological equivalence,

$$\Delta_E = E \times p + \sum_i a_i q_i \times E$$

with p, q_i points on E and $a_i \in \mathbb{Z}$. Then (1) intersecting both sides with $E \times p'$ for another point p', we would find (for homological equivalence) that $\Delta_E = E \times p + q \times E$ with q one point. Then (2) intersecting with the cycle $T = \{(x, x + p_0) \mid x \in E\}$, where $+p_0$ is translation on E by a point p_0 different from zero, gives a contradiction: the left hand side results in a class of degree 0 and the right hand side in a class of degree 2.

Whereas this is well known and easy, it is nevertheless reassuring that we can also deduce it from Proposition 5.3, considering a degeneration $\pi_{\mathscr{V}} \colon \mathscr{V} \to C$ of plane cubic curves into a triangle of lines V (we keep the notation of the preceding Section). Here $\pi_{\mathscr{V}}$ is strictly semistable, but if we form $\mathscr{V} \times_C \mathscr{V} \to C$, the total space has singularities at points where four components of the central fibre intersect (which also then fails to be simple normal crossing).

This gives rise to nine nodes of the total space. Blowing these up, we get the picture schematically depicted on the left of Figure 5: each node gets replaced by a $\mathbb{P}^1 \times \mathbb{P}^1$ that is a component of the central fibre. However, as components of the new scheme-theoretic central fibre, these $\mathbb{P}^1 \times \mathbb{P}^1$'s will have multiplicity 2, and the resulting family is not strictly semistable. Therefore we contract the lines of one ruling of each of these quadrics to arrive at the picture on the right of Figure 5 (we contract in the "North-West South-East direction" in the Figure). Here each hexagon is a $\mathbb{P}^2$ with three points blown up, and the intersection of two such "hexagons" is a (-1)-curve in each of the surfaces. In this way we obtain a strictly semistable modification $\pi \colon \mathscr{X} \to C$ of the product family. Here we do not follow the resolution scheme suggested in [Har01, Prop 2.1] as our method leads to a more

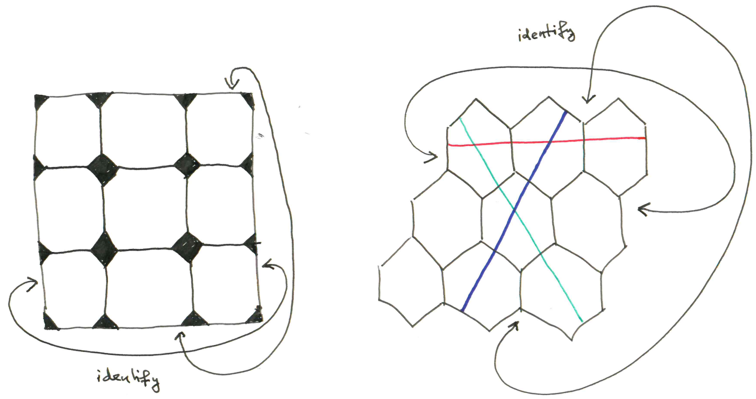

FIGURE 5.

symmetric central fibre. The green line in Figure 5 indicates the specialization of the diagonal, and the red line that of "elliptic curve times a point". The blue line is the specialization of "point times elliptic curve".

On each one of the hexagons, $\mathbb{P}^2$'s blown up in three points, there is thus a basis of the Picard group consisting of the pullback H of the hyperplane class, and the three exceptional divisors E_1, E_2, E_3. One can identify the boundary components of each hexagon, proceeding in clockwise direction, with $E_1, H - E_1 - E_2, E_2, H - E_2 - E_3, E_3, H - E_1 - E_3$.

Proposition 7.1. *The following hold:*

a) The classes of the red, green and blue lines in Figure 5 generate $\mathrm{CH}^1_{\mathrm{prelog}}(X)$ modulo torsion.

b) The classes of the red, green and blue lines in Figure 5 together with the half of their sum generate $\mathrm{CH}^1_{\mathrm{prelog,sat}}(X)$.

Proof. Part a) is a computation of the same type as in the proof of Proposition 6.1 and can be found in [BBG-M2]. We observe that in this case the diagram of Proposition 2.8 is of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathbb{Z}^{11} \arrow[r] \arrow[d, "\varphi"] & \mathrm{CH}^1_{\mathrm{prelog}}(X) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}^{27} \arrow[r, "\delta"] \arrow[d, "\rho'"] & \mathbb{Z}^{36} \arrow[r, "\nu_*"] \arrow[d, "\rho"] & \mathbb{Z}^{11} \oplus \mathbb{Z}/3 \arrow[r] & 0 \\
\mathbb{Z}^{18} \arrow[r, "\delta'"] & \mathbb{Z}^{27} & &
\end{tikzcd}$$

Let $\bar{\nu}_*$ be the composition of ν_* with the projection to $\mathbb{Z}^{11}$. The composition $\varphi \circ \bar{\nu}_*$ is an 11×11 matrix representing $\mathrm{CH}^1_{\mathrm{prelog}}(X)$ modulo torsion. A computation done in [BBG-M2] shows that the rank of this matrix is 3 in every characteristic except char $= 2$ where it is 2. The sum of the red, green and blue lines is divisible by 2, which gives b). □

Corollary 7.2. *There is no prelog decomposition of the diagonal in* $\mathrm{CH}^1_{\mathrm{prelog,sat}}(X)$.

Proof. Every prelog cycle not dominating via the first projection must be a multiple of the blue line. For a decomposition of the diagonal we would need the difference between the green and red lines to be some multiple of the blue line; this is impossible since the three classes are independent over $\mathbb{Z}$. □

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