Bounds on the growth of high Sobolev norms of solutions to nonlinear Schrödinger equations

by

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Abstract

In this thesis, we study the growth of Sobolev norms of global solutions of solutions to nonlinear Schrödinger type equations which we can't bound from above by energy conservation. The growth of such norms gives a quantitative estimate on the low-to-high frequency cascade which can occur due to the nonlinear evolution. In our work, we present two possible frequency decomposition methods which allow us to obtain polynomial bounds on the high Sobolev norms of the solutions to the equations we are considering. The first method is a high regularity version of the *I-method* previously used by Colliander, Keel, Staffilani, Takaoka, and Tao and it allows us to treat a wide range of equations, including the power type NLS equation and the Hartree equation with sufficiently regular convolution potential, as well as the Gross-Pitaevskii equation for dipolar quantum gases in the physically relevant 3D setting. The other method is based on a rough cut-off in frequency and it allows us to bound the growth of fractional Sobolev norms of the completely integrable defocusing cubic NLS on the real line.

Thesis Supervisor: Gigliola Staffilani
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Chapter 1

Introduction

1.1 General setup

Nonlinear Dispersive Partial Differential Equations model nonlinear wave phenomena which arise in various physical systems, such as the limiting dynamics of large Bose systems [92, 103], shallow water waves [79], and geometric optics [103]. These are nonlinear evolution equations whose solutions spread out as waves in the spatial domain if no boundary conditions are imposed. The most famous examples of nonlinear dispersive PDE are the Nonlinear Schrödinger equation (NLS), the Korteweg-de Vries equation (KdV), and the Nonlinear wave equation (NLW). A key feature of these equations is that they are Hamiltonian, and hence they possess an energy functional which is formally conserved under their evolution.

The tools used to study nonlinear dispersive PDE come from harmonic analysis and from Fourier analysis. If one studies the equations on periodic domains, one also has to apply techniques from analytic number theory, as was first done in the work of Bourgain [9]. All of these tools are primarily used in order to understand the dispersive properties of the linear part of the equation. These dispersive properties are manifested through an appropriate class of spacetime estimates known as *Strichartz estimates*, as we will recall below. A family of Strichartz estimates in the non-periodic setting was first proved in the work of Strichartz [98], and the endpoint case was resolved by Keel-Tao [69]. In the work of Bourgain [9], Strichartz estimates were proved in the periodic setting. An appropriate use of Strichartz estimates and a fixed point argument allows one to obtain local well-posedness in the critical or sub-critical regime[1]. In sub-critical regime, if one also has an a priori bound coming from conservation of an energy functional, one can easily obtain global well-posedness. Key contributions to the study of the Cauchy problem for nonlinear dispersive PDE were made by Ginibre-Velo [50, 51], Lin-Strauss [81], Kato [67], Hayashi-Nakamitsu-Tsutsumi [62], Cazenave-Weissler [29], Kenig-Ponce-Vega [72], and Bourgain [9]. These results mostly concern the subcritical regime. There has also been a substantial amount of work done in the critical regime. Global existence results were proved in the *energy-critical regime*, by Struwe [102], Grillakis [57, 58], Shatah-Struwe [94], Kapitanski [66], Bahouri-Shatah [3], Bahouri-Gérard [2], Bourgain [16], Nakanishi [88, 89], Tao [105], Kenig-Merle [70], Ryckman-Visan [91], Visan [109], Colliander-Keel-Staffilani-Takaoka-Tao [37], and in the *mass-critical regime* by Tao-Visan-Zhang [107], Killip-Visan-Zhang [76], Killip-Tao-Visan [74], Dodson [43, 44, 43].

Our work has focused on studying the qualitative properties of global solutions in the case of the NLS equation. The property that we want to understand is the transfer of energy from low to high frequencies. More precisely, we want to start out with initial data which are localized in frequency, and we want to see how fast a substantial part of the frequency support can flow to the high frequencies under the evolution of the NLS. As we will see below, one way to quantify this frequency cascade is through the growth in time of the high Sobolev norms of a solution u.

1.2 Statement of the problem

1.2.1 The NLS Cauchy problem

Given $s \geq 1$, we will consider the following general NLS Cauchy problem:

[1]The criticality of the equation is determined by scaling.

$$\begin{cases} iu_t + \Delta u = K(u), x \in X, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in H^s(X). \end{cases} \tag{1.1}$$

Here, $u = u(x,t)$, X is a spatial domain, t is the time variable, K is a nonlinearity, and $H^s(X)$ is the Sobolev space of index s on X. We will not study the general problem (1.1), but we will specialize to two possibilities in the spatial domain:

(i) $X = \mathbb{T}^d$ (Periodic setting).

(ii) $X = \mathbb{R}^d$ (Non-periodic setting).

We will restrict our attention to $d \leq 3$. For the nonlinearity $K(u)$, we will typically consider two main types:

(i) $K(u) = |u|^{2k}u$, for some $k \in \mathbb{N}$ (Defocusing algebraic nonlinearity of degree $2k+1$).

(ii) $K(u) = (V * |u|^2)u$, for some function $V : X \to \mathbb{R}$ (Hartree nonlinearity).

Here $*$ denotes convolution in the x-variable: $f * g(x) = \int_X f(y)g(x-y)dy$. We note that the first nonlinearity is local, i.e its value at a point x depends only on the value of u at x, whereas the second nonlinearity is non-local. Some $K(u)$ we will consider will be combinations and modifications of the algebraic and Hartree nonlinearities.

1.2.2 Conserved quantities

All the models we will consider will have conserved mass given by:

$$M(u(t)) = \int_X |u(x,t)|^2 dx. \tag{1.2}$$

and conserved energy given by:

$$E(u(t)) = \frac{1}{2}\int_X |\nabla u(x,t)|^2 dx + \int_X P(u(x,t))dx. \tag{1.3}$$

The part $P(u(x,t))$ depends on the nonlinearity. We note that $P(u) = \frac{1}{2k+2}|u|^{2k+2}$ for the algebraic nonlinearity of degree $2k+1$ and $P(u) = \frac{1}{4}(V * |u|^2)|u|^2$ for the Hartree nonlinearity. The first term in $E(u(t))$ is called the *kinetic energy* and the second term is called the *potential energy.* In all of the models that we will be studying, the potential energy will be non-negative. We call this type of problem *defocusing.* One can also study the *focusing* in which the nonlinearity $K(u) = -|u|^{2k-1}u$. For such problems, energy is non longer necessarily a non-negative quantity. In this work, we will restrict our attention to the defocusing problem though.

The fact that mass and energy are conserved can be formally checked by differentiating under the integral sign. A rigorous justification requires a density argument which uses the well-posedness of the Cauchy problem. An overview of these ideas is given in [106].

1.2.3 Global existence and a uniform bound

Existence of solutions to (1.1) locally in time can be shown by using a fixed point argument [50, 81]. Since the initial data lies in H^1, one can use the conservation of mass and energy, as well as the fact that the potential energy is non-negative, one can deduce the existence of global solutions From the conservation of mass and energy and the non-negativity of the potential energy, it follows that the H^1 norm of a global solution is uniformly bounded in time [50, 51, 29, 9]. Namely, the following bound holds:

$$\|u(t)\|_{H^1} \leq C(\Phi). \tag{1.4}$$

1.2.4 Low-to-high frequency cascade

We will be interested in obtaining bounds on the H^s norm of a solution. We recall that:

$$\|u(t)\|_{H^s} = \Big(\int |\widehat{u}(\xi,t)|^2 d\xi \Big)^{\frac{1}{2}}. \tag{1.5}$$

Here $\widehat{\cdot}$ denotes the Fourier transform on X, and the domain of integration is $\mathbb{R}^d$ when $X = \mathbb{R}^d$ and it is $\mathbb{Z}^d$ when $X = \mathbb{T}^d$. If we combine Plancherel's Theorem and the conservation of mass (1.2), it follows that the quantity:

$$\Big(\int |\widehat{u}(\xi,t)|^2 d\xi\Big)^{\frac{1}{2}}. \tag{1.6}$$

is constant in t. Hence, the area under the graph of the function $\xi \mapsto |\widehat{u}(\xi,t)|^2$ is the same for all times t. By (1.5) and (1.6), it follows that the growth of $\|u(t)\|_{H^s}$ for $s \gg 1$ gives us a quantitative estimate how much of the frequency support of u, i.e. support of $\xi \mapsto \widehat{u}(\xi,t)$, has shifted to the high frequencies, i.e to the set where $|\xi| \gg 1$. The latter phenomenon is called the *low-to-high frequency cascade* or *forward cascade*. We must note that it is not possible for the whole frequency support of u to transfer to the high frequencies by (1.4). Hence, the growth of high Sobolev norms effectively estimates how much a only a part of the frequency support of u has moved to the high frequencies. We note that this problem sometimes also goes under the name of *weak turbulence* or *wave turbulence* and it has been studied since the 1960s in the physics literature [60, 82, 117], and in the mathematical literature [6, 7]. The latter two papers were based on methods from probability theory. In the 1990s, the problem was also studied numerically [83]. The aim of all of the mentioned works is to obtain a statistical description of the forward cascade mechanism in weakly interacting dispersive wave models.

1.3 Previously known results

Suppose that u is a solution of (1.1). One can immediately obtain exponential bounds on the growth of Sobolev norms by iterating the local-in-time bounds coming from the local well-posedness of the equation. The main reason is that the increment time coming from local well-posedness is determined by the conserved quantities of the equation. More precisely, one recalls from [9, 15, 106] that there exist $\delta > 0$ and $C > 1$ depending only on the initial data such that for all times t_0:

$$\|u(t_0+\delta)\|_{H^s} \leq C\|u(t_0)\|_{H^s}. \tag{1.7}$$

We iterate (1.7) to obtain the exponential bound:

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi} e^{A|t|}. \tag{1.8}$$

It is, however, possible to obtain polynomial bounds. This was achieved for other nonlinear Schrödinger equations in [12, 27, 98, 99, 118]. The main idea in these papers was to modify (1.7) to obtain an improved iteration bound by which there exists a constant $r \in (0,1)$ depending on k, s and $\delta, C > 0$ depending also on the initial data such that for all times t_0:

$$\|u(t_0+\delta)\|_{H^s}^2 \leq \|u(t_0)\|_{H^s}^2 + C\|u(t_0)\|_{H^s}^{2-r}. \tag{1.9}$$

In [12], (1.9) is proved by the *Fourier multiplier method*, whereas in [98, 99], this bound is proved by using fine multilinear estimates. The key to the latter approach was in the use of smoothing estimates similar to those used in [73]. A slightly different approach, based on the analysis from [22], is used to obtain the same iteration bound in [27, 118].

One can show that (1.9) implies:

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi} (1+|t|)^{\frac{1}{r}}. \tag{1.10}$$

A slightly different approach to bounding $\|u(t)\|_{H^s}$ is given in [13]. In this work, one considers the defocusing cubic NLS on $\mathbb{R}^3$ and by an appropriate use of Strichartz estimates, it is shown that, given $\Phi \in H^1(\mathbb{R}^3)$, one obtains the following uniform bound on the localized Sobolev norms for the solution u:

$$\|u(t)\|_{H^s_{loc,x}(\mathbb{R}^3)} \leq C(\Phi) \tag{1.11}$$

Here $\|f\|_{H^s_{loc,x}(\mathbb{R}^3)} := \sup_{I\subseteq\mathbb{R}^3, I \text{ is a unit cube}} \|f\|_{H^s_x(I)}$, where $\|\cdot\|_{H^s_x(I)}$ denotes the corresponding restriction norm. However, the local bound (1.11) can be improved to a global one by using the results from [35] below, as we will see.

For certain NLS equations, one can deduce uniform bounds on $\|u(t)\|_{H^s}$ from scattering results. We recall that in the context of the NLS equation, the equation (1.1) is said to scatter in H^s if, for all $\Phi \in H^s$, there exist $u_\pm \in H^s$ such that:

$$\lim_{t \to \pm\infty} \|u(t) - S(t)u_\pm\|_{H^s} = 0. \tag{1.12}$$

where $S(t)$ denotes the free Schrödinger evolution operator. By unitarity of $S(\cdot)$ on H^s, it follows that (1.12) implies that $\|u(t)\|_{H^s}$ is uniformly bounded. Several H^{s_1}-scattering results have been shown NLS-type equations in [35, 37, 42, 43, 44, 75]. From H^{s_1}-scattering, one can deduce H^s-scattering if the initial data lies in H^s, for any $s \geq s_1$. This *persistence of regularity* result for scattering is sketched in Chapter 4.

Let us note that all of the mentioned scattering results hold on non-periodic domains. It is not expected to be possible to obtain scattering results on periodic domains due to weaker dispersion. This fact that L^2 scattering doesn't hold was precisely verified for the defocusing cubic NLS on $\mathbb{T}^2$ in [39].

Another special situation occurs when the NLS equation is completely integrable. For our purposes, this means that there exist infinitely many conservation laws, which in turn give bounds on all Sobolev norms of degree a positive integer. More precisely, if (1.1) is completely integrable, $k \in \mathbb{N}$, and $\Phi \in H^k$, then $\|u(t)\|_{H_k}$ is uniformly bounded in time. The most famous NLS equation which is completely is the cubic NLS on $\mathbb{R}$ and on $S^1 = \mathbb{T}^1$ [84]. Another completely integrable model is the derivative nonlinear Schrödinger equation (DNLS), which is defined on $\mathbb{R}$ as the spatial domain with the nonlinearity $K(u) = i\partial_x(|u|^2 u)$ [68]. We note that complete integrability doesn't immediately imply that $\|u(t)\|_{H^s}$ is uniformly bounded when s is not an integer, and the only assumption that we have on the initial data is that it belongs to H^s.

1.3.1 The linear Schrödinger equation with potential

The growth of high Sobolev norms also been studied for the linear Schrödinger equation with potential, namely for a real function $V = V(x,t)$, one considers the following linear PDE:

$$iu_t + \Delta u = Vu. \tag{1.13}$$

The growth of high Sobolev norms for (1.13) has been studied in [18, 17, 41, 111]. Under rather restrictive smoothness assumptions on V (for instance, in [18], V is taken to be jointly smooth in x and t with uniformly bounded partial derivatives with respect to both of the variables), it is shown that solutions to (1.13) satisfy for all $\epsilon > 0$ and all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi,\epsilon} (1+|t|)^{\epsilon}. \tag{1.14}$$

in [18], and, for some $r > 0$

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi} log(1+|t|)^{r}. \tag{1.15}$$

in [17, 111]. The latter result requires even stronger assumptions on V.

The idea of the proof of (1.14),(1.15) is to reduce the problem to one that is periodic in time and then to use localization of eigenfunctions of a certain linear differential operator together with separation properties of the eigenvalues of the Laplace operator on S^1. These separation properties can be deduced by elementary means on S^1. In [18], the bound (1.14) is also proved on S^d, for $d \geq 2$. In this case, the separation properties are proved by a more sophisticated number theoretic argument.

A different proof of (1.14) was later given in [41]. The argument given in [41] is based on an iterative change of variable. In addition to recovering the result (1.14) on any d-dimensional torus, the same bound is proved for the linear Schrödinger equation on any Zoll manifold, i.e. on any compact manifold whose geodesic flow is periodic. Moreover, in [110], it was shown that one can even obtain uniform bounds on $\|u(t)\|_{H^s}$ if one assumes certain spectral properties related to the potential V. These properties can be checked for the special potential $V(x,t) = \delta \cos x \cos t$ when $\delta \ll 1$ is sufficiently small.

It would be an interesting project to obtain bounds of the type (1.14) for an NLS equation evolving from smooth initial data. Here, we have to restrict to an NLS equation for which H^s-scattering is not known. Namely, as we noted above, H^s-scattering implies uniform bounds on $\|u(t)\|_{H^s}$. Since one is considering initial data, one should also consider an NLS equation which is not completely integrable. Hence, a good model to consider would be the defocusing quintic NLS equation on S^1. A possible approach to deduce (1.14) to substitute $V = |u|^{2k}$ into (1.13) and bootstrap polynomial bounds on $\|u(t)\|_{H^s}$ by applying the technique from [18] to obtain better bounds. However, there doesn't seem to be a simple way to implement this approach. The reason is that the reduction to the problem which is periodic in time doesn't work as soon as one has some growth in time of a fixed finite number of Sobolev norms.

The problem of Sobolev norm growth was also recently studied in [39], but in the sense of bounding the growth from below. In this paper, the authors exhibit the existence of smooth solutions of the cubic defocusing nonlinear Schrödinger equation on $\mathbb{T}^2$ whose H^s norm is arbitrarily small at time zero and is arbitrarily large at some large finite time. The work [39] is related to work of Kuksin [80] in which the author considers the case of small dispersion. By an appropriate rescaling, this can be shown to be equivalent to studying the same problem as in [39] with large initial data.

Furthermore, if one starts from a specific initial data containing only five frequencies, an analysis of which Fourier modes become excited has recently been studied in [25] by different methods. One should note that both papers study the behavior of the high Sobolev norms at a large finite time and that behavior at infinity is still an open problem.

Let us remark that in the mentioned works it is not clear if the constructed solution u satisfies $\limsup_{t\to+\infty} \|u(t)\|_{H^s} = \infty$ for $s \gg 1$. The only known constructions of solutions to nonlinear dispersive PDEs with divergent high Sobolev norms are due to Bourgain [10, 11, 12]. In these papers, the KdV, NLS, and nonlinear Wave-type equation are studied respectively. However, one has to modify the original equations to look at a spectrally defined Laplacian or nonlinearity. The result in [12] gives a powerlike lower bound on the growth. The techniques are based on perturbation from the linear equations. It is not clear how to modify these methods to use them for the standard dispersive models. Furthermore, we note that if one considers a linear Schrödinger equation with an appropriate random potential, the H^1 norm grows at least like a power of t almost surely [18].

A different way of modifying the NLS equation leads to the *cubic Szegö equation*:

$$\begin{cases} iu_t = \Pi(|u|^2 u), x \in S^1, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in L^2_+(S^1). \end{cases} \tag{1.16}$$

Here, $L^2(S^1)$ is the closed subspace of $L^2(S^1)$ of functions having Fourier coefficients with only non-negative indices, i.e. of the form $\sum_{k\in\mathbb{N}_0} f_k e^{ikx}$ and $\Pi : L^2(S^1) \to L^2_+(S^1)$ is the projection operator:

$$\Pi(\sum_{k\in\mathbb{Z}} a_k e^{ikx}) := \sum_{k\in\mathbb{N}_0} a_k e^{ikx}.$$

The operator Π is called the *Szegö projection*.

The analogous instability result to the one obtained in [39] for the equation (1.16) was recently obtained by Gérard and Grellier in [47] by using methods from complex analysis. It is also shown that (1.16) is completely integrable. On the other hand, there is no dispersive term in the equation, so the instability result is not unexpected.

1.4 Main ideas of our proofs

The main step in all of our proofs is to obtain a good iteration bound, based on an appropriate frequency decomposition. The iteration bounds we will use will usually not be as dependent on the structure of the nonlinearity as the iteration bound (1.9). As we will see, there will be essentially two different iteration bounds we will use, one with a smooth frequency cut-off, and one with a rough frequency cut-off. We always use the smooth frequency cut-off in the periodic setting, as the smoothness allows us to compensate for the lack of many dispersive estimates. In the non-periodic setting, we can use both types of frequency cut-offs, but in practice the rough cut-off is more useful only in the case of estimating the growth of fractional Sobolev norms of solutions to completely integrable equations such as the Cubic NLS on $\mathbb{R}$.

1.4.1 The smooth cut-off; the *upside-down I-method*

We will use the idea, used in [18, 17, 111], of estimating the high-frequency part of the solution. Let E^1 denote an operator which, after an appropriate rescaling, essentially adds the square L^2 norm of the low frequency part and the square H^s norm of the high frequency part of a function. The threshold between the low and high frequencies is the parameter $N > 1$. With this definition, we want to show that there exist $\beta > 0$, depending on the nonlinearity and spatial domain and $\delta, C > 0$ depending only on Φ such that for all times t_0:

$$E^1(u(t_0+\delta)) \leq (1+\frac{C}{N^\beta})E^1(u(t_0)). \tag{1.17}$$

One observes that (1.17) is more similar to (1.7) than to (1.9). The key fact to observe is that, due to the present decay factor, iteration of (1.17) $O(N^{\frac{1}{2}-})$ times doesn't cause exponential growth in $E^1(u(t))$, as it did for $\|u(t)\|_{H^s}$ in (1.8). We note that it is more difficult to obtain the decay factor in the periodic setting, than in the non-periodic setting.

We take:

$$E^1(f) := \|\mathcal{D}f\|_{L^2}^2. \tag{1.18}$$

Here $\mathcal{D}$ is an appropriate Fourier multiplier. In this paper, we take the $\mathcal{D}$-operator to be an *upside-down I-operator*, corresponding to high regularities. We construct $\mathcal{D}$ in such a way that:

$$\|\mathcal{D}f|_{L^2} \lesssim \|f\|_{H^s} \lesssim N^s\|\mathcal{D}f\|_{L^2}. \tag{1.19}$$

The operator $\mathcal{D}$ is the opposite from the standard *I-operator*, which was first developed in the work of Colliander-Keel-Staffilani-Takaoka-Tao (I-Team) [31, 32, 33, 34, 36].The idea of using an *upside-down I-operator* first appeared in [33], but in the low regularity context. The purpose of such an operator is to control the evolution of a Sobolev norm which is higher than the norm associated to a particular conserved quantity.

We then want to estimate:

$$\int_I \frac{d}{dt} \|\mathcal{D}u(t)\|_{L^2}^2 dt. \tag{1.20}$$

over an appropriate time interval I whose length depends only on the initial data.

Similarly as in the papers by the I-Team, the multiplier θ corresponding to the operator $\mathcal{D}$ is not a rough cut-off. Hence, in frequency regimes where certain cancelation occurs, we can symmetrize the expression and see how the cancelation manifests itself in terms of θ, as in [33]. If there is no cancelation in the symmetrized expression, we need to look at the spacetime Fourier transform. Arguing as in [22, 118], we decompose our solution into components whose spacetime Fourier transform is localized in the parabolic region $\langle \tau + |\xi|^2 \rangle \sim L$. In each of the cases, we obtain a satisfactory decay factor. The mentioned symmetrizations and localizations allow us to compensate for the absence of an *improved Strichartz estimate* when working in the periodic setting. The localization to parabolic regions is particularly useful in the case of quintic and higher order NLS on S^1.

In certain cases, we can add a multilinear correction to the quantity $E^1(u(t))$, as defined in (1.18) to obtain a quantity $E^2(u(t))$ which is equivalent to $E^1(u(t))$, but is even more slowly varying, i.e. for which β in (1.17) is even larger. The idea is to choose the correction to be such that $\frac{d}{dt}E^2(u(t))$ contains the same number of x derivatives as $\frac{d}{dt}E^1(u(t))$, but that these derivatives are distributed over more factors of u, thus making $E^2(u(t))$ even more slowly varying. Heuristically, we can view this is a way of artificially adding more dispersion to the problem. This method is called the *method of higher modified energies*, and was previously in the context of a modification of the I-method in [32, 33]. The method of higher modified energies comes from the general principle is reminiscent of the method of Birkhoff normal forms [4, 112] used in KAM theory. We apply the mentioned method in one and two-dimensional problems, both in the periodic and in the non-periodic setting. The two-dimensional setting is more subtle due to orthogonality issues that arise in studying the resonant frequencies.

1.4.2 The rough cut-off

We again start with a threshold N between the low and the high frequencies. Here, we take the rough projection Q defined by:

$$\widehat{Qf}(\xi) := \chi_{|\xi| \geq N} \hat{f}(\xi). \tag{1.21}$$

The main idea of the method is to look at the high and low-frequency part of the solution u similarly as in [18], and, in addition, to use the bound on the integral Sobolev norms that one obtains from the complete integrability. Namely, for $k \in \mathbb{N}$:

$$\|u(t)\|_{H^k} \leq B_k(\Phi) \tag{1.22}$$

From (1.22), we can deduce that for all times t:

$$\|(I - Q)u(t)\|_{H^s} \leq C(\Phi)N^{\alpha}. \tag{1.23}$$

where $\alpha := s - \lfloor s \rfloor \in [0, 1)$ is the fractional part of s. We note that the power of N is then in $[0, 1)$ and is not s, as in (1.19). We use the estimate (1.23) to bound the low-frequency part of the solution.

The key is then bound $\|Qu(t)\|_{H^s}$. This is the point at which we have to find the appropriate iteration bound. We want to show there exists $\beta > 0$ depending on the equation, an increment $\delta > 0$, and $C > 0$, both depending only on the initial data such that for all $t_0 \in \mathbb{R}$, one has:

$$\|Qu(t_0 + \delta)\|_{H^s}^2 \leq (1 + \frac{C}{N^{\beta-}})\|Qu(t_0)\|_{H^s}^2 + B_1. \tag{1.24}$$

Here, $B_1 = O(N^{\gamma}$, for some $\gamma > 0$ independent of s. The idea now is to iterate (1.24) for times $t_0 = 0, \delta, \ldots, n\delta$, where $n \in \mathbb{N}$ is an integer such that $n \lesssim N^{\beta-}$, and to telescope to obtain bounds on $\|Qu(t)\|_{H^s}$.

This approach has been used to give bounds on the growth of fractional Sobolev norms for the defocusing cubic NLS on $\mathbb{R}$. We have been able to derive only on $\mathbb{R}$ and not on S^1. Our proof relies heavily on the fact that we are working on a non-periodic domain since we have to use *improved bilinear Strichartz estimates*, which are known not to hold in the periodic setting.

1.5 Some notation and conventions

We denote by $A \lesssim B$ an estimate of the form $A \leq CB$, for some $C > 0$. If C depends on d, we write $A \lesssim_d B$. We also write the latter condition as $C = C(d)$. Given a real number r, we denote by $r+$ the number $r + \epsilon$, where we take $0 < \epsilon \ll 1$. The number $r-$ is defined analogously as $r - \epsilon$. For $1 \leq p < \infty$, we define:

$$\|f\|_{L^p_x(X)} := \Big(\int_X |f(x)|^p dx \Big)^{\frac{1}{p}}.$$

and we write:

$$\|f\|_{L^\infty_x(X)} := ess\ sup_{x \in X} |f(x)|.$$

Furthermore, given $1 \leq q, r \leq \infty$, we write:

$$\|w\|_{L^q_t L^r_x(X \times \mathbb{R})} := \| \|w(\cdot, t)\|_{L^q_x(X)} \|_{L^r_t(\mathbb{R})}.$$

If $q = r$, we observe that this is the norm $\| \cdot \|_{L^q_{x,t}(X \times \mathbb{R})}$. We usually write the norms as $\| \cdot \|_{L^p}$ or $\| \cdot \|_{L^q_t L^r_x}$ when there is no confusion. Given $1 \leq p \leq \infty$, we define its *Hölder conjugate exponent* $1 \leq p' \leq \infty$ by the formula: $\frac{1}{p} + \frac{1}{p'} = 1$.

The Fourier transform

Given $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$, and $f \in L^2(X)$, we define the spatial Fourier transform by:

$$\widehat{f}(\xi) := \int_X f(x) e^{-i\langle x,\xi\rangle} dx. \tag{1.25}$$

When $X = \mathbb{R}^d$, the Fourier transform is defined on $\mathbb{R}^d$, and when $X = \mathbb{T}^d$, it is defined on $\mathbb{Z}^d$. In this case, we will usually denote the ξ by n. $\langle \cdot, \cdot \rangle$ is defined to be the L^2-inner product on $\mathbb{R}^d$, and $\mathbb{Z}^d$, when $X = \mathbb{R}^d$ and $X = \mathbb{T}^d$ respectively. A key fact is *Plancherel's Theorem*:

$$\|\widehat{f}\|_{L^2} \sim \|f\|_{L^2} \tag{1.26}$$

Given $w \in L^2(X \times \mathbb{R})$, we also define the spacetime Fourier transform by:

$$\widetilde{u}(\xi, \tau) := \int_X \int_{\mathbb{R}} u(x,t) e^{-i\langle x,\xi\rangle - it\tau} dt dx. \tag{1.27}$$

Sobolev spaces

Let us take the following convention for the Japanese bracket $\langle \cdot \rangle$:

$$\langle x \rangle := \sqrt{1 + |x|^2}. \tag{1.28}$$

Let us recall that we are working in Sobolev Spaces $H^s = H^s(X)$ on the the domain X, whose norms are defined for $s \in \mathbb{R}$ by:

$$\|f\|_{H^s} = \Big(\int |\widehat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}}. \tag{1.29}$$

where $f : X \to \mathbb{C}$. Let us define $H^\infty(X) := \bigcap_{s>0} H^s(X)$.

Free Schrödinger propagator

We let $S(t)$ denote the free Schrödinger propagator. Namely, given $\phi \in L^2(X)$, the solution to:

$$\begin{cases} iu_t + \Delta u = 0, x \in X, t \in \mathbb{R} \\ u(x,0) = \phi(x). \end{cases} \tag{1.30}$$

is denoted by $u(x,t) = (S(t)\phi)(x)$. By using the Fourier transform in x, one can check that:

$$(S(t)\phi)\widehat{\ }(\xi) = e^{-it|\xi|^2}\widehat{\phi}(\xi). \tag{1.31}$$

From Plancherel's Theorem (1.26), it follows that $S(t)$ acts unitarily on L^2-based Sobolev spaces.

$X^{s,b}$ spaces

An important tool in our work will also be $X^{s,b}$ spaces. These spaces come from the norm defined for $s, b \in \mathbb{R}$:

$$\|u\|_{X^{s,b}} := \Big(\int \langle \xi \rangle^{2s} \langle \tau + |\xi|^2 \rangle^{2b} |\widetilde{u}(\xi,\tau)|^2 d\tau d\xi \Big)^{\frac{1}{2}}. \tag{1.32}$$

where $u : X \times \mathbb{R} \to \mathbb{C}$. The $X^{s,b}$ spaces can be defined for general dispersive equations and are sometimes also called *Dispersive Sobolev spaces.* These spaces were first used in their present form in the work of Bourgain [9]. A similar type of space was previously used in the study of the one-dimensional wave equation by Beals [5] and Rauch-Reed [90]. Implicitly, $X^{s,b}$ spaces also appeared in the context of spacetime estimates for null-forms in the work of Klainerman and Machedon [78].

The $X^{s,b}$ spaces obey the structure of the linear Schrödinger equation. If $S(t)$ denotes the linear Schrödinger propagator as above, then one can check that $(S(t)\phi)\widetilde{\ }(\xi,\tau)$ is supported on the paraboloid $\tau + |\xi|^2 = 0$. Hence, the $X^{s,b}$ norm heuristically speaking measures how far the function u is from being a solution to the free Schrödinger equation. Furthermore, we can write the $X^{s,b}$ norm as: $\|u\|_{X^{s,b}} = \|S(-t)u\|_{H^b_t H^s_x}$. If there is possibility of confusion, we write $X^{s,b}$ as $X^{s,b}(X \times \mathbb{R})$ to emphasize that we are working on the spacetime domain $X \times \mathbb{R}$.

Littlewood-Paley decomposition

Given a function $v \in L^2_{x,t}(X \times \mathbb{R})$, and a dyadic integer N, we define the function v_N as the function obtained from v by restricting its spacetime Fourier Transform to the region $|\xi| \sim N$. We refer to this procedure as a *dyadic decomposition* or *Littlewood-Paley decomposition*. In particular, we can write each function as a sum of such dyadically localized components:

$$v \sim \sum_{dyadic\ N} v_N.$$

Multilinear expressions

We give some useful notation for multilinear expressions, which was first used in [31]. Let us first explain the notation when $X = \mathbb{T}^d$.

For $k \geq 2$, an even integer, we define the hyperplane:

$$\Gamma_k := \{(n_1, \dots, n_k) \in (\mathbb{Z}^d)^k : n_1 + \cdots + n_k = 0\},$$

endowed with the measure $\delta(n_1 + \cdots + n_k)$. Given a function $M_k = M_k(n_1, \dots, n_k)$ on Γ_k, i.e. a *k-multiplier*, one defines the *k-linear functional* $\lambda_k(M_k; f_1, \dots, f_k)$ by:

$$\lambda_k(M_k; f_1, \dots, f_k) := \int_{\Gamma_k} M_k(n_1, \dots, n_k) \prod_{j=1}^{k} \widehat{f_j}(n_j).$$

As in [31], we adopt the notation:

$$\lambda_k(M_k; f) := \lambda_k(M_k; f, \bar{f}, \dots, f, \bar{f}). \tag{1.33}$$

We will also sometimes write n_{ij} for $n_i + n_j$, and n_{ijk} for $n_i + n_j + n_k$, etc.

When $X = \mathbb{R}^d$, we analogously define:

$$\Gamma_k := \{(\xi_1, \dots, \xi_k) \in (\mathbb{R}^2)^k : \xi_1 + \cdots + \xi_k = 0\}.$$

In this case, the measure on Γ_k is induced from Lebesgue measure $d\xi_1 \cdots d\xi_{k-1}$ on $(\mathbb{R}^2)^{k-1}$ by pushing forward under the map:

$$(\xi_1, \dots, \xi_{k-1}) \mapsto (\xi_1, \dots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}).$$

1.6 General facts from harmonic analysis

Strichartz estimates in the non-periodic setting

As was mentioned above, a fundamental tool we will have to use will be the *Strichartz estimates*

Theorem 1.6.1. *(Strichartz estimates for the Schrödinger equation) We consider the domain $\mathbb{R}^d$, and we say that a pair (q,r) is admissible if $2 \leq q, r \leq \infty$, $(q,r,d) \neq (2,\infty,2)$, and if the following relation is satisfied: $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. If (q,r) and (q_1,r_1) are admissible exponents, then the following homogeneous Strichartz estimate holds:*

$$\|S(t)\phi\|_{L^q_t L^r_x(\mathbb{R}^d \times \mathbb{R})} \lesssim_{d,q,r} \|\phi\|_{L^2_x(\mathbb{R}^d)}. \tag{1.34}$$

In addition, one has the inhomogeneous Strichartz estimate:

$$\|\int_{\sigma<t} S(t-\sigma)F(\sigma)d\sigma\|_{L^q_t L^r_x(\mathbb{R}^d \times \mathbb{R})} \lesssim_{d,q,r,q_1,r_1} \|F\|_{L^{q_1'}_t L^{r_1'}_x(\mathbb{R}^d \times \mathbb{R})}. \tag{1.35}$$

The non-endpoint case, i.e. when $q, q_1 \neq 2$ was first proved in [53, 114] and was based on the work of Strichartz [101]. The latter, in turn, was motivated by earlier harmonic analysis results in [93, 108]. The endpoint case $q, q_1 = 2$ was resolved in [69]. The reason why the endpoint case is so difficult is that the endpoint version of the Hardy-Littlewood-Sobolev inequality that one would like to use doesn't hold. The authors of [69] get around this difficulty by using an appropriate dyadic decomposition.

In the mentioned papers, the key to prove Theorem 1.6.1 is to use the *dispersive estimate*:

$$\|S(t)\phi\|_{L^\infty_x(\mathbb{R}^d)} \lesssim \frac{1}{t^{\frac{d}{2}}}\|\phi\|_{L^1_x(\mathbb{R}^d)}. \tag{1.36}$$

and combine it with an appropriate TT^* *argument*. The bound (1.36) is shown as a consequence of the convolution representation of $S(t)\phi$ and Young's inequality.

Strichartz estimates in the periodic setting

Strichartz estimates are more difficult to prove on compact domains due to weaker dispersion. We observe that on a compact domain, the dispersive estimate (1.36) can't hold, since we can't have decay of the L^∞ norm and conservation of the L^2 norm. The local-in-time periodic analogue of (1.34), which is:

$$\|S(t)\phi\|_{L^{\frac{2(d+2)}{d}}_{t,x}(\mathbb{T}^d\times[0,1])} \lesssim \|\phi\|_{L^2_x(\mathbb{T}^d)} \tag{1.37}$$

is known not to hold. In [9], it is shown that, when $d = 1$ and $N \in \mathbb{N}$, one has:

$$\|\sum_{n=1}^{N} e^{inx-in^2t}\|_{L^6_{t,x}(\mathbb{T}\times[0,1])} \gtrsim (\log N)^{\frac{1}{6}}N^{\frac{1}{2}}.$$

Hence (1.37) can't hold in general. However, some positive results are known. They either require q to be smaller than $\frac{2(d+2)}{d}$ or that the function ϕ be localized in frequency. In the latter case, one obtains a loss of derivative on the right-hand side of the inequality. More precisely, the bounds that one could expect are:

$$\|S(t)\phi\|_{L^q_{t,x}(\mathbb{T}^d\times[0,1])} \lesssim \|\phi\|_{L^2_x(\mathbb{T}^d)}, \text{ when } q < \frac{2(d+2)}{d}. \tag{1.38}$$

$$\|S(t)\phi\|_{L^q_{t,x}(\mathbb{T}^d\times[0,1])} \lesssim N^{(\frac{d}{2}-\frac{d+2}{q})+}, \text{ when } q \geq \frac{2(d+2)}{d}, \; supp\, \widehat{\phi} \subseteq B(0,N). \qquad (1.39)$$

The first work dealing with these questions was that of Bourgain [9] in which (1.38) was proved in the special case $q = 4$, and (1.39) was proved in the cases $d = 1$, and $d = 2$. When $d = 3$, (1.39) was proved under the additional assumption that $q \geq 4$. Appropriate global-in-time versions were later proved in [58]. In both works, the key tool was to use lattice point counting techniques related to the work of Bombieri and Pila [8]. Let us note that some partial results on Strichartz estimates on the irrational torus have been proved in [20, 27]

Link between $X^{s,b}$ spaces and the Schrödinger equation

The $X^{s,b}$ spaces are well suited to the Schrödinger equation. Let us briefly explain how one can see the connection. All of the facts we will mention now hold equally on $\mathbb{R}^d$ and on $\mathbb{T}^d$. They were already used in [9] and other works which first used $X^{s,b}$ space methods.

Given a Schwartz cut-off function in time $\eta \in \mathcal{S}(\mathbb{R})$, the following localization estimate holds:

$$\|\eta(t)S(t)\phi\|_{X^{s,b}} \lesssim_{\eta,s,b} \|f\|_{H^s}. \qquad (1.40)$$

The following useful fact links $X^{s,b}$ spaces and Strichartz estimates:

Proposition 1.6.2. *(c.f. Lemma 2.9 from [106]) Suppose that Y is a Banach space of functions with the property that:*

$$\|e^{it\tau_0}S(t)\phi\|_Y \lesssim \|f\|_{H^s}$$

for all $f \in H^s$ and all $\tau_0 \in \mathbb{R}$. Then, for all $b > \frac{1}{2}$, it holds that:

$$\|u\|_Y \lesssim \|u\|_{X^{s,b}}.$$

As a consequence, we can deduce that for all pairs (q, r) for which the Strichartz estimate in the $\|\cdot\|_{L^q_t L^r_x}$ norm holds, one has the following estimate:

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{X^{0,b}}\,, \text{whenever}\, b > \frac{1}{2}. \tag{1.41}$$

Improved bilinear Strichartz estimates

Strichartz estimates, and in particular the estimate (1.41) allow us to deduce multilinear estimates by using Hölder's inequality. For example, if we consider $u, v \in L^2_{t,x}(\mathbb{R} \times \mathbb{R})$, we can deduce that:

$$\|uv\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R})} \|v\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R})}. \tag{1.42}$$

However, if we have further assumptions on the support of the Fourier transform of $u(\cdot, t)$ and $v(\cdot, t)$ in the space variable, it is possible to deduce an improved estimate. This key observation was first made by Bourgain in [14]. Later, a simplified proof was given in [31]. The improved bilinear estimate is:

Proposition 1.6.3. *(Improved bilinear Strichartz estimate in the non-periodic setting) Suppose $N_1, N_2 > 0$, with $N_1 \gg N_2$, and suppose that $f, g \in X^{0,\frac{1}{2}+}(\mathbb{R}^d \times \mathbb{R})$ are such that for all $t \in \mathbb{R}$:*

$$\textit{supp}\, \hat{f}(t) \subseteq \{|\xi| \sim N_1\},\ \textit{supp}\, \hat{g}(t) \subseteq \{|\xi| \sim N_2\}.$$

Then, the following bound holds:

$$\|fg\|_{L^2_{t,x}(\mathbb{R}^d\times\mathbb{R})} \lesssim \frac{N_2^{\frac{d-1}{2}}}{N_1^{\frac{1}{2}}} \|f\|_{X^{0,\frac{1}{2}+}(\mathbb{R}^d\times\mathbb{R})} \|g\|_{X^{0,\frac{1}{2}+}(\mathbb{R}^d\times\mathbb{R})}. \tag{1.43}$$

In particular, if $d = 1$, we obtain a decay factor of $\frac{C}{N_1^{\frac{1}{2}}}$ which is an improvement over the bound in (1.42).

Let us remark that the analogue of the Improved bilinear Strichartz estimate with a decay factor doesn't hold in the periodic setting. The following result is also due to Bourgain [9], in the case $d = 2$.

Proposition 1.6.4. *(Improved bilinear Strichartz estimate for free evolution in the periodic setting) Suppose* $N_1, N_2 > 0$, *with* $N_1 \gg N_2$, *and suppose that* $f, g \in X^{0,\frac{1}{2}+}(\mathbb{T}^d \times \mathbb{R})$ *are such that for all* $t \in \mathbb{R}$*:*

$$supp\, \hat{f}(t) \subseteq \{|n| \sim N_1\},\ supp\, \hat{g}(t) \subseteq \{|n| \sim N_2\}.$$

Suppose that $I \subseteq \mathbb{R}$ *is a compact time-interval. Then, given the following bound holds:*

$$\|\chi(t) fg\|_{L^2_{t,x}(\mathbb{T}^2 \times I)} \lesssim \frac{N_2^{\frac{d-1}{2}}}{N_1^{\frac{1}{2}}} \|f\|_{X^{0,\frac{1}{2}+}(\mathbb{T}^2 \times \mathbb{R})} \|g\|_{X^{0,\frac{1}{2}+}(\mathbb{T}^2 \times \mathbb{R})}. \tag{1.44}$$

As a consequence of Proposition 1.6.4, the following bound follows:

Proposition 1.6.5. *(Improved bilinear Strichartz estimate in the periodic setting) Suppose* $N_1, N_2 > 0$, *with* $N_1 \gg N_2$, *and suppose that* $f, g \in X^{0,\frac{1}{2}+}(\mathbb{T}^d \times \mathbb{R})$ *are such that for all* $t \in \mathbb{R}$*:*

$$supp\, \hat{f}(t) \subseteq \{|\xi| \sim N_1\},\ supp\, \hat{g}(t) \subseteq \{|\xi| \sim N_2\}.$$

Suppose that $I \subseteq \mathbb{R}$ *is a compact time-interval. Then, the following bound holds:*

$$\|fg\|_{L^2_{t,x}(\mathbb{T}^2 \times I)} \lesssim N_2^{0+} \|f\|_{X^{0,\frac{1}{2}+}(\mathbb{T}^2 \times \mathbb{R})} \|g\|_{X^{0,\frac{1}{2}+}(\mathbb{T}^2 \times \mathbb{R})}. \tag{1.45}$$

The idea to prove Proposition 1.6.5 from Proposition 1.6.4 is to use the Fourier inversion formula to write:

$$u(x,t) \sim \int_{\mathbb{R}} e^{it\tau} S(t) \mathcal{F}(S(-t)u)(x,\tau) d\tau$$

and similarly for v and then use the Cauchy-Schwarz inequality in the parabolic variable together with the assumption that $b = \frac{1}{2}+ > \frac{1}{2}$. These ideas are explained in more detail in [22, 106]. We note that, when $N_1 \gg N_2$, the estimate (1.45) indeed gives us an improvement of the the bound we would otherwise obtain directly from (1.39). The power of N_2^{0+} comes from a lattice point counting argument and as such can't be less than 1. A comprehensive survey about bilinear improved Strichartz estimates on the torus can be found in [48, 100]. We note that improved Bilinear Strichartz estimates have recently been studied in the case of compact Riemannian manifolds in [22, 23, 59].

In our proofs, we will use the bilinear improved Strichartz estimate to obtain a decay factor in the iteration bound. From the preceding discussion, we note that this estimate will be useful to this end only when we are working in the non-periodic setting. As a result, we will obtain better bounds on non-periodic domains. This is consistent with the heuristic that dispersion is stronger in the non-periodic setting.

1.7 Organization of the Chapters

In Chapter 2, we study the problem on S^1. Here, we consider the defocusing power-type NLS and the Hartree equation, as well as other modifications of the defocusing cubic NLS. In Chapter 3, we study the problem on $\mathbb{R}$. In this chapter, we find bounds on the growth of fractional Sobolev norms of solutions the defocusing cubic NLS. In addition to the cubic NLS, we also consider the Hartree equation. Chapter 4 is devoted to the study of the problem on two-dimensional domains. We consider the problem both on $\mathbb{T}^2$ and on $\mathbb{R}^2$. Results from Chapters 2 through 4 will be published in [97, 96, 95]. In Chapter 5, we study the Gross-Pitaevskii equation for dipolar quantum gases on $\mathbb{R}^3$, which is the physically the most relevant case. The results from Chapter 5 are the first step in a joint work with Kay Kirkpatrick and Gigliola Staffilani in which we plan to study the Gross-Pitaevskii equation for dipolar quantum gases in more detail [77].

Chapter 2

Bounds on S^1

2.1 Introduction

In this chapter, we first study the 1D defocusing periodic nonlinear Schrödinger equation. Namely, given $k \in \mathbb{N}$ and $s \in \mathbb{R}$ with $s \geq 1$, we will first consider the initial value problem:

$$\begin{cases} iu_t + \Delta u = |u|^{2k}u, x \in S^1, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in H^s(S^1). \end{cases} \tag{2.1}$$

The mass and energy are given by:

$$M(u(t)) := \int_{S^1} |u(x,t)|^2 dx \ (Mass). \tag{2.2}$$

and

$$E(u(t)) := \frac{1}{2}\int_{S^1} |\nabla u(x,t)|^2 dx + \frac{1}{2k+2}\int_{S^1} |u(x,t)|^{2k+2} \ (Energy). \tag{2.3}$$

As was noted in [46, 84], the equation (2.1) is completely integrable when $k = 1$. Hence, if we start from smooth initial data, all the Sobolev norms of a solution will be uniformly bounded in time. We consider several modifications of the cubic NLS in which we break the complete integrability. The first modification we consider is the Hartree equation on S^1:

$$\begin{cases} iu_t + \Delta u = (V * |u|^2)u, \;\; x \in S^1, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in H^s(S^1). \end{cases} \tag{2.4}$$

The assumptions that we have on V are:

(i) $V \in L^1(S^1)$.

(ii) $V \geq 0$.

(iii) V is even.

We can also break the integrability by adding an external potential on the right-hand side of the equation to obtain:

$$\begin{cases} iu_t + \Delta u = |u|^2 u + \lambda u, \;\; x \in S^1, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in H^s(S^1). \end{cases} \tag{2.5}$$

Here, we are assuming:

(i) $\lambda \in C^\infty(S^1)$.

(ii) λ is real-valued.

Finally, we can add an inhomogeneity factor λ into the nonlinearity, and obtain:

$$\begin{cases} iu_t + \Delta u = \lambda |u|^2 u, \;\; x \in S^1, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in H^s(S^1). \end{cases} \tag{2.6}$$

Here, the inhomogeneity $\lambda = \lambda(x)$ satisfies:

(i) $\lambda \in C^\infty(S^1)$.

(ii) $\lambda \geq 0$.

2.1.1 Statement of the main results

The results that we prove are:

Theorem 2.1.1. *Let $k \geq 2$ be an integer and let $s \geq 1$ be a real number. Let u be a global solution to (2.1). Then, there exists a continuous function C, depending on $(s, k, E(\Phi), M(\Phi))$ such that, for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq C(s, k, E(\Phi), M(\Phi))(1 + |t|)^{2s+} \|\Phi\|_{H^s}. \tag{2.7}$$

For the modifications of the cubic NLS, we can prove the following results:

Theorem 2.1.2. *Let $s \geq 1$ and let u be a global solution of* (2.4). *Then, there exists a function C as above, such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{\frac{1}{2}s+} \|\Phi\|_{H^s}. \tag{2.8}$$

Furthermore, we prove:

Theorem 2.1.3. *Let $s \geq 1$ and let u be a global solution of* (2.5). *Then, there exists a function C as above, such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{s+} \|\Phi\|_{H^s}. \tag{2.9}$$

Theorem 2.1.4. *Let $s \geq 1$ and let u be a global solution of* (2.6). *Then, there exists a function C as above, such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{2s+} \|\Phi\|_{H^s}. \tag{2.10}$$

It makes sense to consider the case $k = 1$ in Theorem 2.1.1, as long as we are taking s which is not an integer, and if we are assuming only $\Phi \in H^s(S^1)$. It turns out that we can get a better bound, which is the same as the one obtained for (2.4):

Corollary 2.1.5. *Let $s > 1$ be a real number and let u be a global solution of* (2.4). *Then, there exists a function C as above, such that for all $t \in \mathbb{R}$:*

$$\begin{cases} iu_t + \Delta u = |u|^2 u, x \in S^1, t \in \mathbb{R} \\ u(x,0) = \Phi(x) \in H^s(S^1). \end{cases} \tag{2.11}$$

Corollary 2.1.5 will be a consequence of the proof of Theorem 2.1.2. The obtained, however, doesn't allow us to recover the uniform bounds on the integral Sobolev norms of a solution, as we observed, up to a loss of t^{0+} in the non-periodic case, which we will see in the following chapter. The question of bounding the growth of fractional Sobolev norms of solutions to the 1D periodic and non-periodic cubic NLS was posed on [113].

Analogous results hold for focusing-type equations, except that then we need to consider initial data which is sufficiently small in an appropriate norm. As we will see in the proof, the only reason why we are looking at defocusing equations is that we have global existence in H^1, and the a priori bound on the H^1 norm.

We can obtain the same conclusion for the defocusing variant of (2.1) if $\|\Phi\|_{H^1}$ is sufficiently small. On the other hand, in the case of the Hartree equation (2.4), we can change the second assumption on V to just assume that V is real-valued, as long as we suppose that $\|\Phi\|_{L^2}$ is sufficiently small. For such initial data, the conclusion of Theorem 2.1.2 will still hold. Under an analogous L^2-smallness assumption on the initial data, we can consider (2.5) with focusing nonlinearity, and (2.6) with λ which is assumed to be real-valued, but not necessarily non-negative. The conclusions of Theorem 2.1.3 and Theorem 2.1.4 will still hold then. We will henceforth consider only the defocusing-type equations.

2.1.2 Previously known results

The techniques previously used in [12, 27, 98, 99, 118] can be adapted to (2.1). Namely, one can show that there exists a constant $r \in (0,1)$ depending on k, s and $\delta, C > 0$ depending also on the initial data such that for all times t_0:

$$\|u(t_0+\delta)\|_{H^s}^2 \leq \|u(t_0)\|_{H^s}^2 + C\|u(t_0)\|_{H^s}^{2-r}. \tag{2.12}$$

which can, in turn, be shown to imply:

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi} (1+|t|)^{\frac{1}{r}}. \tag{2.13}$$

If one uses (2.12), the bounds one obtains become progressively worse as we increase k since r can be shown to become smaller as k grows. In this way, we see that the iteration bound is dependent on the structure of the nonlinearity. When $k=2$, we can show that (2.12) holds for $r = \frac{1}{18(s-1)}-$, from where we deduce the bound:

$$\|u(t)\|_{H^s} \lesssim_{s,\Phi} (1+|t|)^{18(s-1)+}. \tag{2.14}$$

This is a worse bound than the one obtained by Theorem 2.1.1 . If one tries to apply (2.12) for higher order nonlinearities, one gets an even weaker bound.

It should be noted that a better bound for the quintic equation than the one given by Theorem 5.1.1 was observed by Bourgain in the appendix of [19]. The techniques sketched out in this paper are completely different and come from dynamical systems. In [19], the author uses an appropriate normal form which reduces the nonlinearity to its essential part, i.e. to the frequency configurations which are close to being resonant. The result in [19] is mentioned only for the quintic equation. As we will note, due to the fact that it uses Besov-type spaces, which don't embed into $L^\infty_{t,x}$, we can't seem to modify this method to apply it to (1.1) with $k>2$.

Let us finally remark that after the publication of our result, the techniques that we will present were combined with the techniques from [19] in [40] to obtain a slightly improved bound than Theorem 2.1.1 when $k>2$, and Corollary 2.1.5. We note that the method used in [40] is not sufficient to improve the bound for $k=2$ obtained in [19].

2.1.3 Main ideas of the proofs

The main idea of the proof of Theorem 2.1.1 is to obtain a good iteration bound. We will use the idea, used in [18, 17, 111], of estimating the high-frequency part of the solution. Let E^1 denote an operator which, after an appropriate rescaling, essentially adds the square L^2 norm of the low frequency part and the square H^s norm of the high frequency part of a function. The threshold between the low and high frequencies is the parameter $N > 1$. With this definition, we show that there exist $\delta, C > 0$ depending only on Φ such that for all times t_0:

$$E^1(u(t_0 + \delta)) \leq (1 + \frac{C}{N^{\frac{1}{2}-}})E^1(u(t_0)). \tag{2.15}$$

The key fact to observe is that, due to the present decay factor, iteration of (2.15) $O(N^{\frac{1}{2}-})$ times doesn't cause exponential growth in $E^1(u(t))$.

The crucial point hence is to obtain the decay factor in (2.15). The reason why this is difficult is that we are working in the periodic setting in which we don't have the improved bilinear Strichartz estimates proved in [14, 31]. In [34], one could fix this problem by rescaling the circle to add more dispersion and reproving the estimates in the rescaled setting. Finally, one could scale back to the original circle, keeping in mind the relationship between the scaling parameter, the time interval on which one is working, and the threshold between the "high" and the "low" frequencies. This approach is unsuccessful in our setting since it is impossible to scale back, because the time on which we can obtain nontrivial bounds tends to zero as the rescaling factor tends to infinity [1].

We take:

$$E^1(f) := \|\mathcal{D}f\|^2_{L^2}. \tag{2.16}$$

Here $\mathcal{D}$ is an appropriate Fourier multiplier. In this paper, we take the $\mathcal{D}$-operator to be an *upside-down I-operator*, corresponding to high regularities. The idea of using an *upside-down I-operator* first appeared in [33], but in the low regularity context. The purpose of such an operator is to control the evolution of a Sobolev norm which

[1] We note that this is not the same phenomenon that occurs for super-critical equations. The reason why the rescaling here doesn't give the result is that there are too many constraints on all of the parameters.

is higher than the norm associated to a particular conserved quantity. This is the opposite from the standard *I-operator*, which was first developed in [31, 32, 33, 34, 36].

We then want to estimate:

$$\int_I \frac{d}{dt}\|\mathcal{D}u(t)\|_{L^2}^2 dt. \tag{2.17}$$

over an appropriate time interval I whose length depends only on the initial data.

Similarly as in the papers by the I-Team, the multiplier θ corresponding to the operator $\mathcal{D}$ is not a rough cut-off. Hence, in frequency regimes where certain cancelation occurs, we can symmetrize the expression and see how the cancelation manifests itself in terms of θ, as in [33]. If there is no cancelation in the symmetrized expression, we need to look at the spacetime Fourier transform. Arguing as in [22, 118], we decompose our solution into components whose spacetime Fourier transform is localized in the parabolic region $\langle \tau + n^2 \rangle \sim L$. In each of the cases, we obtain a satisfactory decay factor. The mentioned symmetrizations and localizations allow us to compensate for the absence of an *improved Strichartz estimate*.

The proofs of Theorems 2.1.2, 2.1.3, and 2.1.4 are based on similar techniques. For (2.4), and (2.5), we can use the method of *higher modified energies* as in [32, 33], i.e. we can find an approximation $E^2(u)$ of $\|u\|_{H^s}^2$ that varies in time slower than $E^1(u)$. $E^2(u)$ is obtained as a multilinear correction of $E^1(u)$. We deduce better iteration bounds than the one in (2.15), from which the results in Theorem 2.1.2 and Theorem 2.1.3 follow. The technique of higher modified energies doesn't seem to work for (2.6). Heuristically, this means that adding an inhomogeneity as in (2.6) breaks the integrability of the cubic NLS more than adding the convolution potential in (2.4), or adding the external potential in (2.5). Let us note that the techniques sketched in [19] could in principle be applied to (2.4) to obtain the same result. The techniques from [19] don't seem to apply to (2.5) and (2.6).

2.2 Facts from harmonic analysis

There are some key facts one should note about $X^{s,b}(\mathbb{R}\times S^1)$ spaces: By Plancherel's Theorem, one has:

$$\|u\|_{L^2_t L^2_x} \sim \|u\|_{X^{0,0}}. \tag{2.18}$$

Using Sobolev embedding, one obtains:

$$\|u\|_{L^\infty_t L^\infty_x} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \tag{2.19}$$

and:

$$\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}. \tag{2.20}$$

Interpolating between (2.18) and (2.20), it follows that:

$$\|u\|_{L^4_t L^2_x} \lesssim \|u\|_{X^{0,\frac{1}{4}+}}. \tag{2.21}$$

Two more key $X^{s,b}$ space estimates are the two following Strichartz inequalities:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{3}{8}}}. \tag{2.22}$$

$$\|u\|_{L^6_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}}. \tag{2.23}$$

For the proof of (2.22), one should consult Proposition 2.13. in [106]. A proof of (2.23) can be found in [58]. It is crucial to observe that both estimates are global in time.

Throughout the paper, we will need to consider quantities such as $\|\chi_{[c,d]}(t)f\|_{X^{s,b}}$. We will show the following bound:

Lemma 2.2.1. *If $b\in(0,\frac{1}{2})$ and $s\in\mathbb{R}$, then, for $c,d\in\mathbb{R}$ such that $c<d$, one has:*

$$\|\chi_{[c,d]}(t)u\|_{X^{s,b}(\mathbb{R}\times S^1)} \lesssim \|u\|_{X^{s,b+}(\mathbb{R}\times S^1)} \tag{2.24}$$

where the implicit constant doesn't depend on c, d.

A similar fact was proved in [36], but in slightly different spaces. Furthermore, let us mention that a stronger statement was mentioned in a remark after Proposition 32 in [24]. For completeness, we present the proof of Lemma 2.2.1 in Appendix A of this chapter.

From Lemma 2.2.1, we deduce that in particular:

Corollary 2.2.2. *For c, d as above, one has:*

$$\|\chi_{[c,d]}u\|_{X^{0,\frac{3}{8}}} \lesssim \|u\|_{X^{0,\frac{3}{8}+}} \tag{2.25}$$

This fact will be used later on.

2.3 Quintic and Higher Order NLS

In this section, we will define the *upside-down I-operator* $\mathcal{D}$. In order to use this operator effectively, we need to prove appropriate local-in-time bounds. Finally, we use symmetrization to get good estimates on the growth of $\|\mathcal{D}u(t)\|_{L^2}^2$.

Throughout the first three parts of the section, we will prove the claim in the case $k = 2$, for simplicity of notation. Generalizations to higher nonlinearities are given in the fourth part of this section.

2.3.1 Definition of the $\mathcal{D}$ operator

Suppose $N > 1$ is given. Let $\theta : \mathbb{Z} \to \mathbb{R}$ be given by:

$$\theta(n) := \begin{cases} \left(\frac{|n|}{N}\right)^s, \text{if } |n| \geq N \\ 1, \text{ if } |n| \leq N \end{cases} \tag{2.26}$$

Then, if $f : S^1 \to \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(n) := \theta(n)\hat{f}(n). \tag{2.27}$$

We observe that:

$$\|\mathcal{D}f\|_{L^2} \lesssim_s \|f\|_{H^s} \lesssim_s N^s \|\mathcal{D}f\|_{L^2}. \tag{2.28}$$

Our goal is to then estimate $\|\mathcal{D}u(t)\|_{L^2}$, from which we can estimate $\|u(t)\|_{H^s}$ by (2.28).

2.3.2 A local-in-time estimate and an approximation lemma

From our proof, we will note the key role of good local-in-time and associated approximation results. Here, we collect the statements of these results, whose proofs we give in Appendix B of this chapter. The first result we want to show is that there exist $\delta = \delta(s, E(\Phi), M(\Phi)), C = C(s, E(\Phi), M(\Phi)) > 0$, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : S^1 \times \mathbb{R} \to \mathbb{C}$ such that:

$$v|_{[t_0, t_0+\delta]} = u|_{[t_0, t_0+\delta]}. \tag{2.29}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{2.30}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{2.31}$$

Moreover, δ and C can be chosen to depend continuously on the energy and mass.

Proposition 2.3.1. *Given $t_0 \in \mathbb{R}$, there exists a globally defined function $v : S^1 \times \mathbb{R} \to \mathbb{C}$ satisfying the properties (2.29),(2.30),(2.31).*

In the proof of Proposition 2.3.1, we need to use a "persistence of regularity" argument, which relies on the following fact:

Proposition 2.3.2. *Let $R > 0, s \geq 1, B := \{v : \|v\|_{X^{s,b}} \leq R\}$. Then (B,d) is complete as a metric space if we take:*

$$d(v,w) := \|v - w\|_{X^{1,b}}. \tag{2.32}$$

A related technical fact that we will need to use in the proof of the Theorem 2.1.1, and later, in the proofs of the other Theorems is the following:

Proposition 2.3.3. *(Approximation Lemma)*

If u satisfies:

$$\begin{cases} iu_t + \Delta u = |u|^{2k}u, \\ u(x,0) = \Phi(x). \end{cases} \tag{2.33}$$

and if the sequence $(u^{(n)})$ satisfies:

$$\begin{cases} iu_t^{(n)} + \Delta u^{(n)} = |u^{(n)}|^{2k}u^{(n)}, \\ u^{(n)}(x,0) = \Phi_n(x). \end{cases} \tag{2.34}$$

where $\Phi_n \in C^\infty(S^1)$ and $\Phi_n \xrightarrow{H^s} \Phi$, then, one has for all t:

$$u^{(n)}(t) \xrightarrow{H^s} u(t).$$

The mentioned approximation Lemma allows us to work with smooth solutions and pass to the limit in the end. Namely, we note that if we take initial data Φ_n as earlier, then $u^{(n)}(t)$ will belong to $H^\infty(S^1)$ for all t. On the other hand, by continuity of mass, energy, and the H^s norm on H^s, it follows that:

$$M(\Phi_n) \to M(\Phi),\ E(\Phi_n) \to E(\Phi),\ \|\Phi_n\|_{H^s} \to \|\Phi\|_{H^s}.$$

Suppose that we knew that Theorem 2.1.1 were true in the case of smooth solutions. Then, it would follow that for all $t \in \mathbb{R}$:

$$\|u^{(n)}(t)\|_{H^s} \leq C(s,k,E(\Phi_n),M(\Phi_n))(1+|t|)^{2s+}\|\Phi_n\|_{H^s},$$

The claim for u would now follow by applying the continuity properties of C and the approximation Lemma.

We will henceforth work with $\Phi \in C^\infty(S^1)$. This implies that $u(t) \in H^\infty(S^1)$ for all t. The claimed result is then deduced from this special case by the approximation procedure given earlier. As we will see, the analogue of Proposition 2.3.1 holds for (2.4),(2.5), and for (2.6). A similar argument shows that for these equations, it suffices to consider the case when $\Phi \in C^\infty$. The advantage of working with smooth solutions is that all the formal calculations will then be well-defined.

2.3.3 Control on the increment of $\|\mathcal{D}u(t)\|_{L^2}^2$

For $t \in [t_0, t_0+\delta]$, we can work with $\mathcal{D}v(t)$ instead of with $\mathcal{D}u(t)$, where v is the object we had constructed earlier. By our smoothness assumption, we know $v(t) \in H^\infty(S^1)$.

Now, for $t \in [t_0, t_0+\delta]$, one has [2]:

$$\frac{d}{dt}\|\mathcal{D}v(t)\|_{L^2}^2 = 2Re\,\langle \mathcal{D}v_t, \mathcal{D}v\rangle = 2Re\,\langle i\mathcal{D}\Delta v - i\mathcal{D}(v\bar{v}v\bar{v}v), \mathcal{D}v\rangle$$

Since $Re\,\langle i\mathcal{D}\Delta v, \mathcal{D}v\rangle = 0$, this expression equals:

$$= -2Re\,\langle i\mathcal{D}(v\bar{v}v\bar{v}v), \mathcal{D}v\rangle.$$

After an appropriate symmetrization, by using notation as in Section 2 and arguing as in [33], we get that this expression equals:

$$\frac{1}{3}i \cdot \lambda_6((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_6))^2; v(t)).$$

Let us take:

[2]We are using the fact that $v(t) \in H^\infty(S^1)$ in order to deduce that this quantity is finite!

$$M_6(n_1,n_2,n_3,n_4,n_5,n_6) := (\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_6))^2.$$

We now analyze:

$$\|\mathcal{D}u(t_0+\delta)\|_{L^2}^2 - \|\mathcal{D}u(t_0)\|_{L^2}^2 = \|\mathcal{D}v(t_0+\delta)\|_{L^2}^2 - \|\mathcal{D}v(t_0)\|_{L^2}^2 =$$

$$= \int_{t_0}^{t_0+\delta} \frac{d}{dt} \|\mathcal{D}v(t)\|_{L^2}^2 dt =$$

$$= \frac{1}{3} i \Big(\sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \int_{t_0}^{t_0+\delta} M_6(n_1,n_2,n_3,n_4,n_5,n_6) \hat{v}(n_1) \hat{\bar{v}}(n_2) \hat{v}(n_3) \hat{\bar{v}}(n_4) \hat{v}(n_5) \hat{\bar{v}}(n_6) dt \Big) =$$

$$= \frac{1}{3} i \Big(\sum_{n_1-n_2+n_3-n_4+n_5-n_6=0}$$

$$\int_{t_0}^{t_0+\delta} M_6(n_1,n_2,n_3,n_4,n_5,n_6) \hat{v}(n_1) \overline{\hat{v}(n_2)} \hat{v}(n_3) \overline{\hat{v}(n_4)} \hat{v}(n_5) \overline{\hat{v}(n_6)} dt \Big) =: I \qquad (2.35)$$

We want to prove an appropriate decay bound on the increment. The bound that we will prove is:

Lemma 2.3.4. *(Iteration Bound) For all $t_0 \in \mathbb{R}$, one has:*

$$\big| \|\mathcal{D}v(t_0+\delta)\|_{L^2}^2 - \|\mathcal{D}v(t_0)\|_{L^2}^2 \big| \lesssim \frac{1}{N^{\frac{1}{2}-}} \|\mathcal{D}v(t_0)\|_{L^2}^2.$$

From the proof, it will follow that the implied constant depends only on s, Energy, and Mass, and hence is uniform in time. We call this constant $C = C(s, Energy, Mass) > 0$. In fact, by construction, it will follow that all the implied constants we obtain will depend continuously on energy and mass, and hence will be continuous functions of Φ w.r.t to the H^1 norm. For brevity, we will suppress this fact in our further arguments.

Let us first observe how Lemma 2.3.4 implies Theorem 2.1.1 for $k = 2$. From Lemma 2.3.4, and for the C constructed earlier, it follows that:

$$\|\mathcal{D}u(\delta)\|_{L^2}^2 \leq (1 + \frac{C}{N^{\frac{1}{2}-}})\|\mathcal{D}\Phi\|_{L^2}^2.$$

The same C satisfies:

$$\forall t_0 \in \mathbb{R}\,, \|\mathcal{D}u(t_0 + \delta)\|_{L^2}^2 \leq (1 + \frac{C}{N^{\frac{1}{2}-}})\|\mathcal{D}u(t_0)\|_{L^2}^2. \tag{2.36}$$

Using (2.36) iteratively, we obtain that [3] $\forall T > 1$:

$$\|\mathcal{D}u(T)\|_{L^2}^2 \leq (1 + \frac{C}{N^{\frac{1}{2}-}})^{\lceil \frac{T}{\delta} \rceil}\|\mathcal{D}\Phi\|_{L^2}^2.$$

i.e. there exists $\alpha = \alpha(s, Energy, Mass) > 0$ s.t. for all $T > 1$, one has:

$$\|\mathcal{D}u(T)\|_{L^2}^2 \leq (1 + \frac{C}{N^{\frac{1}{2}-}})^{\alpha T}\|\mathcal{D}\Phi\|_{L^2}^2. \tag{2.37}$$

For $\lambda_1, \lambda_2 > 0$, we know:

$$\lim_{x \to +\infty} \left(1 + \frac{1}{\lambda_1 x}\right)^{\lambda_2 x} = e^{\frac{\lambda_2}{\lambda_1}} < \infty. \tag{2.38}$$

By using (2.37) and (2.38), we can take:

$$T \sim N^{\frac{1}{2}-}. \tag{2.39}$$

Hence:

$$\|\mathcal{D}u(T)\|_{L^2} \lesssim \|\mathcal{D}\Phi\|_{L^2}. \tag{2.40}$$

Recalling (2.28), and using (2.40), (2.39), and the fact that $T > 1$, we obtain:

$$\|u(T)\|_{H^s} \lesssim N^s\|\mathcal{D}u(T)\|_{L^2} \lesssim N^s\|\mathcal{D}\Phi\|_{L^2} \lesssim N^s\|\Phi\|_{H^s}$$

[3] Strictly speaking, we are using (2.31) to deduce that we can get the bound for all such times, and not just those which are a multiple of δ.

$$\lesssim T^{2s+}\|\Phi\|_{H^s} \lesssim (1+T)^{2s+}\|\Phi\|_{H^s}. \tag{2.41}$$

Since for times $t \in [0,1]$, we get the bound of Theorem 2.1.1 just by iterating the local well-posedness construction, the claim for these times follows immediately. Combining this observation, (2.41), recalling the approximation result, and using time-reversibility, we obtain that for all $s \geq 1$, there exists $C = C(s, Energy, Mass)$ such that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{2s+}\|u(0)\|_{H^s}. \tag{2.42}$$

Moreover, C depends continuously on energy and mass. This proves Theorem 2.1.1 when $k=2$. $\square$

We now turn to the proof of Lemma 2.3.4.

Proof. Let us consider WLOG the case when $t_0 = 0$. The general case follows by time translation and by the fact that all of our implied constants are independent of time. The idea is to localize the factors of v into dyadic annuli in frequency dual to x, i.e. to perform the Littlewood-Paley decomposition. Namely, for each j such that $n_j \neq 0$, we find a dyadic integer N_j such that $|n_j| \sim N_j$. If $n_j = 0$, we take the corresponding N_j to be equal to 1.

We let v_{N_j} denote the function obtained from v by localizing in frequency to the dyadic annulus $|n| \sim N_j$. Let $|n_a|, |n_b|$ denote the largest two elements of the set $\{|n_1|, |n_2|, |n_3|, |n_4|, |n_5|, |n_6|\}$.

In our analysis of (2.35), we have to consider two Big Cases:

$\diamondsuit$**Big Case 1:** In the expression for M_6, $(\theta(n_a))^2$ and $(\theta(n_b))^2$ appear with the opposite sign.

$\diamondsuit$**Big Case 2:** In the expression for M_6, $(\theta(n_a))^2$ and $(\theta(n_b))^2$ appear with the same sign.

As we will see, the ways in which we bound the contributions to (2.35) coming from the two Big Cases are quite different.

Let $I^{(1)}$ denote the contribution coming to I (as defined in (2.35)) from Big Case 1, and let $I^{(2)}$ denote the contribution coming from Big Case 2.

Big Case 1: We can assume WLOG that $|n_a| = |n_1|$, and $|n_b| = |n_2|$. In the proof of Big Case 1, we will see that the order of the other four frequencies in absolute value doesn't matter. Namely, the order of the four lower frequencies won't affect any of the multiplier bounds (which depend only on $|n_1|$ and $|n_2|$), and the estimates that we will use on the factors of v corresponding to these four frequencies will not depend on complex conjugates. Hence, it suffices to consider WLOG the case when:

$$|n_1| \geq |n_2| \geq |n_3| \geq |n_4| \geq |n_5| \geq |n_6|. \tag{2.43}$$

We observe that, in this contribution, the N_j satisfy:

$$N_1 \gtrsim N_2 \gtrsim N_3 \gtrsim N_4 \gtrsim N_5 \gtrsim N_6. \tag{2.44}$$

By definition of θ, we observe that

$$M_6(n_1, n_2, n_3, n_4, n_5, n_6) = 0 \text{ if } |n_1|, |n_2|, |n_3|, |n_4|, |n_5|, |n_6| \leq N.$$

Hence, by construction of $|n_1|$, one has $|n_1| > N$ so we obtain the additional localization:

$$N_1 \gtrsim N. \tag{2.45}$$

Finally, since $n_1 - n_2 + n_3 - n_4 + n_5 - n_6 = 0$, (2.43) and the triangle inequality imply that $|n_1| \sim |n_2|$.

From this fact, we can deduce the localization:

$$N_1 \sim N_2. \tag{2.46}$$

The expression we wish to estimate is:

$$I_{N_1,N_2,N_3,N_4,N_5,N_6} :=$$

$$\sum_{n_1-n_2+n_3-n_4+n_5-n_6=0;|n_1|\geq\cdots\geq|n_6|} \int_0^\delta M_6 \widehat{v_{N_1}}(n_1)\overline{\widehat{v_{N_2}}(n_2)}\widehat{v_{N_3}}(n_3)\overline{\widehat{v_{N_4}}(n_4)}\widehat{v_{N_5}}(n_5)\overline{\widehat{v_{N_6}}(n_6)}dt.$$

Let $\widetilde{I}$ denote the contribution to I, as defined in (2.35), coming from (2.43). Then $\widetilde{I}$ satisfies:

$$|\widetilde{I}| \lesssim \sum_{N_j \text{ satisfying } (2.44),(2.45),(2.46)} |I_{N_1,N_2,N_3,N_4,N_5,N_6}|.$$

Within Big Case 1, we consider two cases:

$\diamond$**Case 1:**$N_3, N_4, N_5, N_6 \ll N_1^{\frac{1}{2}}$.

$\diamond$**Case 2:**$N_3 \gtrsim N_1^{\frac{1}{2}}$.

Case 1:

The key step in this case is the following bound on M_6, which comes from cancelation.

$$M_6 = O(N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)). \tag{2.47}$$

Before we prove (2.47), let us see how it gives us a good bound. Assuming (2.47) for the moment, we observe that:

$$|I_{N_1,N_2,N_3,N_4,N_5,N_6}| =$$

$$= \Big| \sum_{n_1-n_2+n_3-n_4+n_5-n_6=0;|n_1|\geq\cdots\geq|n_6|} \int_{\mathbb{R}} M_6 (\chi_{[0,\delta]}v_{N_1})\widehat{\ }(n_1)\overline{\widehat{v_{N_2}}(n_2)}\widehat{v_{N_3}}(n_3)\overline{\widehat{v_{N_4}}(n_4)}\widehat{v_{N_5}}(n_5)\overline{\widehat{v_{N_6}}(n_6)}dt \Big| =$$

$$= \Big| \sum_{n_1-n_2+n_3-n_4+n_5-n_6=0; |n_1|\geq \cdots \geq |n_6|} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5-\tau_6=0}$$

$$M_6(\chi_{[0,\delta]}v_{N_1})\tilde{}(n_1,\tau_1)\overline{\widetilde{v_{N_2}}(n_2,\tau_2)}\widetilde{v_{N_3}}(n_3,\tau_3)\overline{\widetilde{v_{N_4}}(n_4,\tau_4)}\widetilde{v_{N_5}}(n_5,\tau_5)\overline{\widetilde{v_{N_6}}(n_6,\tau_6)}d\tau_j\Big| \lesssim$$

$$\lesssim N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2) \sum_{n_1-n_2+n_3-n_4+n_5-n_6=0; |n_1|\geq \cdots \geq |n_6|} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5-\tau_6=0}$$

$$\{|(\chi_{[0,\delta]}v_{N_1})\tilde{}(n_1,\tau_1)||\widetilde{v_{N_2}}(n_2,\tau_2)||\widetilde{v_{N_3}}(n_3,\tau_3)||\widetilde{v_{N_4}}(n_4,\tau_4)||\widetilde{v_{N_5}}(n_5,\tau_5)||\widetilde{v_{N_6}}(n_6,\tau_6)|\}d\tau_j \leq$$

Since the integrand is non-negative, we can eliminate the restriction in the sum that $|n_1| \geq \cdots |n_6|$, so the expression is:

$$\leq N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2) \sum_{n_1-n_2+n_3-n_4+n_5-n_6=0} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5-\tau_6=0}$$

$$\{|(\chi_{[0,\delta]}v_{N_1})\tilde{}(n_1,\tau_1)||\widetilde{v_{N_2}}(n_2,\tau_2)||\widetilde{v_{N_3}}(n_3,\tau_3)||\widetilde{v_{N_4}}(n_4,\tau_4)||\widetilde{v_{N_5}}(n_5,\tau_5)||\widetilde{v_{N_6}}(n_6,\tau_6)|\}d\tau_j.$$

Let us define:

$$F_1(x,t) := \sum_n \int_{\mathbb{R}} |(\chi_{[0,\delta]}v_{N_1})\tilde{}(n_1,\tau_1)| e^{i(nx+t\tau)} d\tau. \tag{2.48}$$

For $j = 2,3,4,5,6$, we let:

$$F_j(x,t) := \sum_n \int_{\mathbb{R}} |\widetilde{v_{N_j}}(n,\tau)| e^{i(nx+t\tau)} d\tau. \tag{2.49}$$

We now recall a fact from Fourier analysis. For simplicity, let us suppose that $f_1,\ldots,f_6$ are functions on $\mathbb{R}$. Let us suppose that all $\widehat{f_j}$ are real-valued.

Then one has:

$$\int f_1\overline{f_2}f_3\overline{f_4}f_5\overline{f_6}dx =$$

$$= \int_{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5-\xi_6=0} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\widehat{f_4}(\xi_4)\widehat{f_5}(\xi_5)\widehat{f_6}(\xi_6)d\xi_j. \tag{2.50}$$

Using the analogue of (2.50) for the spacetime Fourier transform on $S^1 \times \mathbb{R}$, together with (2.48) and (2.49), and the previous bound we obtained on $|I_{N_1,N_2,N_3,N_4,N_5,N_6}|$, we deduce that:

$$|I_{N_1,N_2,N_3,N_4,N_5,N_6}| \lesssim N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\int_{\mathbb{R}}\int_{S^1} F_1\overline{F_2}F_3\overline{F_4}F_5\overline{F_6}dxdt =$$

$$= N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\Big|\int_{\mathbb{R}}\int_{S^1} F_1\overline{F_2}F_3\overline{F_4}F_5\overline{F_6}dxdt\Big| \leq$$

Which by Hölder's inequality is:

$$\leq N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\|F_1\|_{L^4_{t,x}}\|\overline{F_2}\|_{L^4_{t,x}}\|F_3\|_{L^4_{t,x}}\|\overline{F_4}\|_{L^4_{t,x}}\|F_5\|_{L^\infty_{t,x}}\|\overline{F_6}\|_{L^\infty_{t,x}} =$$

$$= N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\|F_1\|_{L^4_{t,x}}\|F_2\|_{L^4_{t,x}}\|F_3\|_{L^4_{t,x}}\|F_4\|_{L^4_{t,x}}\|F_5\|_{L^\infty_{t,x}}\|F_6\|_{L^\infty_{t,x}}$$

By using (2.22) and (2.19), this is:

$$\lesssim N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\|F_1\|_{X^{0,\frac{3}{8}}}\|F_2\|_{X^{0,\frac{3}{8}}}\|F_3\|_{X^{0,\frac{3}{8}}}\|F_4\|_{X^{0,\frac{3}{8}}}\|F_5\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|F_6\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} =$$

$$= N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\|\chi_{[0,\delta]}v_{N_1}\|_{X^{0,\frac{3}{8}}}\|v_{N_2}\|_{X^{0,\frac{3}{8}}}\|v_{N_3}\|_{X^{0,\frac{3}{8}}}\|v_{N_4}\|_{X^{0,\frac{3}{8}}}\|v_{N_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

By using (2.25) to bound the first factor, this expression is:

$$\lesssim N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\|v_{N_1}\|_{X^{0,\frac{3}{8}+}}\|v_{N_2}\|_{X^{0,\frac{3}{8}}}\|v_{N_3}\|_{X^{0,\frac{3}{8}}}\|v_{N_4}\|_{X^{0,\frac{3}{8}}}\|v_{N_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \leq$$

$$\leq N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\|v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_3}\|_{X^{1,\frac{1}{2}+}}\|v_{N_4}\|_{X^{1,\frac{1}{2}+}}\|v_{N_5}\|_{X^{1,\frac{1}{2}+}}\|v_{N_6}\|_{X^{1,\frac{1}{2}+}} \lesssim$$

$$\lesssim N_1^{-\frac{1}{2}}\|\mathcal{D}v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}} \leq N_1^{-\frac{1}{2}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}} \lesssim$$

$$\lesssim N_1^{-\frac{1}{2}} \|\mathcal{D}\Phi\|_{L^2}^2 \|\Phi\|_{H^1}^4 \lesssim N_1^{-\frac{1}{2}} \|\mathcal{D}\Phi\|_{L^2}^2. \tag{2.51}$$

In the last two inequalities, we used Proposition 2.3.1 , followed by the uniform bound on the H^1 norm of the solution to our equation given by the conservation of energy and mass.

This is the bound that we can obtain from (2.47). We now prove (2.47).

We must consider three possible subcases:

Subcase 1: $|n_2| < N$.

Subcase 2: $|n_2| \geq N$ and $|n_3| < N$.

Subcase 3: $|n_3| \geq N$.

Subcase 1:

Here, we have:

$$N_1 \gtrsim N, N_2 \sim |n_2| < N, N_1 \sim N_2.$$

So, one obtains:

$$N_1 \sim N_2 \sim N.$$

Also, we know:

$$n_1 - n_2 + n_3 - n_4 + n_5 - n_6 = 0, \text{ and } n_3, n_4, n_5, n_6 = O(N_1^{\frac{1}{2}}) = O(N^{\frac{1}{2}}).$$

Consequently:

$$|n_1| = N + r_1, |n_2| = N - r_2, \text{ where } r_1, r_2 > 0, \text{ and } r_1, r_2 = O(N^{\frac{1}{2}}).$$

$$\Rightarrow (\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_6))^2 =$$

$$= \frac{|n_1|^{2s}}{N^{2s}} - 1 + 1 - 1 + 1 - 1 = \frac{|n_1|^{2s} - N^{2s}}{N^{2s}} = \frac{(N + O(N^{\frac{1}{2}}))^{2s} - N^{2s}}{N^{2s}} =$$

$$= O\Big(\frac{N^{2s-\frac{1}{2}}}{N^{2s}}\Big) = O(N^{-\frac{1}{2}}) = O(N_1^{-\frac{1}{2}}) = O(N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)).$$

In the last inequality, we used the fact that $\theta(N_1), \theta(N_2) \geq 1$.

Subcase 2:

Here: $n_2 = n_1 + (n_3 - n_4 + n_5 - n_6)$, from where it follows that:

$$n_2 = n_1 + O(|n_1|^{\frac{1}{2}})$$

We observe:

$$(\theta(n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_6))^2 = 1 - 1 + 1 - 1 = 0.$$

So:

$$M_6 = (\theta(n_1))^2 - (\theta(n_2))^2 = \frac{|n_1|^{2s}}{N^{2s}} - \frac{|n_2|^{2s}}{N^{2s}} = \frac{|n_1|^{2s} - |n_1 + O(|n_1|^{\frac{1}{2}})|^{2s}}{N^{2s}} =$$

$$= O\Big(\frac{|n_1|^{2s-\frac{1}{2}}}{N^{2s}}\Big) = O\Big(\frac{N_1^{2s-\frac{1}{2}}}{N^{2s}}\Big) = O\Big(N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\Big).$$

Here, we used the fact that: $\theta(N_1), \theta(N_2) \sim \frac{N_1^s}{N^s}$.

Subcase 3:

In this subcase, we can no longer use the cancelation coming from

$$(\theta(n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_6))^2.$$

The way one gets around this problem is as follows:

We first note that:

$(\theta(n_1))^2 - (\theta(n_2))^2 = O\Big(\frac{|n_1|^{2s-\frac{1}{2}}}{N^{2s}}\Big) = O\Big(N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\Big)$, as before.

Also $|n_3| = O(|n_1|^{\frac{1}{2}})$, so:

$$(\theta(n_3))^2 = O\Big(\frac{|n_3|^{2s}}{N^{2s}}\Big) = O\Big(\frac{|n_1|^s}{N^{2s}}\Big) = O\Big(\frac{|n_1|^{2s-\frac{1}{2}}}{N^{2s}}\Big).$$

Hence, by monotonicity properties of θ, we deduce:

$$(\theta(n_3))^2, (\theta(n_4))^2, (\theta(n_5))^2, (\theta(n_6))^2 = O\Big(\frac{|n_1|^{2s-\frac{1}{2}}}{N^{2s}}\Big).$$

Combining the previous estimates, we obtain:

$$M_6 = O\Big(\frac{|n_1|^{2s-\frac{1}{2}}}{N^{2s}}\Big) = O\Big(N_1^{-\frac{1}{2}}\theta(N_1)\theta(N_2)\Big).$$

The estimate (2.47) now follows.

Case 2:

We recall that in this case, one has $N_3 \gtrsim N_1^{\frac{1}{2}}$. Here, we don't expect to get cancelation coming from M_6, so we just bound:

$$|M_6| = |(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_5))^2|$$

$$\lesssim (\theta(n_1))^2 \lesssim (\theta(N_1))^2 \lesssim \theta(N_1)\theta(N_2). \tag{2.52}$$

With notation as in Case 1, we use (2.52) and arguments analogous to those used to derive (2.51) to deduce:

$$|I_{N_1,N_2,N_3,N_4,N_5,N_6}| \lesssim$$

$$\lesssim \theta(N_1)\theta(N_2)\|v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_3}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{0,\frac{1}{2}+}}\|v_{N_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \lesssim$$

$$\lesssim \|\mathcal{D}v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}(\frac{1}{N_3}\|v_{N_3}\|_{X^{1,\frac{1}{2}+}})\|v_{N_4}\|_{X^{0,\frac{1}{2}+}}\|v_{N_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \lesssim$$

$$\lesssim N_1^{-\frac{1}{2}}\|\mathcal{D}v\|^2_{X^{0\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}} \lesssim N_1^{-\frac{1}{2}}\|\mathcal{D}\Phi\|^2_{L^2}. \tag{2.53}$$

The last bound follows from Proposition 2.3.1. We note that this is the same bound we obtained in (2.51). Combining (2.51) and (2.53), and recalling that $I^{(1)}$ denotes the contribution of I from Big Case 1, it follows that:

$$|I^{(1)}| \lesssim \sum_{N_j \text{ satisfying } (2.44),(2.45),(2.46)} N_1^{-\frac{1}{2}} \|\mathcal{D}\Phi\|_{L^2}^2 \lesssim$$

$$\lesssim \sum_{N_j \text{ satisfying } (2.44),(2.45),(2.46)} N_1^{-\frac{1}{2}+} N_2^{-0+} N_3^{-0+} N_4^{-0+} N_5^{-0+} N_6^{-0+} \|\mathcal{D}\Phi\|_{L^2}^2 \lesssim$$

$$\lesssim \frac{1}{N^{\frac{1}{2}-}} \|\mathcal{D}\Phi\|_{L^2}^2. \tag{2.54}$$

By construction, the implied constant depends only on $(s, Energy, Mass)$, and is continuous in energy and mass.

Big Case 2:

We recall that in this Big Case, in the expression for M_6, $(\theta(n_a))^2$ and $(\theta(n_b))^2$ appear with the same sign. Arguing as in Big Case 1, we observe that the order of the four lower frequencies doesn't matter. Let us reorder the variables so that the hyperplane over which we are summing becomes $n_1 + n_2 + n_3 - n_4 - n_5 - n_6 = 0$. It suffices to consider the case when:

$$|n_1| \geq |n_2| \geq |n_3| \geq |n_4| \geq |n_5| \geq |n_6|.$$

The expression we want to bound is:

$$\sum_{n_1+n_2+n_3-n_4-n_5-n_6=0, |n_1|\geq|n_2|\geq|n_3|\geq|n_3|\geq|n_4|\geq|n_5|\geq|n_6|} \{\int_0^\delta M_6' \hat{v}(n_1)\hat{v}(n_2)\hat{v}(n_3)\overline{\hat{v}(n_4)}\,\overline{\hat{v}(n_5)}\,\overline{\hat{v}(n_6)}dt\}.$$

Here, we are taking:

$$M_6'(n_1,n_2,n_3,n_4,n_5,n_6) := (\theta(n_1))^2+(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2-(\theta(n_5))^2-(\theta(n_6))^2.$$

As before, we dyadically localize the factors of v in the Fourier domain.

In this Big Case, we want to estimate:

$$J_{N_1,N_2,N_3,N_4,N_5,N_6} := \sum_{n_1+n_2+n_3-n_4-n_5-n_6=0, |n_1|\geq|n_2|\geq|n_3|\geq|n_3|\geq|n_4|\geq|n_5|\geq|n_6|}$$

$$\{\int_0^\delta M_6' \widehat{v_{N_1}}(n_1)\widehat{v_{N_2}}(n_2)\widehat{v_{N_3}}(n_3)\overline{\widehat{v_{N_4}}(n_4)}\,\overline{\widehat{v_{N_5}}(n_5)}\,\overline{\widehat{v_{N_6}}(n_6)}dt\}.$$

One has the additional localizations on the N_j's:

$$N_1 \gtrsim N_2 \gtrsim N_3 \gtrsim N_4 \gtrsim N_5 \gtrsim N_6. \tag{2.55}$$

$$N_1 \sim N_2. \tag{2.56}$$

$$N_1 \gtrsim N. \tag{2.57}$$

In this Big Case, we don't necessarily obtain any cancelation in M_6', so we just write:

$$|M_6'| \lesssim (\theta(n_1))^2 \lesssim (\theta(N_1))^2 \lesssim \theta(N_1)\theta(N_2). \tag{2.58}$$

Let us now estimate $J_{N_1,N_2,N_3,N_4,N_5,N_6}$.

Our analysis of this contribution will use techniques similar to those used in [22, 118]. As we will see, when one can't deduce decay estimates just from looking at the Fourier transform in x, one can look at the Fourier transform in t.

We consider two cases:

⋄ **Case 1:** $N_3, N_4, N_5, N_6 \ll N_1^{\frac{1}{2}}$.

We observe that:

$$J_{N_1,N_2,N_3,N_4,N_5,N_6} = \sum_{n_1+n_2+n_3-n_4-n_5-n_6=0, |n_1|\geq\cdots\geq|n_6|}$$

$$\int_{\mathbb{R}} \{M_6'(\chi_{[0,\delta]}\widehat{v_{N_1}}(n_1))\widehat{v_{N_2}}(n_2)\widehat{v_{N_3}}(n_3)\overline{\widehat{v_{N_4}}(n_4)}\,\overline{\widehat{v_{N_5}}(n_5)}\,\overline{\widehat{v_{N_6}}(n_6)}\}dt =$$

$$= \sum_{n_1+n_2+n_3-n_4-n_5-n_6=0, |n_1|\geq \cdots \geq |n_6|} \int_{\tau_1+\tau_2+\tau_3-\tau_4-\tau_5-\tau_6=0}$$

$$M_6'(\chi_{[0,\delta]}v_{N_1})^{\sim}(n_1,\tau_1)\widetilde{v_{N_2}}(n_2,\tau_2)\widetilde{v_{N_3}}(n_3,\tau_3)\overline{\widetilde{v_{N_4}}(n_4,\tau_4)}\,\overline{\widetilde{v_{N_5}}(n_5,\tau_5)}\,\overline{\widetilde{v_{N_6}}(n_6,\tau_6)}d\tau_j.$$

Now, as in [22, 118], we localize in parabolic regions determined by $\langle \tau + n^2 \rangle$.

Namely, given a dyadic integer L_1, we let $(\chi_{[0,\delta]}v_{N_1})_{L_1} = (\chi_{[0,\delta]}v)_{N_1,L_1}$ denote the function obtained from $\chi_{[0,\delta]}v_{N_1} = (\chi_{[0,\delta]}v)_{N_1}$ by restricting its spacetime Fourier transform to the region where $\langle \tau + n^2 \rangle \sim L_1$.

Likewise, for $j \geq 2$, and for L_j a dyadic integer, we denote by v_{N_j,L_j} the function obtained from v_{N_j} by localizing its spacetime Fourier transform to $\langle \tau + n^2 \rangle \sim L_j$.

So, now, we want to estimate:

$$J_{\bar{L},\bar{N}} :=$$

$$= \sum_{n_1+n_2+n_3-n_4-n_5-n_6=0, |n_1|\geq \cdots \geq |n_6|} \int_{\tau_1+\tau_2+\tau_3-\tau_4-\tau_5-\tau_6=0}$$

$$M_6'\widetilde{(\chi_{[0,\delta]}v)}_{N_1,L_1}(n_1,\tau_1)\widetilde{v_{N_2,L_2}}(n_2,\tau_2)\widetilde{v_{N_3,L_3}}(n_3,\tau_3)\overline{\widetilde{v_{N_4,L_4}}(n_4,\tau_4)}\,\overline{\widetilde{v_{N_5,L_5}}(n_5,\tau_5)}\,\overline{\widetilde{v_{N_6,L_6}}(n_6,\tau_6)}d\tau_j.$$

We have to consider two subcases w.r.t. the τ_j:

Subcase 1: $|\tau_3|, |\tau_4|, |\tau_5|, |\tau_6| \ll N_1^2$.

Subcase 2: $max\,\{|\tau_3|, |\tau_4|, |\tau_5|, |\tau_6|\} \gtrsim N_1^2$.

Subcase 1:

Let us denote by $J^1_{\bar{L},\bar{N}}$ the contribution to $J_{\bar{L},\bar{N}}$ coming from this subcase.

Take

$$(n_1, \tau_1) \in supp\,\widetilde{(\chi_{[0,\delta]}v)}_{N_1,L_1},$$

and:

$$(n_2,\tau_2) \in supp\,\widetilde{v_{N_2,L_2}}.$$

keeping in mind the assumptions of the subcase.

We then obtain:

$$L_1 + L_2 \gtrsim |\tau_1 + n_1^2| + |\tau_2 + n_2^2| \geq |\tau_1 + \tau_2 + n_1^2 + n_2^2| \geq |n_1^2 + n_2^2| - |\tau_1 + \tau_2| =$$

$$= |n_1^2 + n_2^2| - |\tau_3 - \tau_4 - \tau_5 - \tau_6| \geq |n_1|^2 - |\tau_3| - |\tau_4| - |\tau_5| - |\tau_6| \gtrsim N_1^2$$

In the last inequality, we used the fact that:

$$|n_1| \gtrsim N_1, |\tau_3|, |\tau_4|, |\tau_5|, |\tau_6| \ll N_1^2.$$

In the calculation, we observe the crucial role of the inequality:

$$|n_1^2 + n_2^2| \geq |n_1|^2.$$

Since $L_1, L_2 \geq 1$, the previous calculation gives us that:

$$L_1 L_2 \gtrsim N_1^2. \tag{2.59}$$

We now note that:

$$|J^1_{\bar{L},\bar{N}}| \leq$$

$$\leq \sum_{n_1+n_2+n_3-n_4-n_5-n_6=0, |n_1|\geq \cdots \geq |n_6|} \int_{\tau_1+\tau_2+\tau_3-\tau_4-\tau_5-\tau_6=0; |\tau_3|,|\tau_4|,|\tau_5|,|\tau_6| \ll N_1^2}$$

$$\{|M_6'||\widetilde{(\chi_{[0,\delta]}v)}_{N_1,L_1}(n_1,\tau_1)||\widetilde{v_{N_2,L_2}}(n_2,\tau_2)||\widetilde{v_{N_3,L_3}}(n_3,\tau_3)|$$

$$|\widetilde{\overline{v_{N_4,L_4}}}(n_4,\tau_4)|\,|\widetilde{\overline{v_{N_5,L_5}}}(n_5,\tau_5)|\,|\widetilde{\overline{v_{N_6,L_6}}}(n_6,\tau_6)|\}d\tau_j \leq$$

$$\leq \sum_{n_1+n_2+n_3-n_4-n_5-n_6=0} \int_{\tau_1+\tau_2+\tau_3-\tau_4-\tau_5-\tau_6=0}$$

$$\{|M_6'||\widetilde{(\chi_{[0,\delta]}v)}_{N_1,L_1}(n_1,\tau_1)||\widetilde{v_{N_2,L_2}}(n_2,\tau_2)||\widetilde{v_{N_3,L_3}}(n_3,\tau_3)|$$

$$|\widetilde{v_{N_4,L_4}}(n_4,\tau_4)|\,|\widetilde{v_{N_5,L_5}}(n_5,\tau_5)|\,|\widetilde{v_{N_6,L_6}}(n_6,\tau_6)|\}d\tau_j.$$

Similarly as in Big Case 1, let us define:

$$G_1(x,t) := \sum_n \int e^{inx+it\tau}|\widetilde{(\chi_{[0,\delta]}v)}_{N_1,L_1}(n,\tau)|d\tau.$$

For $j = 2, \ldots, 6$, we let:

$$G_j(x,t) := \sum_n \int e^{inx+it\tau}|\widetilde{v_{N_j,L_j}}(n,\tau)|d\tau.$$

Arguing as in Big Case 1, using Hölder's inequality and (2.58), we get [4]:

$$|J^1_{\bar{L},\bar{N}}| \lesssim \theta(N_1)\theta(N_2)\|G_1\|_{L^4_tL^2_x}\|G_2\|_{L^4_tL^2_x}\|G_3\|_{L^4_tL^\infty_x}\|G_4\|_{L^4_tL^\infty_x}\|G_5\|_{L^\infty_{t,x}}\|G_6\|_{L^\infty_{t,x}}$$

which is by Sobolev embedding:

$$\lesssim \theta(N_1)\theta(N_2)\|G_1\|_{L^4_tL^2_x}\|G_2\|_{L^4_tL^2_x}\|G_3\|_{L^4_tH^{\frac{1}{2}+}_x}\|G_4\|_{L^4_tH^{\frac{1}{2}+}_x}\|G_5\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|G_6\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

Since $supp\ \widehat{G_3} \subseteq \{-cN_3,\ldots,cN_3\}, supp\ \widehat{G_4} \subseteq \{-cN_4,\ldots,cN_4\}$, this expression is:

[4] Strictly speaking, we should be truncating G_3, G_4, G_5, and G_6 to $|\tau| \ll N_1^2$, but we ignore this for simplicity of notation since we will later reduce to estimating these factors in $X^{s,b}$ norms, which don't increase if we localize the spacetime Fourier transform.

$$\lesssim \theta(N_1)\theta(N_2)\|G_1\|_{L_t^4 L_x^2}\|G_2\|_{L_t^4 L_x^2}(N_3^{\frac{1}{2}+}\|G_3\|_{L_t^4 L_x^2})(N_4^{\frac{1}{2}+}\|G_4\|_{L_t^4 L_x^2})\|G_5\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|G_6\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

which is furthermore by using (2.21):

$$\lesssim \theta(N_1)\theta(N_2)N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}\|G_1\|_{X^{0,\frac{1}{4}+}}\|G_2\|_{X^{0,\frac{1}{4}+}}\|G_3\|_{X^{0,\frac{1}{4}+}}\|G_4\|_{X^{0,\frac{1}{4}+}}\|G_5\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|G_6\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} =$$

$$= \theta(N_1)\theta(N_2)N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}\|(\chi_{[0,\delta]}v)_{N_1,L_1}\|_{X^{0,\frac{1}{4}+}}\|v_{N_2,L_2}\|_{X^{0,\frac{1}{4}+}}$$

$$\|v_{N_3,L_3}\|_{X^{0,\frac{1}{4}+}}\|v_{N_4,L_4}\|_{X^{0,\frac{1}{4}+}}\|v_{N_5,L_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6,L_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \lesssim$$

$$\lesssim N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}\|(\mathcal{D}(\chi_{[0,\delta]}v))_{N_1,L_1}\|_{X^{0,\frac{1}{4}+}}\|(\mathcal{D}v)_{N_2,L_2}\|_{X^{0,\frac{1}{4}+}}$$

$$\|v_{N_3,L_3}\|_{X^{0,\frac{1}{4}+}}\|v_{N_4,L_4}\|_{X^{0,\frac{1}{4}+}}\|v_{N_5,L_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6,L_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

$$\lesssim N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}\frac{1}{L_1^{\frac{1}{4}-}}\|(\mathcal{D}(\chi_{[0,\delta]}v))_{N_1,L_1}\|_{X^{0,\frac{1}{2}-}}\frac{1}{L_2^{\frac{1}{4}}}\|(\mathcal{D}v)_{N_2,L_2}\|_{X^{0,\frac{1}{2}+}}$$

$$\frac{1}{N_3L_3^{\frac{1}{4}}}\|v_{N_3,L_3}\|_{X^{1,\frac{1}{2}+}}\frac{1}{N_4L_4^{\frac{1}{4}}}\|v_{N_4,L_4}\|_{X^{1,\frac{1}{2}+}}\frac{1}{N_5^{\frac{1}{2}-}L_5^{0+}}\|v_{N_5,L_5}\|_{X^{1,\frac{1}{2}+}}\frac{1}{N_6^{\frac{1}{2}-}L_6^{0+}}\|v_{N_6,L_6}\|_{X^{1,\frac{1}{2}+}}$$

By (2.24) and the definition of the localizations w.r.t. N_j, L_j, this quantity is:

$$\lesssim N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}\frac{1}{L_1^{\frac{1}{4}-}}\frac{1}{L_2^{\frac{1}{4}}}\frac{1}{N_3L_3^{\frac{1}{4}}}\frac{1}{N_4L_4^{\frac{1}{4}}}\frac{1}{N_5^{\frac{1}{2}-}L_5^{0+}}\frac{1}{N_6^{\frac{1}{2}-}L_6^{0+}}$$

$$\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}} \lesssim$$

$$\lesssim \frac{1}{(L_1L_2)^{\frac{1}{4}-}}\frac{1}{N_3^{\frac{1}{2}-}N_4^{\frac{1}{2}-}N_5^{\frac{1}{2}-}N_6^{\frac{1}{2}-}}\frac{1}{L_1^{0+}L_2^{0+}L_3^{\frac{1}{4}}L_3^{\frac{1}{4}}L_5^{0+}L_6^{0+}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}}$$

From (2.59) and Proposition 2.3.1, this is:

$$\lesssim \frac{1}{N_1^{\frac{1}{2}-}}\frac{1}{N_3^{\frac{1}{2}-}N_4^{\frac{1}{2}-}N_5^{\frac{1}{2}-}N_6^{\frac{1}{2}-}}\frac{1}{L_1^{0+}L_2^{0+}L_3^{\frac{1}{4}}L_3^{\frac{1}{4}}L_5^{0+}L_6^{0+}}\|\mathcal{D}\Phi\|^2_{L^2}\|\Phi\|^4_{H^1} \lesssim$$

$$\lesssim \frac{1}{N_1^{\frac{1}{2}-}}\frac{1}{N_2^{0+}N_3^{\frac{1}{2}-}N_4^{\frac{1}{2}-}N_5^{\frac{1}{2}-}N_6^{\frac{1}{2}-}}\frac{1}{L_1^{0+}L_2^{0+}L_3^{\frac{1}{4}}L_3^{\frac{1}{4}}L_5^{0+}L_6^{0+}}\|\mathcal{D}\Phi\|^2_{L^2}. \tag{2.60}$$

In order to deduce the last bound, we used the fact that: $N_1 \sim N_2$ and $\|\Phi\|_{H^1} \lesssim 1$.

Subcase 2:

We recall that in this subcase, one has:

$$max\{|\tau_3|, |\tau_4|, |\tau_5|, |\tau_6|\} \gtrsim N_1^2.$$

Let us consider the case: $|\tau_3| = max\{|\tau_3|, |\tau_4|, |\tau_5|, |\tau_6|\}$. We can analogously consider the other cases, but we have to group the factors in Hölder's Inequality then [5]. Let us localize as in the previous subcase, and let us denote by $J^2_{\bar{L},\bar{N}}$ the contribution to $J_{\bar{L},\bar{N}}$ coming from this subcase.

Suppose now that $(\tau_3, n_3) \in supp\ \widetilde{v_{N_3,L_3}}$, keeping in mind the assumptions of the subcase. Then:

$$|n_3| \sim N_3 \ll N_1^{\frac{1}{2}}, |\tau_3| \gtrsim N_1^2 \Rightarrow |\tau_3 + n_3^2| \gtrsim N_1^2 - N_1 \gtrsim N_1^2.$$

Consequently:

$$L_3 \gtrsim N_1^2. \tag{2.61}$$

Arguing analogously as in the previous subcase, we obtain:

[5] We take the $L^4_t L^2_x$ norm of the factor with highest $|\tau|$.

$$|J^2_{\bar{L},\bar{N}}| \lesssim \theta(N_1)\theta(N_2)N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}$$

$$\|(\chi_{[0,\delta]}v)_{N_1,L_1}\|_{X^{0,\frac{1}{4}+}}\|v_{N_2,L_2}\|_{X^{0,\frac{1}{4}+}}\|v_{N_3,L_3}\|_{X^{0,\frac{1}{4}+}}\|v_{N_4,L_4}\|_{X^{0,\frac{1}{4}+}}\|v_{N_5,L_5}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|v_{N_6,L_6}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \lesssim$$

$$\lesssim N_3^{\frac{1}{2}+}N_4^{\frac{1}{2}+}\frac{1}{L_1^{\frac{1}{4}-}}\|\mathcal{D}v_{N_1,L_1}\|_{X^{0,\frac{1}{2}+}}\frac{1}{L_2^{\frac{1}{4}}}\|\mathcal{D}v_{N_2,L_2}\|_{X^{0,\frac{1}{2}+}}$$

$$\frac{1}{N_3L_3^{\frac{1}{4}}}\|v_{N_3,L_3}\|_{X^{1,\frac{1}{2}+}}\frac{1}{N_4L_4^{\frac{1}{4}}}\|v_{N_4,L_4}\|_{X^{1,\frac{1}{2}+}}\frac{1}{N_5^{\frac{1}{2}-}L_5^{0+}}\|v_{N_5,L_5}\|_{X^{1,\frac{1}{2}+}}\frac{1}{N_6^{\frac{1}{2}-}L_6^{0+}}\|v_{N_6,L_6}\|_{X^{1,\frac{1}{2}+}} \lesssim$$

$$\lesssim \frac{1}{L_3^{\frac{1}{4}-}}\frac{1}{N_3^{\frac{1}{2}-}N_4^{\frac{1}{2}-}N_5^{0+}N_6^{0+}}\frac{1}{L_1^{\frac{1}{4}-}L_2^{\frac{1}{4}}L_3^{0+}L_4^{\frac{1}{4}}L_5^{0+}L_6^{0+}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}}$$

which by (2.61) is:

$$\lesssim \frac{1}{N_1^{\frac{1}{2}-}}\frac{1}{N_3^{\frac{1}{2}-}N_4^{\frac{1}{2}-}N_5^{0+}N_6^{0+}}\frac{1}{L_1^{\frac{1}{4}-}L_2^{\frac{1}{4}}L_3^{0+}L_4^{\frac{1}{4}}L_5^{0+}L_6^{0+}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{N_1^{\frac{1}{2}-}}\frac{1}{N_2^{0+}N_3^{\frac{1}{2}-}N_4^{\frac{1}{2}-}N_5^{0+}N_6^{0+}}\frac{1}{L_1^{\frac{1}{4}-}L_2^{\frac{1}{4}}L_3^{0+}L_4^{\frac{1}{4}}L_5^{0+}L_6^{0+}}\|\mathcal{D}\Phi\|^2_{L^2}. \tag{2.62}$$

$\diamond$ **Case 2:** $N_3 \gtrsim N_1^{\frac{1}{2}}$. Let us recall that we want to estimate:

$$J_{N_1,N_2,N_3,N_4,N_5,N_6} =$$

$$\sum_{n_1+n_2+n_3-n_4-n_5-n_6=0,|n_1|\geq\ldots\geq|n_6|}\int_{\mathbb{R}} M_6'(\chi_{[0,\delta]}\widehat{v_{N_1}}(n_1))\widehat{v_{N_2}}(n_2)\widehat{v_{N_3}}(n_3)\overline{\widehat{v_{N_4}}(n_4)}\,\overline{\widehat{v_{N_5}}(n_5)}\,\overline{\widehat{v_{N_6}}(n_6)}dt.$$

Let us note that:

$$|M_6'| = |(\theta(n_1))^2+(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2-(\theta(n_5))^2-(\theta(n_6))^2| \lesssim \theta(N_1)\theta(N_2).$$

We note that this Case is analogous to Case 2 of Big Case 1. Hence, arguing exactly as we did in this Case, we obtain:

$$|J_{N_1,N_2,N_3,N_4,N_5,N_6}| \lesssim \frac{1}{N_1^{\frac{1}{2}-}(N_2N_3N_4N_5N_6)^{0+}}\|\mathcal{D}\Phi\|_{L^2}^2. \tag{2.63}$$

We combine (2.60),(2.62),(2.63) and sum in N_j, L_j to deduce that the contribution to I from Big Case 2, which we denoted by $I^{(2)}$ has the property that:

$$|I^{(2)}| \lesssim \frac{1}{N^{\frac{1}{2}-}}\|\mathcal{D}\Phi\|_{L^2}^2. \tag{2.64}$$

This gives us a good bound in Big Case 2. Combining (2.54) and (2.64), we finally obtain:

$$|I| = |\|\mathcal{D}u(\delta)\|_{L^2}^2 - \|\mathcal{D}u(0)\|_{L^2}^2| \lesssim \frac{1}{N^{\frac{1}{2}-}}\|\Phi\|_{L^2}^2.$$

By construction, the implied constant here depends only on $(s, Energy, Mass)$. Let us denote it by $C = C(s, Energy, Mass)$. We use Proposition 2.3.1 and the fact that the H^1 norm can be bounded by a continuous function of energy and mass to deduce that C is continuous in energy and mass. Lemma 2.3.4 now follows.

□

2.3.4 Proof of Theorem 2.1.1 for $k \geq 3$

We finally note that for $k \geq 3$, we can bound the increment of $\|\mathcal{D}u(t)\|_{L^2}^2$ in an analogous way as we did for $k = 2$. Namely, we observe that all the estimates on M_6, M_6' we used depended only on the two highest frequencies and not on how many more frequencies there were. Furthermore, in the later estimates, when we had to use Hölder's inequality, we just estimate the $k-2$ extra factors in $L^\infty_{t,x}$, and use the fact that $X^{\frac{1}{2}+,\frac{1}{2}+} \hookrightarrow L^\infty_{t,x}$. At the end, this only results in a "0+ loss" in the dyadic decay factor, and we get the same increment bound (2.36) as before.

This finishes the proof of Theorem 2.1.1 for $k \geq 2$. □

2.3.5 Remarks on the result of Bourgain

As was mentioned in Section 1.3., in the appendix of [19], Bourgain gives a sketch of how one should be able to deduce a better bound in the case $k = 2$ though. The methods he indicates there don't seem to apply to the higher nonlinearities $k > 2$. The problem lies in the fact that the inductive procedure from [19] is linked to the quintic structure of the nonlinearity.

Bourgain starts by defining the following Besov-type norms:

$$\|u\|_{0,p} := \Big(\int_{\mathbb{R}} \Big(\sum_j |\widetilde{u}(j, j^2 + \xi)|^2\Big)^{\frac{p}{2}} d\xi\Big)^{\frac{1}{p}}.$$

This space is similar to the $X^{s,b}$ space we are using, but $X^{s,b}$ spaces were not used in [19]. The estimate one starts from is the following Strichartz Estimate: Assuming that *supp* $\hat{\phi} \subseteq \{-N, \dots, N\}$, one has:

$$\|S(t)\phi\|_{L^6_{t,x}} \lesssim N^{0+} \|\phi\|_{L^2_x}. \tag{2.65}$$

Suppose now that $q = q(x, t)$ has the property that *supp* $\widehat{q(t)} \subseteq \{-N, \dots, N\}$. By writing u as a superposition of modulated free solutions (c.f. Lemma 2.9 in [106]), (2.65) implies:

$$\|q\|_{L^6_{t,x}} \lesssim N^{0+} \|q\|_{0,1}. \tag{2.66}$$

By using Hölder's inequality, one then deduces:

$$\int_{\mathbb{R}} \int_{S^1} |q(x,t)|^6 dx dt \lesssim \|q\|^6_{L^6_{t,x}} \lesssim N^{0+} \|q\|^6_{0,1}. \tag{2.67}$$

The estimate (2.67) is used as the base of the induction in the paper. At each step, the Hamiltonian is modified using a symplectic transformation of the phase space $l^2(\mathbb{Z})$ in such a way that the nonlinearity is reduced to its essential part. In each iteration, it is shown inductively that the analogue of (2.67) holds for the modified Hamiltonian.

The reason why one doesn't seem to be able to apply these methods to the case $k > 2$ is that the Besov-type norms introduced earlier don't allow us to control the spacetime L^∞ norm in a satisfactory way. On the other hand, we recall that for $X^{s,b}$ spaces, we used the bound: $\|u\|_{L^\infty_{t,x}} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}$. It appears that the only estimate, one can use for the spacetime L^∞ norm is obtained as follows:

Suppose $q = q(x,t)$ satisfies *supp* $\widehat{q(t)} \subseteq \{-N, \ldots, N\}$.

Then:

$$\|q\|_{L^\infty_{t,x}} \lesssim \|q\|_{L^\infty_t H^{\frac{1}{2}+}_x} \lesssim N^{\frac{1}{2}+}\|q\|_{L^\infty_t L^2_x} \lesssim N^{\frac{1}{2}+}\|q\|_{0,1}. \tag{2.68}$$

Here, in the first step, we used Sobolev embedding and in the last step, we used the triangle inequality.

From Hölder's inequality, (2.66), and (2.68) we can deduce that for $k \geq 3$, one has:

$$\int_{\mathbb{R}}\int_{S^1} |q(x,t)|^{2k+2} dxdt \lesssim \|q\|^6_{L^6_{t,x}} \|q\|^{2k-4}_{L^\infty_{t,x}}.$$

$$\lesssim N^{0+}\|q\|^6_{0,1} N^{\frac{2k-4}{2}+}\|q\|^{2k-4}_{0,1} \lesssim N^{(k-2)+}\|q\|^{2k+2}_{0,1}. \tag{2.69}$$

We observe that this no longer gives us a N^{0+} factor on the right hand side, which was crucial in the proof in [19].

2.4 Modifications of the Cubic NLS

2.4.1 Modification 1: Hartree Equation

Let us now consider the Hartree equation on S^1, i.e. the equation (2.4). The equation (2.4) has the following conserved quantities:

$$M(u(t)) = \int |u(x,t)|^2 dx \ \ \textit{(Mass)}$$

and

$$E(u(t)) = \frac{1}{2}\int |\nabla u(x,t)|^2 dx + \frac{1}{4}\int (V * |u|^2)(x,t)|u(x,t)|^2 dx \ \ \textit{(Energy)}$$

The fact that the mass is conserved follows from the fact that V is real-valued. The fact that the energy is conserved can be checked by using the equation and integrating by parts. The calculation crucially relies on the fact that V is even, see [28]. Furthermore, since $V \geq 0$, we immediately obtain uniform bounds on $\|u(t)\|_{H^1}$. M is clearly continuous on H^1. By using Young's inequality, Hölder's inequality and Sobolev embedding, it follows that E is also continuous on H^1.

Local-in-time estimates for the Hartree Equation

Let u denote a global solution of (2.4). Recalling the definition of the operator $\mathcal{D}$ in (2.27), we have:

Proposition 2.4.1. *Given $t_0 \in \mathbb{R}$, there exists a globally defined function $v : S^1 \times \mathbb{R} \to \mathbb{C}$ satisfying the properties:*

$$v|_{[t_0,t_0+\delta]} = u|_{[t_0,t_0+\delta]}. \tag{2.70}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{2.71}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi))\|\mathcal{D}u(t_0)\|_{L^2}. \tag{2.72}$$

Moreover, δ and C can be chosen to depend continuously on the energy and mass.

Proof. The proof of Proposition 2.4.1 is analogous to the Proof of Proposition 2.3.1 (see Appendix B of this chapter). The only modification we have to make is to note that $V \in L^1(S^1)$ implies that $\widehat{V} \in L^\infty(S^1)$. Instead of estimating an expression of the form $||v_\delta|^2 v_\delta|$ as in the proof of Proposition 3.1, we have to estimate: $|(V * |v_\delta|^2)v_\delta|$.

However,

$$|(V*|v_\delta|^2)v_\delta| = \Big|\sum_{n_1+n_2+n_3=n}\int_{\tau_1+\tau_2+\tau_3=\tau} d\tau_j \widehat{V}(n_1+n_2)\widetilde{v_\delta}(n_1,\tau_1)\widetilde{\overline{v}_\delta}(n_2,\tau_2)\widetilde{v_\delta}(n_3,\tau_3)\Big| \leq$$

$$\leq \sum_{n_1+n_2+n_3=n}\int_{\tau_1+\tau_2+\tau_3=\tau} d\tau_j |\widehat{V}(n_1+n_2)||\widetilde{v_\delta}(n_1,\tau_1)||\widetilde{\overline{v}_\delta}(n_2,\tau_2)||\widetilde{v_\delta}(n_3,\tau_3)| \lesssim$$

$$\lesssim \sum_{n_1+n_2+n_3=n}\int_{\tau_1+\tau_2+\tau_3=\tau} d\tau_j |\widetilde{v_\delta}(n_1,\tau_1)||\widetilde{\overline{v}_\delta}(n_2,\tau_2)||\widetilde{v_\delta}(n_3,\tau_3)|.$$

This is the same expression that we obtain in the proof of Proposition 3.1. The existence part (i.e. the analogue of properties (2.30) and (2.31)) now follows in the same way as in the mentioned Proposition. On the other hand, for the uniqueness part (i.e. the analogue of (2.29)), let $v(t), w(t)$ solve (2.4) with the same initial data on the time interval $[0,\delta]$. We also suppose that $\|v(t)\|_{H^1}, \|w(t)\|_{H^1}$ are uniformly bounded on this interval . By Minkowski's inequality, and by unitarity of the Linear Schrödinger propagator, we obtain, for all $0 \leq t \leq \delta$:

$$\|v(t)-w(t)\|_{L^2} \leq \int_0^t \|S(t-t')((V*|v|^2)v(t') - (V*|w|^2)w(t'))\|_{L^2} dt' =$$

$$= \int_0^t \|(V*|v|^2)v(t') - (V*|w|^2)w(t')\|_{L^2} dt'$$

If we combine Hölder's inequality, Young's inequality and Sobolev embedding, we deduce:

$$\|(V*(u_1u_2))u_3\|_{L^2} \leq \|V\|_{L^1}\|u_1u_2\|_{L^\infty}\|u_3\|_{L^2} \lesssim \|u_1\|_{H^1}\|u_2\|_{H^1}\|u_3\|_{L^2}.$$

Similarly:

$$\|(V*(u_1u_2))u_3\|_{L^2} \leq \|V\|_{L^1}\|u_1u_2\|_{L^2}\|u_3\|_{L^\infty} \lesssim \|u_1\|_{L^2}\|u_2\|_{H^1}\|u_3\|_{H^1}.$$

Hence:

$$\|v(t)-w(t)\|_{L^2} \lesssim \int_0^t (\|v(t')|_{H^1}+\|w(t')\|_{H^1})^2\|v(t')-w(t')\|_{L^2}dt' \lesssim \int_0^t \|v(t')-w(t')\|_{L^2}dt'.$$

Uniqueness now follows from Gronwall's inequality. □

We will now use the method of *higher modified energies* as in [34, 32]. The key is to obtain a better approximation to $\|u(t)\|_{H^s}^2$ than $\|\mathcal{D}u(t)\|_{L^2}^2$ by using a multilinear correction term.

Introduction of the Higher Modified Energy

Before we define the multilinear correction to $E^1(u) := \|\mathcal{D}u(t)\|_{L^2}^2$, let us first find $\frac{d}{dt}\|\mathcal{D}u(t)\|_{L^2}^2$.

$$\frac{d}{dt}\|\mathcal{D}u(t)\|_{L^2}^2 \sim \frac{d}{dt}\big(\sum_{n_1+n_2=0} \widehat{\mathcal{D}u}(n_1)\widehat{\mathcal{D}\bar{u}}(n_2)\big) =$$

$$= \sum_{n_1+n_2=0} (\theta(n_1)(i\Delta u - i(V*|u|^2)u)\widehat{\ }(n_1)\theta(n_2)\widehat{\bar{u}}(n_2)+\theta(n_1)\widehat{u}(n_1)(-i\Delta\bar{u}+i(V*|u|^2)\bar{u})\widehat{\ }(n_2)\theta(n_2) =$$

$$= \sum_{n_1+n_2=0} \big(-i\big((\theta(n_1))^2n_1^2 - (\theta(n_2))^2n_2^2\big)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)$$
$$-i\big((\theta(n_2))^2((V*|u|^2)u)\widehat{\ }(n_1)\widehat{\bar{u}}(n_2) - (\theta(n_1))^2\widehat{u}(n_1)((V*|u|^2)\bar{u})\widehat{\ }(n_2)\big)\big) =$$

$$= -i \sum_{n_1+n_2+n_3+n_4=0} ((\theta(n_2))^2\widehat{V}(n_3+n_4)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)$$
$$-(\theta(n_1))^2\widehat{V}(n_3+n_4)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)) =$$

$$= \frac{1}{2}i \sum_{n_1+n_2+n_3+n_4=0} ((\theta(n_1))^2\widehat{V}(n_3+n_4)+(\theta(n_3))^2\widehat{V}(n_1+n_2)$$
$$-(\theta(n_2))^2\widehat{V}(n_3+n_4)-(\theta(n_4))^2\widehat{V}(n_1+n_2))\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)$$

Since V is even, so is $\widehat{V}$. Hence, when $n_1+n_2+n_3+n_4=0$, we have that: $\widehat{V}(n_1+n_2) = \widehat{V}(n_3+n_4)$. So, we deduce that:

$$\frac{d}{dt}\|\mathcal{D}u(t)\|_{L^2}^2 = ci \sum_{n_1+n_2+n_3+n_4=0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)$$

$$\widehat{V}(n_3+n_4)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4), \tag{2.73}$$

where c is a real constant.

Recalling the notation from Section 2, we consider the following *higher modified energy*

$$E^2(u) := E^1(u) + \lambda_4(M_4; u). \tag{2.74}$$

The quantity M_4 will be determined soon.

The modified energy E^2 comes as a "multilinear correction" of the modified energy E^1 considered earlier:

In order to find $\frac{d}{dt}E^2(u)$, we need to find $\frac{d}{dt}\lambda_4(M_4; u)$. Thus, if we fix a multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4; u) =$$

$$\frac{d}{dt}\Big(\sum_{n_1+n_2+n_3+n_4=0} M_4(n_1,n_2,n_3,n_4)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\Big) =$$

$$= -i\lambda_4(M_4(n_1^2 - n_2^2 + n_3^2 - n_4^2); u)$$

$$-i \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \big[M_4(n_{123}, n_4, n_5, n_6)\widehat{V}(n_1+n_2)$$

$$-M_4(n_1, n_{234}, n_5, n_6)\widehat{V}(n_2 + n_3) + M_4(n_1, n_2, n_{345}, n_6)\widehat{V}(n_3 + n_4)$$

$$-M_4(n_1, n_2, n_3, n_{456})\widehat{V}(n_4 + n_5)\big]\widehat{u}(n_1)\widehat{\overline{u}}(n_2)\widehat{u}(n_3)\widehat{\overline{u}}(n_4)\widehat{u}(n_5)\widehat{\overline{u}}(n_6) \tag{2.75}$$

From (2.73), (2.75), it follows that if we take:

$$M_4 := \Psi, \tag{2.76}$$

where Ψ is defined by:

$$\Psi : \Gamma_4 \to \mathbb{R}$$

$$\Psi := \begin{cases} c\frac{((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2)\widehat{V}(n_3+n_4)}{n_1^2 - n_2^2 + n_3^2 - n_4^2}, \text{ if } n_1^2 - n_2^2 + n_3^2 - n_4^2 \neq 0 \\ 0, \text{ otherwise.} \end{cases} \tag{2.77}$$

for an appropriate real constant c. One then has:

$$\frac{d}{dt}E^2(u) = -i\lambda_6(M_6; u). \tag{2.78}$$

where:

$$M_6(n_1, n_2, n_3, n_4, n_5, n_6) := M_4(n_{123}, n_4, n_5, n_6)\widehat{V}(n_1 + n_2)$$

$$-M_4(n_1, n_{234}, n_5, n_6)\widehat{V}(n_2 + n_3) + M_4(n_1, n_2, n_{345}, n_6)\widehat{V}(n_3 + n_4)$$

$$-M_4(n_1, n_2, n_3, n_{456})\widehat{V}(n_4 + n_5) \tag{2.79}$$

Heuristically, we expect this expression to be smaller than $\frac{d}{dt}E^1(u)$ since the derivatives are distributed over six factors of u and $\bar{u}$, whereas before we only had four factors. The key to continue our study of $E^2(u)$ is to deduce bounds on Ψ.

Pointwise bounds on the multiplier Ψ

As in the previous section, we dyadically localize the frequencies as $|n_j| \sim N_j$. We then order the N_j's in decreasing order, to obtain $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. Let us show that the following result holds:

Lemma 2.4.2. *Under the previous assumptions, one has:*

$$\Psi = O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*\big). \tag{2.80}$$

Proof. From the triangle inequality and from the definition of θ, it follows that we need to consider only:

$$N_1^* \sim N_2^* \gtrsim N. \tag{2.81}$$

Furthermore, by construction of Ψ, we just need to prove the bound when $n_1^2 - n_2^2 + n_3^2 - n_4^2 \neq 0$.

We recall that:

$$|\widehat{V}| \lesssim 1 \tag{2.82}$$

Hence, the factor of $\widehat{V}(n_3 + n_4)$ will not affect the estimate.

In the proof of Lemma 2.4.2, it is crucial to observe that, for $(n_1, n_2, n_3, n_4) \in \Gamma_4$:

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 = (n_1 - n_2)(n_1 + n_2) + (n_3 - n_4)(n_3 + n_4) = (n_1 - n_2)(n_1 + n_2) - (n_3 - n_4)(n_1 + n_2) =$$

$$= (n_1 + n_2)(n_1 - n_2 - n_3 + n_4) = 2(n_1 + n_2)(n_1 + n_4) \tag{2.83}$$

In particular, when $n_1^2 - n_2^2 + n_3^2 - n_4^2 \neq 0$, one has: $n_1 + n_2, n_1 + n_4 \neq 0$.

We must consider several cases:

$\diamondsuit$**Case 1:** $N_2^* \gg N_3^*$.

$\diamondsuit$**Case 2:** $N_2^* \sim N_3^*$.

Case 1: Let's suppose WLOG that: $|n_1| \geq |n_3|, |n_2| \geq |n_4|$, and $|n_1| \sim N_1^*$.

One needs to consider two Subcases:

$\diamond$**Subcase 1:** $|n_2| \sim N_2^*$.

$\diamond$**Subcase 2:** $|n_3| \sim N_2^*$.

Subcase 1:

Since $n_1 + n_2 + n_3 + n_4 = 0, |n_1|, |n_2| \gg |n_3|, |n_4|$, it follows that n_1 and n_2 have the opposite sign.

Consequently:

$$|n_1 + n_2| = ||n_1| - |n_2||.$$

However, $|n_1 + n_2| = |n_3 + n_4|$ so:

$$||n_1| - |n_2|| = |n_3 + n_4|.$$

From (2.83), one obtains:

$$|n_1^2 - n_2^2 + n_3^2 - n_4^2| = 2|(n_1 + n_2)(n_1 + n_4)| \sim N_1^* |n_3 + n_4|. \tag{2.84}$$

In the last estimate, we used the fact that $|n_1| \gg |n_4|$ and $|n_1 + n_2| = |n_3 + n_4|$.

Let us now analyze the numerator. We start by observing that [6]:

$$|(\theta(n_1))^2 - (\theta(n_2))^2| \leq$$

$$\leq \frac{1}{N^{2s}}(|n_1|^{2s} - |n_2|^{2s}) \lesssim \frac{1}{N^{2s}}|n_1|^{2s-1}||n_1| - |n_2|| = \frac{1}{N^{2s}}|n_1|^{2s-1}|n_3 + n_4|. \tag{2.85}$$

We now have to consider $(\theta(n_3))^2 - (\theta(n_4))^2$.

One must consider three possibilities:

[6]We are considering $|n_1| \geq |n_2|, |n_1| \geq N$; it's possible that $|n_2| < N$, but this is accounted for by the "$\leq$".

Sub-subcase 1: $|n_3|, |n_4| < N$.

Sub-subcase 2: $|n_4| < N \leq |n_3|$ or $|n_3| < N \leq |n_4|$.

Sub-subcase 3: $|n_3|, |n_4| \geq N$.

Sub-subcase 1: In this sub-subcase, one has:$(\theta(n_3))^2 - (\theta(n_4))^2 = 0$.

Sub-subcase 2: Let's consider WLOG the case when $|n_4| < N \leq |n_3|$. The case $|n_3| < N \leq |n_4|$ is analogous.

We obtain:

$$|(\theta(n_3))^2 - (\theta(n_4))^2| = \frac{1}{N^{2s}} ||n_3|^{2s} - N^{2s}| \leq \frac{1}{N^{2s}} ||n_3|^{2s} - |n_4|^{2s}| =$$

$$= \frac{1}{N^{2s}} ||n_3|^{2s} - |-n_4|^{2s}| \lesssim \frac{1}{N^{2s}} |n_3|^{2s-1} |n_3 + n_4|.$$

We note that the first inequality follows from the assumptions of the sub-subcase.

Sub-subcase 3:

We note:

$$|(\theta(n_3))^2 - (\theta(n_4))^2| = \frac{1}{N^{2s}} ||n_3|^{2s} - |n_4|^{2s}|.$$

Arguing as in the previous sub-subcase, we obtain:

$$|(\theta(n_3))^2 - (\theta(n_4))^2| \lesssim \frac{1}{N^{2s}} |n_3|^{2s-1} |n_3 + n_4|.$$

So, we obtain that in Subcase 1, one has the bound:

$$|(\theta(n_3))^2 - (\theta(n_4))^2| \lesssim \frac{1}{N^{2s}} |n_3|^{2s-1} |n_3 + n_4| \lesssim \frac{1}{N^{2s}} (N_1^*)^{2s-1} |n_3 + n_4|. \tag{2.86}$$

Combining (2.85) and (2.86), one obtains:

$$|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2| \lesssim \frac{1}{N^{2s}} (N_1^*)^{2s-1} |n_3 + n_4|. \tag{2.87}$$

From (2.82), (2.84), and (2.87), it follows that in Subcase 1:

$$\Psi = O\big(\frac{1}{N^{2s}}(N_1^*)^{2s-2}\big) = O\big(\frac{1}{(N_1^*)^2}\frac{(N_1^*)^{2s}}{N^{2s}}\big) = O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{2.88}$$

Subcase 2:

Here one has $|n_3| \sim N_2^*$. In this Subcase, we don't expect to obtain any cancelation in the numerator or in the denominator. We get:

$$|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2| = O((\theta(n_1))^2) = O(\frac{1}{N^{2s}}(N_1^*)^{2s})$$

$$|n_1^2 - n_2^2 + n_3^2 - n_4^2| \sim (N_1^*)^2.$$

So, again using (2.82), we deduce:

$$\Psi = O\big(\frac{1}{N^{2s}}(N_1^*)^{2s-2}\big) = O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{2.89}$$

Case 2:

Subcase 1:

We first consider the subcase when: $N_1^* \sim N_2^* \sim N_3^* \gg N_4^*$.

Let us assume WLOG that $|n_4| \sim N_4^*$.

Then, by (2.83), one has:

$$||n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2| = 2|(n_1 + n_2)(n_1 + n_4)| \gtrsim N_1^*. \tag{2.90}$$

Here, we also used the fact that $|n_1 + n_4| \sim N_1^*$ and $|n_1 + n_2| \geq 1$. The latter observation follows from the fact that the problem is periodic.

We bound the numerator by:

$$|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2| \lesssim (\theta(n_1))^2 \lesssim \frac{1}{N^{2s}}(N_1^*)^{2s}. \tag{2.91}$$

It follows from (2.90), (2.91), and (2.82) that:

$$\Psi = O\big(\frac{1}{N_1^*}\frac{(N_1^*)^{2s}}{N^{2s}}\big) = O\big(\frac{1}{(N_1^*)^2}\frac{(N_1^*)^{2s}}{N^{2s}}N_1^*\big) =$$

$$= O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)N_3^*\big). \tag{2.92}$$

Subcase 2:

In this case, all the N_j^*'s are equivalent:

$$N_1^* \sim N_2^* \sim N_3^* \sim N_4^*.$$

By using (2.83), and the fact that $|n_1 + n_2| \geq 1, |n_1 + n_4| \geq 1$, it follows that:

$$||n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2| = 2|(n_1 + n_2)(n_1 + n_4)| \gtrsim 1. \tag{2.93}$$

As before:

$$|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2| \lesssim (\theta(n_1))^2 \lesssim \frac{1}{N^{2s}}N_1^{*2s}. \tag{2.94}$$

(2.82), (2.94), and (2.93) now imply:

$$\Psi = O\big(\frac{(N_1^*)^{2s}}{N^{2s}}\big) = O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)(N_1^*)^2\big) =$$

$$= O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*\big). \tag{2.95}$$

Lemma 2.4.2 now follows from (2.88),(2.89),(2.92), and (2.95). $\square$

An approximation result for the higher modified energies

Let us now show that $E^2(u)$ is a good approximation of $E^1(u)$ in a certain precise sense. The result that we prove is:

Lemma 2.4.3. *If we take N to be sufficiently large, then:*

$$E^2(u) \sim E^1(u),$$

where the implied constant no longer depends on N, but depends continuously on energy and mass.

Proof. By construction, we have that: $|E^2(u(t)) - E^1(u(t))| = |\lambda_4(M_4; u(t))|$, where M_4 has been defined in (2.76).Let us WLOG consider the contribution to $\lambda_4(M_4; u(t))$ in which $|n_1| \geq |n_2| \geq |n_3| \geq |n_4|$. The other contributions are bounded analogously. With notation from before, we obtain the following localization:

$$N_1^* \geq N_2^* \geq N_3^* \geq N_4^*;\ N_1^* \gtrsim N. \tag{2.96}$$

Using Lemma 2.4.2 we note that the corresponding contribution to $|E^2(u) - E^1(u)|$ is:

$$\lesssim \sum_{n_1+n_2+n_3+n_4=0, |n_1| \geq \ldots \geq |n_4|} \ \sum_{N_j^* \text{ satisfying } (2.96)}$$

$$\frac{1}{(N_1^*)^2} \theta(N_1^*)\theta(N_1^*) N_3^* N_4^* |\widehat{u_{N_1^*}}(n_1)||\widehat{\overline{u_{N_2^*}}}(n_2)||\widehat{u_{N_3^*}}(n_3)||\widehat{\overline{u_{N_4^*}}}(n_4)|.$$

By taking inverse Fourier transforms, using an $L^2_x, L^2_x, L^\infty_x, L^\infty_x$ Hölder inequality, the $H^{\frac{1}{2}+}_x \hookrightarrow L^\infty_x$ Sobolev embedding and the fact that $\|\cdot\|_{L^2_x}, \|\cdot\|_{H^{\frac{1}{2}+}_x}$ are invariant under change of sign in the Fourier transform, we obtain that the previous quantity is:

$$\lesssim \sum_{N_j^* \text{ satisfying } (2.96)} \frac{1}{(N_1^*)^{1-}} \|\theta(N_1^*) u_{N_1^*}\|_{L^2} \|\theta(N_2^*) u_{N_2^*}\|_{L^2} \|(N_3^*)^{\frac{1}{2}-} u_{N_3^*}\|_{H^{\frac{1}{2}+}} \|(N_4^*)^{\frac{1}{2}-} u_{N_4^*}\|_{H^{\frac{1}{2}+}} \lesssim$$

$$\lesssim \sum_{N_j^* \text{ satisfying } (2.96)} \frac{1}{(N_1^*)^{1-}} \|\mathcal{D}u\|_{L^2}^2 \|u\|_{H^1}^2$$

$$\lesssim \frac{1}{N^{1-}} \|\mathcal{D}u\|_{L^2}^2 = \frac{1}{N^{1-}} E^1(u).$$

The other contributions are bounded in an analogous way. Hence,

$$|E^2(u) - E^1(u)| \lesssim \frac{1}{N^{1-}} E^1(u).$$

Thus, if we take N sufficiently large, we get for the fixed time t:

$$E^2(u(t)) \sim E^1(u(t)). \tag{2.97}$$

The implied constant above doesn't depend on N as long as we choose N to be sufficiently large. It also doesn't depend on t. We see that it depends on the uniform bound on $\|u(t)\|_{H^1}$, hence it depends continuously on energy and mass.

□

Hence, in order to bound $E^1(u)$, it suffices to bound $E^2(u)$.

Estimate on the increment of $E^2(u)$ and proof of Theorem 2.1.2

For $t_0 \in \mathbb{R}$,we now want to estimate the increment:

$$E^2(u(t_0 + \delta)) - E^2(u(t_0)).$$

The bound that we will prove is:

Lemma 2.4.4. *For all $t_0 \in \mathbb{R}$, one has:*

$$|E^2(u(t_0 + \delta)) - E^2(u(t_0))| \lesssim \frac{1}{N^{2-}} E^2(u(t_0)).$$

Let us observe how Lemma 2.4.4 implies Theorem 2.1.2:

Proof. (*of Theorem 2.1.2 assuming Lemma 2.4.4*) We argue similarly as in the proof of Theorem 2.1.1. Namely, from Lemma 2.4.4, together with (2.28) and Lemma 2.4.3, we deduce that:

$$E^2(u(T)) \lesssim E^2(\Phi) \lesssim E^1(\Phi) = \|\mathcal{D}\Phi\|_{L^2}^2 \lesssim \|\Phi\|_{H^s}^2, \tag{2.98}$$

whenever $T \lesssim N^{2-}$.

So, for such T, one has, from (2.28), Lemma 2.4.3, and (2.98):

$$\|u(T)\|_{H^s} \lesssim N^s \sqrt{E^1(u(T))} \lesssim N^s \sqrt{E^2(u(T))} \lesssim N^s \|\Phi\|_{H^s}. \tag{2.99}$$

Since $T \lesssim N^{2-}$, we can take $N = T^{\frac{1}{2}+}$. Substituting this into (2.99), we obtain:

$$\|u(T)\|_{H^s} \lesssim T^{\frac{1}{2}s+} \|\Phi\|_{H^s}. \tag{2.100}$$

Here the implied constants depend only on $(s, Energy, Mass)$, and they depend continuously on energy and mass.

Using (2.100), and arguing as in the proof of Theorem 2.1.1, we obtain that, for $s \geq 1$, there exists $C = C(s, Energy, Mass)$, depending continuously on energy and mass such that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{\frac{1}{2}s+} \|\Phi\|_{H^s}. \tag{2.101}$$

□

We now prove Lemma 2.4.4:

Proof. (*of Lemma 2.4.4*) From Proposition 2.4.1, given t_0, we can construct a global function v which agrees with u on $[t_0, t_0 + \delta]$ and which satisfies appropriate $X^{s,b}$ bounds. Let's WLOG suppose that $t_0 = 0$. We note that all the constants depend only on conserved quantities of the equation, and hence will be independent of t_0. From Lemma 2.4.3, one obtains for $t \in [0, \delta]$:

$$E^2(v) \sim E^1(v).$$

Furthermore, from (2.78) and the construction of v, we recall for $t \in [0, \delta]$:

$$\frac{d}{dt}E^2(v(t)) = -i\lambda_6(M_6; v(t)).$$

We want to estimate $\int_0^\delta \frac{d}{dt}E^2(v)(t)dt$. In order to do this, we just consider the contribution:

$$\int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6)\widehat{V}(n_1+n_2)$$
$$\widehat{v}(n_1)\widehat{\overline{v}}(n_2)\widehat{v}(n_3)\widehat{\overline{v}}(n_4)\widehat{v}(n_5)\widehat{\overline{v}}(n_6)dt =: K \tag{2.102}$$

By symmetry, the other contributions are bounded in an analogous way, since as we will see, our argument won't depend on which factor comes with a complex conjugate, and which factor doesn't.

Let us now dyadically localize in frequency, with the following localizations:

$$|n_1+n_2+n_3| \sim N_1,\ |n_4| \sim N_2,\ |n_5| \sim N_3,\ |n_6| \sim N_4.$$

As before, we introduce the dyadic integers $N_1^*, N_2^*, N_3^*, N_4^*$. It is then the case that:

$$N_1^* \geq N_2^* \geq N_3^* \geq N_4^*,\ N_1^* \gtrsim N. \tag{2.103}$$

The latter fact follows from the fact that the only nonzero contribution comes from the case where $(\theta(n_1+n_2+n_3))^2 - (\theta(n_4))^2 + (\theta(n_5))^2 - (\theta(n_6))^2 \neq 0$. Let's fix an admissible configuration (N_1, N_2, N_3, N_4) and let's denote its contribution to K by:

$$K_{N_1,N_2,N_3,N_4} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_1+n_2+n_3, n_4, n_5, n_6)$$
$$\widehat{(v\bar{v}v)}_{N_1}(n_1+n_2+n_3)\widehat{\overline{v_{N_2}}}(n_2)\widehat{v_{N_3}}(n_5)\widehat{\overline{v_{N_4}}}(n_6)dt.$$

We must consider several cases:

$\diamondsuit$**Case 1:** $N_1 = N_1^*$ or $N_1 = N_2^*$.

$\diamondsuit$**Case 2:** $N_1 = N_3^*$ or $N_1 = N_4^*$.

Case 1:

By symmetry, we consider the case $N_1 = N_1^*$. We will also consider the case when $N_2 = N_2^*, N_3 = N_3^*, N_4 = N_4^*$. The other cases are bounded in a similar way (we just group the terms differently). We obtain:

$$|K_{N_1,N_2,N_3,N_4}| = |\int_{\mathbb{R}} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0}$$

$$M_4(n_1+n_2+n_3,n_4,n_5,n_6)\widehat{(v\bar{v}v)}_{N_1}(n_1+n_2+n_3)(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4)\widehat{v_{N_3}}(n_5)\widehat{\overline{v}_{N_4}}(n_6)dt\Big| =$$

$$= |\int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_1+n_2+n_3,n_4,n_5,n_6)$$

$$\widetilde{(v\bar{v}v)}_{N_1}(n_1+n_2+n_3,\tau_1+\tau_2+\tau_3)(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4,\tau_4)\widetilde{v_{N_3}}(n_5,\tau_5)\widetilde{\overline{v}_{N_4}}(n_6,\tau_6)d\tau_j\Big|$$

Using the triangle inequality, Lemma 2.4.2, and (2.82), this expression is:

$$\lesssim \int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*$$

$$|\widetilde{(v\bar{v}v)}_{N_1}(n_1+n_2+n_3,\tau_1+\tau_2+\tau_3)||(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4,\tau_4)||\widetilde{v_{N_3}}(n_5,\tau_5)||\widetilde{\overline{v}_{N_4}}(n_6,\tau_6)|d\tau_j$$

Since $\theta(N_1^*) \sim \theta(n_1+n_2+n_3)$, by localization, and since $|\widetilde{(v\bar{v}v)}_{N_1}| \leq |\widetilde{v\bar{v}v}|$ by restriction, this expression is:

$$\lesssim \int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \frac{1}{(N_1^*)^2}\theta(n_1+n_2+n_3)\theta(N_2)N_3N_4$$

$$|\widetilde{v\bar{v}v}(n_1+n_2+n_3,\tau_1+\tau_2+\tau_3)||(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4,\tau_4)||\widetilde{v_{N_3}}(n_5,\tau_5)||\widetilde{\overline{v}_{N_4}}(n_6,\tau_6)|d\tau_j$$

$$\lesssim \int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \frac{1}{(N_1^*)^2}\theta(n_1+n_2+n_3)\theta(N_2)N_3N_4$$

$$|\widetilde{v}(n_1,\tau_1)||\widetilde{\bar{v}}(n_2,\tau_2)||\widetilde{v}(n_3,\tau_3)||(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4,\tau_4)||\widetilde{v_{N_3}}(n_5,\tau_5)||\widetilde{\overline{v}_{N_4}}(n_6,\tau_6)|d\tau_j$$

Since one has the *"Fractional Leibniz Rule"*: $\theta(n_1+n_2+n_3)\lesssim\theta(n_1)+\theta(n_2)+\theta(n_3)$, we bound this expression by:

$$\lesssim \int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \frac{1}{(N_1^*)^2}(\theta(n_1)+\theta(n_2)+\theta(n_3))|\widetilde{v}(n_1,\tau_1)||\widetilde{\bar{v}}(n_2,\tau_2)||\widetilde{v}(n_3,\tau_3)|$$

$$(\theta(N_2)|(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4,\tau_4)|)(N_3|\widetilde{v_{N_3}}(n_5,\tau_5)|)(N_4|\widetilde{\overline{v}_{N_4}}(n_6,\tau_6)|)d\tau_j.$$

By symmetry, it suffices to consider:

$$K^1_{N_1,N_2,N_3,N_4}:=\int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0} \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \frac{1}{(N_1^*)^2}\theta(n_1)|\widetilde{v}(n_1,\tau_1)||\widetilde{\bar{v}}(n_2,\tau_2)||\widetilde{v}(n_3,\tau_3)|$$

$$(\theta(N_2)|(\overline{\chi_{[0,\delta]}v}_{N_2})\widetilde{\ }(n_4,\tau_4)|)(N_3|\widetilde{v_{N_3}}(n_5,\tau_5)|)(N_4|\widetilde{\overline{v}_{N_4}}(n_6,\tau_6)|)d\tau_j=$$

We now use an $L^4_{t,x},L^4_{t,x},L^4_{t,x},L^4_{t,x},L^\infty_{t,x},L^\infty_{t,x}$ Hölder's inequality, and argue as in previous sections to deduce that this term is:

$$\lesssim \frac{1}{(N_1^*)^2}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{\frac{1}{2}+,\frac{1}{2}+}}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}}\leq$$

$$\leq \frac{1}{(N_1^*)^2} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^4_{X^{1,\frac{1}{2}+}}$$

which by the $X^{s,b}$ bounds on v is:

$$\lesssim \frac{1}{(N_1^*)^2} \|\mathcal{D}\Phi\|^2_{L^2} \|\Phi\|^4_{H^1} \lesssim \frac{1}{(N_1^*)^2} \|\mathcal{D}\Phi\|^2_{L^2}.$$

One gets the same bound for the other contributions to K_{N_1,N_2,N_3,N_4} in this Case by symmetry.

Case 2: We recall that here $N_1 = N_3^*$ or $N_1 = N_4^*$. By symmetry, we consider the case $N_1 = N_3^*$.

Arguing analogously as in the previous Case, we get the same bound as before. The only difference is that now, in the appropriate bound for M_4, we replace N_3^* by $\langle n_1 + n_2 + n_3 \rangle$ and we then use the inequality:

$$\langle n_1 + n_2 + n_3 \rangle \lesssim \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle$$

as the "*Fractional Leibniz Rule*". So, in any case, we may conclude that:

$$|K_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{(N_1^*)^2} \|\mathcal{D}\Phi\|^2_{L^2}. \tag{2.104}$$

The implied constant depends only on $(s, Energy, Mass)$. Using (2.104),(2.103) and summing, it follows that:

$$|K| \lesssim \frac{1}{N^{2-}} \|\mathcal{D}\Phi\|^2_{L^2} = \frac{1}{N^{2-}} E^1(\Phi).$$

By using Lemma 2.4.3, it follows that:

$$|K| \lesssim \frac{1}{N^{2-}} E^2(\Phi).$$

In an analogous way, we show that the other three terms in $E^2(u(\delta)) - E^2(\Phi)$ satisfy this same bound. The same bound holds for arbitrary t_0. □

A note on Corollary 2.1.5

The same bound that we obtain for the Hartree Equation holds also for the Defocusing Cubic NLS on S^1 with the same proof. We formally take $V = \delta$. The cubic NLS is, however, completely integrable [84], so we see that the obtained bound is far from optimal. If we consider the defocusing cubic NLS on the real line, in Chapter 3, we show bounds which allow us to recover uniform bounds on the integral Sobolev norms of a solution, up to a loss of $(1+|t|)^{0+}$. The proof of this result relies on the improved Strichartz estimate and is at the moment possible only on the real line.

Further remarks

Remark 2.4.5. *The equation* (2.4) *possesses solutions all of whose Sobolev norms are uniformly bounded in time. Namely, if we take $n \in \mathbb{Z}$, and $\alpha \in \mathbb{C}$, then:*

$$u(x,t) := \alpha e^{-i\widehat{V}(0)|\alpha|^2 t} e^{i(nx - n^2 t)}$$

is a solution to (2.4). Since our assumptions on V imply that $\widehat{V}(0) = \int V(x)dx$ is real, it follows that for all $s, t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} = \|u(0)\|_{H^s}.$$

A similar ansatz was used to show instability of the cubic NLS on S^1 in Sobolev spaces of negative index in [21].

Remark 2.4.6. *We could try to construct a third modified energy E^3, in hope of obtaining a better bound. The algebra, however becomes quite complicated so we have not pursued this approach. Several iterations of the higher modified energies were previously used in [34].*

Remark 2.4.7. *The method of higher modified energies doesn't work for the equations we considered in Theorem 2.1.1, i.e. if the nonlinearity is $|u|^k$ for $k \geq 2$. The reason why this is so is that the analogue of the multiplier Ψ on Γ_{2k}, which we again call Ψ,* *is not pointwise bounded. Namely, if we consider the case $k = 2$, we should take:*

$$\psi \sim \frac{(\theta(n_1))^{2s} - (\theta(n_2))^{2s} + (\theta(n_3))^{2s} - (\theta(n_4))^{2s} + (\theta(n_5))^{2s} - (\theta(n_6))^{2s}}{|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 - |n_6|^2}.$$

Let us assume that s is such that:

$$6^{2s} - 2^{2s} + 5^{2s} - 3^{2s} + 1^{2s} - 7^{2s} \neq 0.$$

Then, we know that:

$$(n_1, n_2, n_3, n_4, n_5, n_6) = (6N, -2N, 5N, -3N, N, -7N) \in \Gamma_6.$$

For this frequency configuration, we have:

$$(\theta(n_1))^{2s} - (\theta(n_2))^{2s} + (\theta(n_3))^{2s} - (\theta(n_4))^{2s} + (\theta(n_5))^{2s} - (\theta(n_6))^{2s} =$$

$$6^{2s} - 2^{2s} + 5^{2s} - 3^{2s} + 1^{2s} - 7^{2s} \neq 0$$

and

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 - |n_6|^2 = 36N^2 - 4N^2 + 25N^2 - 9N^2 + N^2 - 49N^2 = 0.$$

Hence, $\Psi(n_1, n_2, n_3, n_4, n_5, n_6)$ is not well-defined. In particular, in this case, we can no longer prove a pointwise multiplier bound as in Lemma 2.4.2. A similar construction can be adapted to the case $k > 2$, if we just take the remaining 2k-4 frequencies to be equal to zero. We note that the phenomenon that the multiplier ψ is unbounded in the case of the quintic and higher order nonlinearities is linked to the fact that the factorization property (2.83) *no longer holds in this context.*

2.4.2 Modification 2: Defocusing Cubic NLS with a potential

Let us now consider the equation (2.5).The equation (2.5) has conserved mass as before, since $|u|^2 + \lambda$ is real-valued. On the other hand, by integrating by parts, one can check that the quantity:

$$E(u(t)) := \frac{1}{2}\int |\nabla u(x,t)|^2 dx + \frac{1}{4}\int |u(x,t)|^4 dx + \frac{1}{2}\int \lambda(x)|u(x,t)|^2 dx$$

is conserved in time. $E(u(t))$ is the conserved energy. By using Hölder's inequality and Sobolev embedding, it follows that E is continuous on H^1.

We note that E is not necessarily non-negative and that it doesn't give an a priori bound on $\|u(t)\|_{\dot{H}^1}$. However, since λ is bounded from below, we obtain:

$$\|u(t)\|_{H^1}^2 \lesssim E(u(t)) + M(u(t)).$$

Hence $\|u(t)\|_{H^1}$ is uniformly bounded.

Local-in-time estimates for (2.5)

Let u be a global solution of (2.5).

Proposition 2.4.8. *Given $t_0 \in \mathbb{R}$, there exists a globally defined function $v : S^1 \times \mathbb{R} \to \mathbb{C}$ satisfying the properties:*

$$v|_{[t_0,t_0+\delta]} = u|_{[t_0,t_0+\delta]}. \tag{2.105}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{2.106}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi))\|\mathcal{D}u(t_0)\|_{L^2}. \tag{2.107}$$

Moreover, δ and C can be chosen to depend continuously on the energy and mass.

Proof. The proof is similar to the proof of Proposition 2.3.1 and Proposition 2.4.1. For the existence part, we argue by a fixed-point method. Let us take $\delta \in (0,1)$, and let $f \in C_0^\infty(\mathbb{R})$ be such that $f = 1$ on $[0,1]$. Let $\mu(x,t) := f(t)\lambda(x)$.

With notation as in Appendix B of this chapter, we consider:

$$Lv := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t)\int_0^\delta S(t-t')(|v_\delta|^2 v_\delta + \mu v_\delta)(t')dt'.$$

So:

$$\|Lv\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} + c\delta^{\frac{1-2b}{2}}\||v_\delta|^2 v_\delta\|_{X^{s,b-1}} + c\delta^{\frac{1-2b}{2}}\|\mu v_\delta\|_{X^{s,b-1}}.$$

The new term that we have to estimate now is $\|\mu v_\delta\|_{X^{s,b-1}}$. We argue by duality; let $c = c(n,\tau)$ be such that:

$$\sum_n \int d\tau |c(n,\tau)|^2 \leq 1.$$

By using the Fractional Leibniz Rule:

$$|\sum_n \int d\tau \langle n\rangle^s \langle \tau + n^2\rangle^{b-1} (\mu v_\delta)\widetilde{}(n,\tau)c(n,\tau)| \lesssim$$

$$\lesssim \sum_n \int d\tau \Big(\sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} d\tau_j \frac{|c(n,\tau)|}{\langle \tau + n^2\rangle^{1-b}} \langle n_1\rangle^s |\widetilde{\mu}(n_1,\tau_1)||\widetilde{v_\delta}(n_2,\tau_2)|\Big)$$

$$+ \sum_n \int d\tau \Big(\sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} d\tau_j \frac{|c(n,\tau)|}{\langle \tau + n^2\rangle^{1-b}} |\widetilde{\mu}(n_1,\tau_1)|\langle n_2\rangle^s |\widetilde{v_\delta}(n_2,\tau_2)|\Big) =: I_1 + I_2$$

Using Parseval's identity, an $L^2_{t,x}, L^4_{t,x}, L^4_{t,x}$ Hölder inequality, and (2.22), arguing as in the proof of Proposition 2.3.1, it follows that:

$$I_1 \lesssim \|c\|_{L^2_\tau l^2_n} \|\mu\|_{X^{s,\frac{3}{8}}} \|v_\delta\|_{X^{0,\frac{3}{8}}} \lesssim \delta^{r_0}\|v\|_{X^{0,b}} \leq \delta^{r_0}\|v\|_{X^{s,b}},$$

for some $r_0 > 0$. Here, we also used the smoothness of μ to deduce that $\|\mu\|_{X^{s,\frac{3}{8}}} \lesssim 1$. An analogous argument gives the same bound for I_2. The existence part of the proof now follows as in the proof of Proposition 2.3.1.

For the uniqueness part, suppose that v,w are two solutions of (2.5) on the time interval $[0,\delta]$ with the same initial data and whose H^1 norms are uniformly bounded on this interval. By using Minkowski's inequality and unitarity of the Schrödinger operator, we deduce that, for all $t \in [0,\delta]$:

$$\|v(t) - w(t)\|_{L^2} \leq \int_0^\delta (\|(|v|^2 v - |w|^2 w)(t')\|_{L^2} + \|(\lambda v - \lambda w)(t')\|_{L^2}) dt'$$

$$\lesssim \int_0^\delta ((\|v\|_{H^1} + \|w\|_{H^1})^2 + \|\lambda\|_{L^\infty}) \|v - w\|_{L^2} dt'$$

$$\lesssim \int_0^\delta \|v(t') - w(t')\|_{L^2} dt'.$$

Uniqueness now follows from Gronwall's inequality. □

Definition of $E^2(u)$ for (2.5)

As in the case of the Hartree Equation, we will use *higher modified energies*. Let:

$$E^1(u) := \|\mathcal{D}u\|_{L^2}^2, E^2(u) := E^1(u) + \lambda_4(M_4; u)$$

As before, we have to determine the multiplier M_4, so that we cancel the quadrilinear terms in $\frac{d}{dt}E^2(u(t))$. We note:

$$\frac{d}{dt}E^1(u(t)) \sim \frac{d}{dt}\big(\sum_{n_1+n_2=0} \widehat{\mathcal{D}u}(n_1)\widehat{\mathcal{D}\bar{u}}(n_2)\big) =$$

$$= \frac{1}{2} i\lambda_4((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2; u)$$

$$+ i \sum_{n_1+n_2+n_3=0} ((\theta(n_1))^2 - (\theta(n_2))^2) \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{\lambda}(n_3) \tag{2.108}$$

On the other hand, we compute that:

$$\frac{d}{dt}\lambda_4(M_4; u) = i\lambda_4(M_4(-n_1^2 + n_2^2 - n_3^2 + n_4^2); u) - i\lambda_6(M_6; u)$$

$$-i \sum_{n_1+n_2+n_3+n_4=0} M_4\big((\lambda u)\widehat{\ }(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) - \widehat{u}(n_1)(\lambda\bar{u})\widehat{\ }(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)$$

$$+\widehat{u}(n_1)\widehat{\bar{u}}(n_2)(\lambda u)\widehat{\ }(n_3)\widehat{\bar{u}}(n_4) - \widehat{u}\,(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)(\lambda\bar{u})\widehat{\ }(n_4)\big) \tag{2.109}$$

Here:

$$M_6(n_1,n_2,n_3,n_4) := M_4(n_{123},n_4,n_5,n_6) - M_4(n_1,n_{234},n_5,n_6)$$

$$+M_4(n_1,n_2,n_{345},n_6) - M_4(n_1,n_2,n_3,n_{456}). \tag{2.110}$$

From (2.108) and (2.109), it follows that we have to choose:

$$M_4 := \Psi_2. \tag{2.111}$$

where Ψ_2 is defined by:

$$\Psi_2 : \Gamma_4 \to \mathbb{R}$$

$$\Psi_2 := \begin{cases} c\frac{(\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2}{n_1^2-n_2^2+n_3^2-n_4^2}, \text{ if } n_1^2-n_2^2+n_3^2-n_4^2 \neq 0 \\ 0, \text{ otherwise.} \end{cases} \tag{2.112}$$

for an appropriate real constant c.

Hence, for such a choice of M_4, we obtain:

$$\frac{d}{dt}E^2(u) = ci \sum_{n_1+n_2+n_3=0} ((\theta(n_1))^2 - (\theta(n_2))^2)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{\lambda}(n_3)$$

$$-i\lambda_6(M_6;u) - i \sum_{n_1+n_2+n_3+n_4=0} M_4\big[(\lambda u)\widehat{\ }(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) - \widehat{u}(n_1)(\lambda\bar{u})\widehat{\ }(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)$$

$$+\widehat{u}(n_1)\widehat{\bar{u}}(n_2)(\lambda u)\widehat{\ }(n_3)\widehat{\bar{u}}(n_4) - \widehat{u}\,(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)(\lambda \bar{u})\widehat{\ }(n_4)\big] \tag{2.113}$$

We dyadically localize the frequencies as $|n_j| \sim N_j$. As before, we define: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. The proof of Lemma 2.4.2 gives us that:

$$M_4 = O(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*). \tag{2.114}$$

As we saw earlier, (2.114) implies:

$$E^2(u) \sim E^1(u). \tag{2.115}$$

Estimate on the increment of $E^2(u)$ for (2.5) and proof of Theorem 2.1.3.

We want to estimate $E^2(u(t_0+\delta)) - E^2(u(t_0)) = E^2(v(t_0+\delta)) - E^2(v(t_0))$. The bound that we will prove is:

Lemma 2.4.9. *For all $t_0 \in \mathbb{R}$, one has:*

$$|E^2(u(t_0+\delta)) - E^2(u(t_0))| \lesssim \frac{1}{N^{1-}}E^2(u(t_0)).$$

Arguing as in the proof of Theorem 2.1.2, Theorem 2.1.3 will then follow immediately from Lemma 2.4.9. We now prove Lemma 2.4.9.

Proof. As before, it suffices to consider $t_0 = 0$. We have to consider three possible types of terms that come from integrating over $[0,\delta]$ the right hand side of (2.113).

1) By a slight modification of our work on the Hartree Equation, we have:

$$|\int_0^\delta \lambda_6(M_6;u)dt| \lesssim \frac{1}{N^{2-}}E^2(\Phi) \tag{2.116}$$

2) In order to estimate the time integral of the quadrilinear term on the right hand side of (2.113), it suffices to estimate:

$$|\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0} M_4(n_1,n_2,n_3,n_4)(\lambda u)\widehat{\ }(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)dt|$$

Here M_4 is the multiplier we defined in (2.111). Let v be as in Proposition 2.4.8, and let $\mu(x,t) = f(t)\lambda(x)$ be as in the proof of Proposition 2.4.8. Let $\chi = \chi(t) := \chi_{[0,\delta]}(t)$. Then, we want to estimate:

$$|\int_{\mathbb{R}} \sum_{n_1+n_2+n_3+n_4=0} M_4(n_1,n_2,n_3,n_4)(\chi\mu v)\widehat{\ }(n_1)\widehat{\bar{v}}(n_2)\widehat{v}(n_3)\widehat{\bar{v}}(n_4)dt|$$

Let N_1, N_2, N_3, N_4 be dyadic integers. We define:

$$I_{N_1,N_2,N_3,N_4} :=$$

$$|\int_{\mathbb{R}} \sum_{n_1+n_2+n_3+n_4=0} M_4(n_1,n_2,n_3,n_4)\widehat{(\chi\mu v)}_{N_1}(n_1)\widehat{\bar{v}}_{N_2}(n_2)\widehat{v}_{N_3}(n_3)\widehat{\bar{v}}_{N_4}(n_4)dt|$$

$$\sim |\sum_{n_1+n_2+n_3+n_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j$$

$$\{M_4(n_1,n_2,n_3,n_4)\widetilde{(\chi\mu v)}_{N_1}(n_1,\tau_1)\widetilde{\bar{v}}_{N_2}(n_2,\tau_2)\widetilde{v}_{N_3}(n_3,\tau_3)\widetilde{\bar{v}}_{N_4}(n_4,\tau_4)\}|$$

$$\leq \sum_{n_1+n_2+n_3+n_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j$$

$$\{|M_4(n_1,n_2,n_3,n_4)||\widetilde{(\chi\mu v)}_{N_1}(n_1,\tau_1)||\widetilde{\bar{v}}_{N_2}(n_2,\tau_2)||\widetilde{v}_{N_3}(n_3,\tau_3)||\widetilde{\bar{v}}_{N_4}(n_4,\tau_4)|\}$$

We define the dyadic integers N_j^* as before. By using the fact that we are summing over the set where $n_1+n_2+n_3+n_4 = 0$, and by using the definition of M_4, we know that:

$$N_1^* \gtrsim N,\ N_1^* \sim N_2^* \tag{2.117}$$

We will consider the case when:

$$N_1 = N_1^*, N_2 = N_2^*, N_3 = N_3^*, N_4 = N_4^*.$$

The other cases are similar. Namely, in the other cases, we use the Fractional Leibniz Rule differently, as we did in order to bound the term K occurring in (2.102).

From (2.114), it follows that:

$$I_{N_1,N_2,N_3,N_4} \lesssim \sum_{n_1+n_2+n_3+n_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j \frac{1}{(N_1^*)^2} \theta(N_1^*)\theta(N_2^*) N_3^* N_4^*$$

$$|\widetilde{(\chi\mu v)}_{N_1}(n_1,\tau_1)||\widetilde{\bar{v}}_{N_2}(n_2,\tau_2)||\widetilde{v}_{N_3}(n_3,\tau_3)||\widetilde{\bar{v}}_{N_4}(n_4,\tau_4)|$$

$$\lesssim \sum_{n_0+n_1+n_2+n_3+n_4=0} \int_{\tau_0+\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j \frac{1}{(N_1^*)^2} \theta(n_0+n_1)\theta(N_2^*) N_3^* N_4^*$$

$$|\widetilde{(\chi\mu)}(n_0,\tau_0)||\widetilde{v}(n_1,\tau_1)||\widetilde{\bar{v}}_{N_2}(n_2,\tau_2)||\widetilde{v}_{N_3}(n_3,\tau_3)||\widetilde{\bar{v}}_{N_4}(n_4,\tau_4)|$$

$$\lesssim \sum_{n_0+n_1+n_2+n_3+n_4=0} \int_{\tau_0+\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j \frac{1}{(N_1^*)^{2-}} (\theta(n_0)+\theta(n_1))$$

$$|\widetilde{(\chi\mu)}(n_0,\tau_0)||\widetilde{v}(n_1,\tau_1)|(\frac{1}{(N_2^*)^{0+}}|\widetilde{(\mathcal{D}\bar{v})}_{N_2}(n_2,\tau_2)|)(\frac{1}{(N_3^*)^{0+}}|\widetilde{\nabla v}_{N_3}(n_3,\tau_3)|)(\frac{1}{(N_4^*)^{0+}}|\widetilde{\nabla \bar{v}}_{N_4}(n_4,\tau_4)|)$$

$$\lesssim I^1_{N_1,N_2,N_3,N_4} + I^2_{N_1,N_2,N_3,N_4}$$

Here:

$$I^1_{N_1,N_2,N_3,N_4} := \sum_{n_0+n_1+n_2+n_3+n_4=0} \int_{\tau_0+\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j \frac{1}{(N_1^*)^{2-}}$$

$$|\widetilde{(\chi\mu)}(n_0,\tau_0)||\widetilde{\mathcal{D}v}(n_1,\tau_1)|(\frac{1}{(N_2^*)^{0+}}|\widetilde{(\mathcal{D}\bar{v})}_{N_2}(n_2,\tau_2)|)(\frac{1}{(N_3^*)^{0+}}|\widetilde{\nabla v}_{N_3}(n_3,\tau_3)|)(\frac{1}{(N_4^*)^{0+}}|\widetilde{\nabla \bar{v}}_{N_4}(n_4,\tau_4)|)$$

and:

$$I^2_{N_1,N_2,N_3,N_4} := \sum_{n_0+n_1+n_2+n_3+n_4=0} \int_{\tau_0+\tau_1+\tau_2+\tau_3+\tau_4=0} d\tau_j \frac{1}{(N_1^*)^{2-}}$$

$$|\widetilde{(\chi \mathcal{D}\mu)}(n_0,\tau_0)||\widetilde{v}(n_1,\tau_1)|(\frac{1}{(N_2^*)^{0+}}|\widetilde{(\mathcal{D}\bar{v})}_{N_2}(n_2,\tau_2)|)(\frac{1}{(N_3^*)^{0+}}|\widetilde{\nabla v}_{N_3}(n_3,\tau_3)|)(\frac{1}{(N_4^*)^{0+}}|\widetilde{\nabla \bar{v}}_{N_4}(n_4,\tau_4)|)$$

We estimate $I^1_{N_1,N_2,N_3,N_4}$. The expression $I^2_{N_1,N_2,N_3,N_4}$ is estimated analogously.

Suppose $F_j : j = 0,1,2,3,4$ are such that:

$$\widetilde{F_0} = |\widetilde{(\chi\mu)}|, \widetilde{F_1} = |\widetilde{\mathcal{D}v}|, \widetilde{F_2} = |\widetilde{(\mathcal{D}\bar{v})}_{N_2}|, \widetilde{F_3} = |\widetilde{\nabla v}_{N_3}|, \widetilde{F_4} = |\widetilde{\nabla v}_{N_4}|$$

By Parseval's identity, and then by Hölder's inequality, we deduce:

$$I^1_{N_1,N_2,N_3,N_4} \lesssim \frac{1}{N_1^{2-}N_2^{0+}N_3^{0+}N_4^{0+}} \int\int F_0 F_1 \bar{F}_2 F_3 \bar{F}_4 dx dt$$

$$\leq \frac{1}{N_1^{2-}N_2^{0+}N_3^{0+}N_4^{0+}} \|F_0\|_{L^4_{t,x}} \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^6_{t,x}} \|F_3\|_{L^6_{t,x}} \|F_4\|_{L^6_{t,x}}$$

By using (2.22), (2.23), and the construction of the functions F_j, this expression is:

$$\lesssim \frac{1}{N_1^{2-}N_2^{0+}N_3^{0+}N_4^{0+}} \|F_0\|_{X^{0,\frac{3}{8}}} \|F_1\|_{X^{0,\frac{3}{8}}} \|F_2\|_{X^{0+,\frac{1}{2}+}} \|F_3\|_{X^{0+,\frac{1}{2}+}} \|F_4\|_{X^{0+,\frac{1}{2}+}}$$

$$= \frac{1}{N_1^{2-}} \|\chi\mu\|_{X^{0,\frac{3}{8}}} \|\mathcal{D}v\|_{X^{0,\frac{3}{8}}} (\frac{1}{N_2^{0+}} \|\mathcal{D}v_{N_2}\|_{X^{0+,\frac{1}{2}+}}) (\frac{1}{N_3^{0+}} \|\nabla v_{N_3}\|_{X^{0+,\frac{1}{2}+}}) (\frac{1}{N_4^{0+}} \|\nabla v_{N_4}\|_{X^{0+,\frac{1}{2}+}})$$

Using Lemma 2.2.1, and Proposition 2.4.8, we deduce that this expression is:

$$\lesssim \frac{1}{N_1^{2-}} \|\mu\|_{X^{0,\frac{1}{2}+}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|\nabla v\|^2_{X^{0,\frac{1}{2}+}}$$

By the smoothness of μ, this is:

$$\lesssim \frac{1}{N_1^{2-}} \|\mathcal{D}\Phi\|_{L^2}^2 = \frac{1}{N_1^{2-}} E^1(\Phi)$$

By (2.115), we obtain that the above term is:

$$\lesssim \frac{1}{N_1^{2-}} E^2(\Phi).$$

An analogous argument shows that $I^2_{N_1,N_2,N_3,N_4}$ is bounded by the same quantity. Hence:

$$I_{N_1,N_2,N_3,N_4} \lesssim \frac{1}{N_1^{2-}} E^2(\Phi)$$

We sum in the N_j and use (2.117) to deduce that:

$$|\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0} M_4\big((\lambda u)\widehat{\ }(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) - \widehat{u}(n_1)(\lambda \bar{u})\widehat{\ }(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)$$

$$+\widehat{u}(n_1)\widehat{\bar{u}}(n_2)(\lambda u)\widehat{\ }(n_3)\widehat{\bar{u}}(n_4) - (\lambda u)\widehat{\ }(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)(\lambda \bar{u})\widehat{\ }(n_4)\big)dt|$$

$$\lesssim \frac{1}{N^{2-}} E^2(\Phi). \tag{2.118}$$

3) We now estimate the time integral of the bilinear term on the right hand side of (2.113). Namely, we bound:

$$|\int_0^\delta \sum_{n_1+n_2+n_3=0} \big((\theta(n_1))^2 - (\theta(n_2))^2\big)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{\lambda}(n_3)dt\big| =$$

$$|\int_{\mathbb{R}} \sum_{n_1+n_2+n_3=0} \big((\theta(n_1))^2 - (\theta(n_2))^2\big)(\chi v)\widehat{\ }(n_1)\widehat{\bar{v}}(n_2)\widehat{\mu}(n_3)dt\big| \sim$$

$$\sim |\sum_{n_1+n_2+n_3=0} \int_{\tau_1+\tau_2+\tau_3=0} d\tau_j \big((\theta(n_1))^2 - (\theta(n_2))^2\big)(\chi v)\widetilde{\ }(n_1,\tau_1)\widetilde{\bar{v}}(n_2,\tau_2)\widetilde{\mu}(n_3,\tau_3)\big|$$

Given dyadic integers N_1, N_2, N_3, we define:

$$J_{N_1,N_2,N_3} := \sum_{n_1+n_2+n_3=0} \int_{\tau_1+\tau_2+\tau_3=0} d\tau_j |(\theta(n_1))^2 - (\theta(n_2))^2| |\widetilde{(\chi v)}_{N_1}(n_1,\tau_1)| |\widetilde{\overline{v}}_{N_2}(n_2,\tau_2)| |\widetilde{\mu}_{N_3}(n_3,\tau_3)|$$

Let's order the frequencies as before to obtain:

$$N_1^* \geq N_2^* \geq N_3^*.$$

By construction of θ, and by the fact that we are integrating over $n_1 + n_2 + n_3 = 0$, we again have that:

$$N_1^* \gtrsim N, N_1^* \sim N_2^* \tag{2.119}$$

We now consider two cases, depending on the relationship between N_1^* and N_3.

Case 1: $N_1^* \sim N_3$.

In this case, one has:

$$(\theta(n_1))^2 - (\theta(n_2))^2 = O((\theta(N_3))^2).$$

We find G_1, G_2, G_3 such that:

$$\widetilde{G_1} = |\widetilde{(\chi v)}_{N_1}|, \widetilde{G_2} = |\widetilde{\overline{v}}_{N_2}|, \widetilde{G_3} = |\widetilde{(\mathcal{D}^2 \mu)}_{N_3}|.$$

So:

$$J_{N_1,N_2,N_3} \lesssim$$

$$\sum_{n_1+n_2+n_3=0} \int_{\tau_1+\tau_2+\tau_3=0} d\tau_j |\widetilde{(\chi v)}_{N_1}(n_1,\tau_1)| |\widetilde{\overline{v}}_{N_2}(n_2,\tau_2)| (\theta(N_3))^2 |\widetilde{\mu}_{N_3}(n_3,\tau_3)|$$

$$\lesssim \sum_{n_1+n_2+n_3=0} \int_{\tau_1+\tau_2+\tau_3=0} d\tau_j |\widetilde{(\chi v)}_{N_1}(n_1,\tau_1)||\widetilde{\overline{v}}_{N_2}(n_2,\tau_2)||\widetilde{(\mathcal{D}^2\mu)}_{N_3}(n_3,\tau_3)|$$

$$\sim \int\int G_1\overline{G_2}G_3 dxdt \leq \|G_1\|_{L^4_{t,x}}\|G_2\|_{L^4_{t,x}}\|G_3\|_{L^2_{t,x}}$$

$$\lesssim \|G_1\|_{X^{0,\frac{3}{8}}}\|G_2\|_{X^{0,\frac{3}{8}}}\|G_3\|_{X^{0,0}} = \|(\chi v)_{N_1}\|_{X^{0,\frac{3}{8}}}\|v_{N_2}\|_{X^{0,\frac{3}{8}}}\|(\mathcal{D}^2\mu)_{N_3}\|_{X^{0,0}}$$

$$\lesssim \|v\|^2_{X^{0,\frac{1}{2}+}}\frac{1}{N_3^M}\|\mathcal{D}^2\mu\|_{X^{M,0}} \lesssim \frac{1}{(N_1^*)^M}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}.$$

The previous bound holds for all $M>0$, by the smoothness properties of μ. From (2.115), we have that the contribution from Case 1 is, in particular:

$$\lesssim \frac{1}{N_1^*}E^2(\Phi). \tag{2.120}$$

Case 2: $N_1^* \gg N_3$.

Subcase 1: $N_3 \leq (N_1^*)^\epsilon$ ($\epsilon>0$ is small).

We recall from (2.85) that:

$$|(\theta(x))^2-(\theta(y))^2| \leq \frac{1}{N^{2s}}||x|^{2s}-|y|^{2s}|.$$

Now, in this subcase:

$$|n_1|\sim|n_2|\sim N_1^*, ||n_1|-|n_2|| = O((N_1^*)^\epsilon).$$

So, by the Mean Value Theorem:

$$||n_1|^{2s}-|n_2|^{2s}| \lesssim (N_1^*)^{2s-1}(N_1^*)^\epsilon.$$

Consequently:

$$|(\theta(n_1))^2 - (\theta(n_2))^2| \lesssim \frac{1}{(N_1^*)^{1-\epsilon}} \theta(N_1)\theta(N_2).$$

With notation as in Case 1, we obtain that:

$$J_{N_1,N_2,N_3} \lesssim$$

$$\sum_{n_1+n_2+n_3=0} \int_{\tau_1+\tau_2+\tau_3=0} d\tau_j \frac{1}{(N_1^*)^{1-\epsilon}} |\widetilde{(\chi \mathcal{D} v)}_{N_1}(n_1,\tau_1)||\widetilde{\mathcal{D}\overline{v}}_{N_2}(n_2,\tau_2)||\widetilde{\mu}_{N_3}(n_3,\tau_3)|$$

We now argue, using an $L^4_{t,x}, L^4_{t,x}, L^2_{t,x}$ Hölder inequality as in Case 1, to deduce:

$$J_{N_1,N_2,N_3} \lesssim \frac{1}{(N_1^*)^{1-\epsilon}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|\mu\|_{X^{0,0}} \lesssim \frac{1}{(N_1^*)^{1-\epsilon}} E^2(\Phi) \tag{2.121}$$

Subcase 2: $N_3 > (N_1^*)^{\epsilon}$(for the same $\epsilon > 0$ as before).

In this subcase, we estimate $|(\theta(n_1))^2 - (\theta(n_2))^2| \lesssim \theta(N_1)\theta(N_2)$, and hence:

$$J_{N_1,N_2,N_3} \lesssim$$

$$\sum_{n_1+n_2+n_3=0} \int_{\tau_1+\tau_2+\tau_3=0} d\tau_j |\widetilde{(\chi \mathcal{D} v)}_{N_1}(n_1,\tau_1)||\widetilde{\mathcal{D}\overline{v}}_{N_2}(n_2,\tau_2)||\widetilde{\mu}_{N_3}(n_3,\tau_3)|$$

We now argue similarly as in Case 1 to deduce that for all $M > 0$:

$$J_{N_1,N_2,N_3} \lesssim \frac{1}{(N_1^*)^{\epsilon M}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}.$$

Hence, if we choose M sufficiently large so that $\epsilon M \geq 1$, we obtain:

$$J_{N_1,N_2,N_3} \lesssim \frac{1}{N_1^*} E^2(\Phi). \tag{2.122}$$

Combining (2.120), (2.121), (2.122), and summing in the N_j, we obtain that:

$$|\int_0^\delta \sum_{n_1+n_2+n_3=0} ((\theta(n_1))^2 - (\theta(n_2))^2)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{\lambda}(n_3)dt| \lesssim \frac{1}{N^{1-}} E^2(\Phi). \qquad (2.123)$$

Lemma 2.4.9 now follows from (2.116), (2.118), and (2.123).

□

Further remarks

Remark 2.4.10. *If λ is a constant function, then $u = e^{-i\lambda t} v$, where v is a solution to the cubic NLS. Since $\|v(t)\|_{H^s}$ is then uniformly bounded in time, the same holds for $\|u(t)\|_{H^s}$. If λ depends on x, one can't argue in this way.*

Remark 2.4.11. *Heuristically, the reason why we get a weaker bound for* (2.5) *than we did for* (2.4) *is the fact that we have bilinear terms which occur in $\frac{d}{dt}E^2(u)$. Hence, the derivatives have to be distributed among fewer factors of u and $\bar{u}$ than there were before.*

2.4.3 Modification 3: Defocusing Cubic NLS with an inhomogeneous nonlinearity

We now consider the equation (2.6). The equation (2.6) has conserved mass. By integration by parts, one can check that energy:

$$E(u(t)) := \frac{1}{2}\int |\nabla u(x,t)|^2 dx + \frac{1}{4}\int \lambda(x)|u(x,t)|^4 dx$$

is conserved in time. Both quantities are continuous on H^1. Since $\lambda \geq 0$, conservation of mass and energy gives us uniform bounds on $\|u(t)\|_{H^1}$.

Local-in-time estimates for (2.6)

Let u be a global solution to (2.6). Let us observe the following fact:

Proposition 2.4.12. *Given $t_0 \in \mathbb{R}$, there exists a globally defined function $v : S^1 \times \mathbb{R} \to \mathbb{C}$ satisfying the properties:*

$$v|_{[t_0,t_0+\delta]} = u|_{[t_0,t_0+\delta]}. \tag{2.124}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{2.125}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{2.126}$$

Moreover, δ and C can be chosen to depend continuously on the energy and mass.

The proof of Proposition 2.4.12 is analogous to the proof of Proposition 2.3.1, so we omit the details.

Estimate on the increment of $E^1(u)$ for (2.6) and proof of Theorem 2.1.4

The presence of the inhomogeneity λ in the nonlinearity makes it impossible to use E^2, as in the case of the previous two equations. The difficulty lies in the fact that the numerators we obtain in the correction terms no longer factorize, so we can't obtain bounds such as (2.80). This is analogous to the situation that occurs for the quintic and higher order NLS. For details, see Remark 2.4.7. Hence, we have to work with E^1. Theorem 2.1.4 will follow if we prove that:

Lemma 2.4.13. *For all $t_0 \in \mathbb{R}$, one has:*

$$|E^1(u(t_0 + \delta)) - E^1(t_0)| \lesssim \frac{1}{N^{\frac{1}{2}-}} E^1(\Phi)$$

Proof. As before, it suffices to consider the case $t_0 = 0$. Arguing as in previous sections, we obtain:

$$\frac{d}{dt}E^1(u) = ci \sum_{n_0+n_1+\cdots+n_4=0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big) \widehat{\lambda}(n_0) \widehat{u}(n_1) \widehat{\bar{u}}(n_2) \widehat{u}(n_3) \widehat{\bar{u}}(n_4)$$

Let $N_0, N_1, \ldots, N_4$ be dyadic integers. We define $\mu(x,t) := f(t)\lambda(x)$ as in the proof of Proposition 2.4.8. The expression we want to estimate is:

$$I_{N_0,N_1,N_2,N_3,N_4} := \Big| \int_0^\delta \sum_{n_0+n_1+\cdots+n_4=0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)$$

$$\widehat{\mu}_{N_0}(n_0)\widehat{v}_{N_1}(n_1)\widehat{\overline{v}}_{N_2}(n_2)\widehat{v}_{N_3}(n_3)\widehat{\overline{v}}_{N_4}(n_4)dt\Big|$$

If $\chi = \chi(t) = \chi_{[0,\delta]}(t)$, then I_{N_0,N_1,N_2,N_3,N_4} is:

$$\lesssim \Big| \sum_{n_0+n_1+\cdots+n_4=0} \int_{\tau_0+\tau_1+\cdots\tau_4=0} d\tau_j \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)$$

$$\widetilde{\mu}_{N_0}(n_0,\tau_0)\widetilde{(\chi v)}_{N_1}(n_1,\tau_1)\widetilde{\overline{v}}_{N_2}(n_2,\tau_2)\widetilde{v}_{N_3}(n_3,\tau_3)\widetilde{\overline{v}}_{N_4}(n_4,\tau_4)\Big|$$

We define N_j^* for $j = 1, \ldots 5$ to be the ordering of $\{N_0, N_1, N_2, N_3, N_4\}$. With this notation, we have the following bounds:

$$N_1^* \gtrsim N, N_1^* \sim N_2^* \tag{2.127}$$

We consider two cases:

Case 1: $N_0 \gtrsim (N_1^*)^\epsilon$ (Here $\epsilon > 0$ is small.)

We use the fact that the multiplier is $O((\theta(N_1^*))^2)$, and an $L^\infty_{t,x}, L^4_{t,x}, L^4_{t,x}, L^4_{t,x}, L^4_{t,x}$ Hölder's inequality to deduce that:

$$I_{N_0,N_1,N_2,N_3,N_4} \lesssim (\theta(N_1^*))^2 \|\mu_{N_0}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \|v_{N_1}\|_{X^{0,\frac{1}{2}+}} \|v_{N_2}\|_{X^{0,\frac{1}{2}+}} \|v_{N_3}\|_{X^{0,\frac{1}{2}+}} \|v_{N_4}\|_{X^{0,\frac{1}{2}+}}.$$

Considering separately the cases when $N_0 \sim N_1^*$ and when $N_0 \ll N_1^*$, this expression is:

$$\lesssim \|\mathcal{D}\mu_{N_0}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \|v\|^3_{X^{0,\frac{1}{2}+}} + \|\mu_{N_0}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^2_{X^{0,\frac{1}{2}+}}$$

$$\lesssim (\|\mathcal{D}\mu_{N_0}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} + \|\mu_{N_0}\|_{X^{\frac{1}{2}+,\frac{1}{2}+}})\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{0,\frac{1}{2}+}}$$

For $M > 0$, this quantity is:

$$\lesssim \frac{1}{(N_0)^M}(\|\mathcal{D}\mu_{N_0}\|_{X^{M+\frac{1}{2}+,\frac{1}{2}+}} + \|\mu_{N_0}\|_{X^{M+\frac{1}{2}+,\frac{1}{2}+}})\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{0,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{\epsilon M}}\|\mathcal{D}\Phi\|^2_{L^2} = \frac{1}{(N_1^*)^{\epsilon M}}E^1(\Phi).$$

In particular, if we choose M sufficiently large so that $\epsilon M \geq \frac{1}{2}$, we get:

$$I_{N_0,N_1,N_2,N_3,N_4} \lesssim \frac{1}{(N_1^*)^{\frac{1}{2}}}E^1(\Phi). \tag{2.128}$$

Case 2: $N_0 \ll (N_1^*)^{\epsilon}$ (for the same ϵ as before)

If we take $\epsilon < \frac{1}{2}$, we note that the same arguments we used to prove Theorem 2.1.1 allow us to deduce that in Case 2:

$$I_{N_0,N_1,N_2,N_3,N_4} \lesssim \frac{1}{(N_1^*)^{\frac{1}{2}}}E^1(\Phi). \tag{2.129}$$

More precisely, we recall the proof of Lemma 2.3.4. The only place in which one can't immediately adapt the proof of Lemma 2.3.4 to (2.6) is in Case 1 of Big Case 1. If one has the additional assumption that $N_0 \ll (N_1^*)^{\frac{1}{2}}$, the proof then follows as before.

Using (2.128), (2.129), and summing in the N_j, the Lemma follows. □

Further remarks

Remark 2.4.14. *If λ is constant, we can obtain* (2.6) *by rescaling the cubic NLS, so Theorem 2.1.4 can be improved in this case.*

2.4.4 Comments on (2.4), (2.5), and (2.6)

The reason why we considered the three equations in this section was because they were obtained from the cubic NLS by breaking the complete integrability. Different ways of breaking the complete integrability of the cubic NLS manifested themselves in the bounds we obtained, and the methods we could use to obtain them. As we saw, the least drastic change happened when we added the convolution potential in the case of the Hartree Equation, whereas the most drastic change happened when we multiplied the nonlinearity with the inhomogeneity in (2.6).

2.5 Appendix A: Proof of Lemma 2.2.1

Proof. We argue by duality. Let us consider v s.t.$\|v\|_{X^{-s,-b}} \leq 1$. We want to prove that:

$$|\int_c^d \int_{S^1} u(x,t)\overline{v(x,t)}dxdt| \lesssim \|u\|_{X^{s,b+}}\|v\|_{X^{-s,-b}}. \tag{2.130}$$

We observe:

$$\chi_{[c,d]}(t) = \frac{sign(t-c) - sign(t-d)}{2}.$$

By symmetry, we just need to get the bound:

$$|\int_{\mathbb{R}} \int_{S^1} sign(t-c)u(x,t)\overline{v(x,t)}dxdt| \lesssim \|u\|_{X^{s,b+}}\|v\|_{X^{-s,-b}}. \tag{2.131}$$

Let us first prove, the claim when $c = 0$, i.e.

$$|\int_{\mathbb{R}} \int_{S^1} sign(t)u(x,t)\overline{v(x,t)}dxdt| \lesssim \|u\|_{X^{s,b+}}\|v\|_{X^{-s,-b}}. \tag{2.132}$$

The key to prove (2.132) is to use the Hilbert transform in the time variable.

We recall that the Hilbert transform on the real line is defined by:

$$Hf := cf * (p.v.\frac{1}{x}). \tag{2.133}$$

The constant c is chosen so that H is an isometry on L^2. It can be shown [45], that one then has the identity:

$$\widehat{Hf}(\xi) \sim -isign(\xi)\hat{f}(\xi). \tag{2.134}$$

From Parseval's identity and from (2.133),(2.134), we obtain:

$$\int_{\mathbb{R}}\int_{S^1} sign(t)u(x,t)\overline{v(x,t)}dxdt \sim \int_{\mathbb{R}}\sum_n \widetilde{u}(n,\tau)(p.v.\int_{\mathbb{R}}\overline{\widetilde{v}(n,\tau')}\frac{1}{\tau-\tau'}d\tau')d\tau =: J.$$

Let us consider three cases:

Case 1: $\langle \tau + n^2 \rangle \sim \langle \tau' + n^2 \rangle$.

Case 2: $\langle \tau + n^2 \rangle \gg \langle \tau' + n^2 \rangle$.

Case 3: $\langle \tau + n^2 \rangle \ll \langle \tau' + n^2 \rangle$.

Let J_1, J_2, J_3 denote the contributions to J coming from the three cases respectively. We estimate these contributions separately.

Case 1: In this case, we perform a dyadic decomposition. Let $\widetilde{u_k}$,$\widetilde{v_k}$ respectively denote the localizations of $\widetilde{u}$, $\widetilde{v}$ to $\langle \tau + n^2 \rangle \sim \langle \tau' + n^2 \rangle \sim 2^k$. Then, since in this case $|j-k| = O(1)$, we get:

$$|J_1| = |\sum_{|j-k|=O(1)}\int_{\mathbb{R}}\sum_n \widetilde{u_j}(n,\tau)(p.v.\int_{\mathbb{R}}\overline{\widetilde{v_k}(n,\tau')}\frac{1}{\tau-\tau'}d\tau')d\tau| \sim$$

$$\sim |\sum_{|j-k|=O(1)}\int_{\mathbb{R}}\sum_n \widetilde{u_j}(n,\tau)H_\tau\overline{\widetilde{v_k}}(n,\tau)d\tau|$$

$$\leq \sum_{|j-k|=O(1)}|\int_{\mathbb{R}}\sum_n \langle n\rangle^s\widetilde{u_j}(n,\tau)\langle n\rangle^{-s}H_\tau\overline{\widetilde{v_k}}(n,\tau)d\tau|.$$

Here, we denoted by $H_\tau(\cdot)$ the Hilbert transform in the τ variable. We then use the Cauchy-Schwarz inequality in (n,τ) to see that the previous expression is:

$$\leq \sum_{|j-k|=O(1)} \|\langle n\rangle^s \widetilde{u_j}\|_{l^2_n L^2_\tau} \|\langle n\rangle^{-s} H_\tau \overline{\widetilde{v_k}}\|_{l^2_n L^2_\tau}.$$

We then recall that the Hilbert transform is bounded on L^2 by (2.134) to deduce that:

$$|J_1| \lesssim \sum_{|j-k|=O(1)} \|\langle n\rangle^s \widetilde{u_j}\|_{l^2_n L^2_\tau} \|\langle n\rangle^{-s} \overline{\widetilde{v_k}}\|_{l^2_n L^2_\tau}$$

Since $|j-k| = O(1)$, and by definition of u_j, v_k, this is:

$$\lesssim \sum_{|j-k|=O(1)} \|\langle n\rangle^s \langle \tau + n^2\rangle^b \widetilde{u_j}\|_{l^2_n L^2_\tau} \|\langle n\rangle^{-s} \langle \tau + n^2\rangle^{-b} \overline{\widetilde{v_k}}\|_{l^2_n L^2_\tau}$$

We use the Cauchy-Schwarz inequality in the sum of j, k to bound this by:

$$\lesssim \|u\|_{X^{s,b}} \|v\|_{X^{-s,-b}} \leq \|u\|_{X^{s,b+}} \|v\|_{X^{-s,-b}}.$$

Case 2: Since in this case $\langle \tau + n^2\rangle \gg \langle \tau' + n^2\rangle$, we have that:

$$|\tau - \tau'| \sim \langle \tau + n^2\rangle \gg \langle \tau' + n^2\rangle.$$

It follows that for all $\theta \in [0,1]$, one has:

$$\frac{1}{|\tau - \tau'|} \lesssim \frac{1}{\langle \tau + n^2\rangle^\theta \langle \tau' + n^2\rangle^{1-\theta}}.$$

We deduce that:

$$|J_2| \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_n |\widetilde{u}(n,\tau)||\widetilde{v}(n,\tau')| \frac{1}{\langle \tau + n^2\rangle^\theta \langle \tau' + n^2\rangle^{1-\theta}} d\tau d\tau' =$$

$$= \sum_n \Big(\int_{\mathbb{R}} |\widetilde{u}(n,\tau)| \langle \tau+n^2\rangle^{\frac{1}{2}+\delta-\theta} \langle \tau+n^2\rangle^{-\frac{1}{2}-\delta} \langle n\rangle^s d\tau\Big)\Big(\int_{\mathbb{R}} |\widetilde{v}(n,\tau')| \langle \tau'+n^2\rangle^{\frac{1}{2}+\delta-(1-\theta)} \langle \tau'+n^2\rangle^{-\frac{1}{2}-\delta} \langle n\rangle^{-s} d\tau'\Big).$$

Here, $\delta > 0$ was arbitrary. Now, we first use the Cauchy-Schwarz inequality in τ, τ', together with the fact that:

$$\|\langle \tau + n^2 \rangle^{-\frac{1}{2}-\delta}\|_{l^\infty_n L^2_\tau} \lesssim 1$$

and

$$\|\langle \tau' + n^2 \rangle^{-\frac{1}{2}-\delta}\|_{l^\infty_n L^2_{\tau'}} \lesssim 1$$

followed by the Cauchy-Schwarz inequality in n to deduce that:

$$|J_2| \lesssim \|u\|_{X^{s,\frac{1}{2}+\delta-\theta}} \|v\|_{X^{-s,\frac{1}{2}+\delta-(1-\theta)}}.$$

Let us take $\delta > 0$ sufficiently small so that $b + \delta < \frac{1}{2}$. We then take: $\theta := \frac{1}{2} - b - \delta$, which is positive, and $b+ := b + 2\delta$. With such a choice, we get that:

$$|J_2| \lesssim \|u\|_{X^{s,b+}} \|v\|_{X^{-s,-b}}.$$

Case 3:

In this case, we again have: $|\tau - \tau'| \gtrsim \langle \tau + n^2 \rangle, \langle \tau' + n^2 \rangle$, and we argue to get the same bound as in the previous case.

The bound (2.132) now follows.

Let us now observe that the bound (2.132) implies (2.131).

Let M_a denote the *modulation operator*

$$M_a f(x) = e^{iax} f(x).$$

Then, one obtains that:

$$(M_a H M_{-a} f)\widehat{\ }(\xi) \sim -i sign(\xi - a) \widehat{f}(\xi).$$

Let $\Phi^{-1}(\cdot)$ denote the inverse spacetime Fourier transform. Then, by Parseval's Identity, we obtain:

$$\int_{\mathbb{R}}\int_{S^1} sign(t-c)u(x,t)\overline{v(x,t)}dxdt \sim$$

$$\sim \int_{\mathbb{R}}\sum_n (\Phi^{-1}u)(n,\tau)\overline{M_c H M_{-c}(\Phi^{-1}v)(n,\tau)}d\tau \sim$$

$$\sim \int_{\mathbb{R}}\sum_n (\Phi^{-1}u)(n,\tau)e^{-ic\tau}p.v.(\int_{\mathbb{R}} \frac{e^{ic\tau'}\overline{(\Phi^{-1}v)(n,\tau')}}{\tau-\tau'}d\tau')d\tau \sim$$

$$\sim \int_{\mathbb{R}}\sum_n \widetilde{u}(n,\tau)e^{ic\tau}p.v.(\int_{\mathbb{R}} \frac{e^{-ic\tau'}\overline{\widetilde{v}(n,\tau')}}{\tau-\tau'}d\tau')d\tau.$$

In the last step, we use the Fourier inversion formula which gives us that:

$$\Phi^{-1}w(n,\tau) \sim \widetilde{w}(-n,-\tau).$$

Multiplication by the unimodular factors $e^{ic\tau}, e^{ic\tau'}$ doesn't change the rest of the argument used to derive (2.132). Hence, the proof of (2.131) follows as before.

□

Remark 2.5.1. *We deduce from the proof that none of the implied constants depend on c and d.*

2.6 Appendix B: Proofs of Propositions 2.3.1, 2.3.2, and 2.3.3

In order to prove Proposition 2.3.1, we recall several facts. One of the key ingredients of the proof is the following set of localization estimates in $X^{s,b}$ spaces. We start with $f \in C_0^\infty(\mathbb{R}), \delta > 0$ arbitrary, and we assume that $b > \frac{1}{2}$. Let $S(t)$ denote the linear Schrödinger propagator. Then, there exists a constant $C > 0$ depending only on f, s, b such that:

$$\|f(\frac{t}{\delta})S(t)\Phi\|_{X^{s,b}} \leq C\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s}. \tag{2.135}$$

$$\|f(\frac{t}{\delta})h\|_{X^{s,b}} \leq C\delta^{\frac{1-2b}{2}}\|h\|_{X^{s,b}}. \tag{2.136}$$

$$\|f(\frac{t}{\delta})\int_0^t S(t-t')w(t')dt'\|_{X^{s,b}} \leq C\delta^{\frac{1-2b}{2}}\|w\|_{X^{s,b-1}}. \tag{2.137}$$

The analogous fact is proved for the $X^{s,b}$ spaces corresponding to the Korteweg-de Vries equation in [71] in the non-periodic case. However, all the bounds for the periodic Schrödinger equation follow in the same way, because we are estimating the integral in the variable dual to time. These bounds for general dispersive equations can be found in [106]. We also note that in (2.137), we can translate time so that our initial time is arbitrary t_0 and not necessarily 0.

If, on the other hand $b' < \frac{1}{2}$,one has:

$$\|f(\frac{t}{\delta})w\|_{X^{s,b'}} \lesssim_f \|w\|_{X^{s,b'}}. \tag{2.138}$$

We observe that the implied constant is independent of $\delta > 0$.

For the proof of the inequality (2.138), one should consult Lemma 1.2. in [58]. We note that the proof from the paper holds if $b = b'$ in the given notation. One can also refer to Lemma 2.11 in [106]

Proof. (*of Proposition 2.3.1*)

Let us WLOG assume that $t_0 = 0$ for simplicity of notation. Later, we will see that the δ we obtain is indeed independent of time. Throughout the proof, we take $\delta > 0$ small which we will determine later. Let $b = \frac{1}{2}+ = \frac{1}{2} + \epsilon$ for ϵ sufficiently small which we also determine later.

Let us start by taking $\chi, \phi, \psi \in C_0^\infty(\mathbb{R})$, with $0 \leq \chi, \phi, \psi \leq 1$, such that:

$$\chi = 1 \text{ on } [-1,1]\ , \chi = 0 \text{ outside } [-2,2]. \tag{2.139}$$

$$\phi = 1 \text{ on } [-2,2]\ , \phi = 0 \text{ on } [-4,4]. \tag{2.140}$$

$$\psi = 1 \text{ on } [-4,4]\ , \psi = 0 \text{ on } [-8,8]. \tag{2.141}$$

We let:

$$\chi_\delta := \chi(\frac{\cdot}{\delta}), \phi_\delta := \phi(\frac{\cdot}{\delta}), \psi_\delta := \psi(\frac{\cdot}{\delta}). \tag{2.142}$$

Then:

$$\chi_\delta = 1 \text{ on } [-\delta, \delta] \ , \chi_\delta = 0 \text{ outside } [-2\delta, 2\delta]. \tag{2.143}$$

$$\phi_\delta = 1 \text{ on } [-2\delta, 2\delta] \ , \phi_\delta = 0 \text{ outside } [-4\delta, 4\delta]. \tag{2.144}$$

$$\psi_\delta = 1 \text{ on } [-4\delta, 4\delta] \ , \psi_\delta = 0 \text{ outside } [-8\delta, 8\delta]. \tag{2.145}$$

For $v : S^1 \times \mathbb{R} \mapsto \mathbb{C}$, we define:

$$Lv := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t) \int_0^t S(t-t')|v|^4 v(t')dt'.$$

By (2.143) and (2.144), and denoting $\phi_\delta v$ by v_δ,we obtain:

$$Lv = \chi_\delta(t)S(t)\Phi - i\chi_\delta(t) \int_0^t S(t-t')|v_\delta|^4 v_\delta(t')dt.'$$

Using (2.135) and (2.137), we obtain:

$$\|Lv\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} + c\delta^{\frac{1-2b}{2}} \||v_\delta|^4 v_\delta\|_{X^{s,b-1}}. \tag{2.146}$$

We estimate the quantity $\||v_\delta|^4 v_\delta\|_{X^{s,b-1}}$ by duality. Let us take:

$$c : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}, \text{ such that } \sum_n \int d\tau |c(n,\tau)|^2 = 1.$$

Let us consider the quantity:

$$\sum_n \int d\tau (1+|n|)^s \, (1+|\tau+n^2|)^{b-1} \, (\widetilde{|v_\delta|^4 v_\delta})(n,\tau) \, c(n,\tau) \ =: I \tag{2.147}$$

Since we know:

$$(\widetilde{|v_\delta|^4 v_\delta})(n,\tau) = \sum_{n_1-n_2+n_3-n_4+n_5=n} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} d\tau_j \widetilde{v_\delta}(n_1,\tau_1)\overline{\widetilde{v_\delta}(n_2,\tau_2)}\widetilde{v_\delta}(n_3,\tau_3)\overline{\widetilde{v_\delta}(n_4,\tau_4)}\widetilde{v_\delta}(n_5,\tau_5).$$

it follows that:

$$|I| \leq \sum_{n} \sum_{n_1-n_2+n_3-n_4+n_5=n} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} d\tau_j \{(1+|n|)^s (1+|\tau+n^2|)^{b-1} |c(n,\tau)|$$

$$|\widetilde{v_\delta}(n_1,\tau_1)||\overline{\widetilde{v_\delta}(n_2,\tau_2)}||\widetilde{v_\delta}(n_3,\tau_3)||\overline{\widetilde{v_\delta}(n_4,\tau_4)}||\widetilde{v_\delta}(n_5,\tau_5)|\}.$$

Since $n = n_1 - n_2 + n_3 - n_4 + n_5$, it follows that:

$$|n|^s \lesssim \max\{|n_1|^s, |n_2|^s, |n_3|^s, |n_4|^s, |n_5|^s\}.$$

By symmetry, it suffices to bound the expression:

$$I_1 := \sum_{n} \sum_{n_1-n_2+n_3-n_4+n_5=n} \int_{\tau_1-\tau_2+\tau_3-\tau_4+\tau_5=\tau} d\tau_j d\tau \{ \frac{|c(n,\tau)|}{(1+|\tau+n^2|)^{1-b}}$$

$$(1+|n_1|)^s |\widetilde{v_\delta}(n_1,\tau_1)||\overline{\widetilde{v_\delta}(n_2,\tau_2)}||\widetilde{v_\delta}(n_3,\tau_3)||\overline{\widetilde{v_\delta}(n_4,\tau_4)}||\widetilde{v_\delta}(n_5,\tau_5)|\} =$$

$$= \sum_{n} \sum_{n_1+n_2+n_3+n_4+n_5=n} \int_{\tau_1+\tau_2+\tau_3+\tau_4+\tau_5=\tau} d\tau_j d\tau \{ \frac{|c(n,\tau)|}{(1+|\tau+n^2|)^{1-b}}$$

$$(1+|n_1|)^s |\widetilde{v_\delta}(n_1,\tau_1)||\overline{\widetilde{v_\delta}(-n_2,-\tau_2)}||\widetilde{v_\delta}(n_3,\tau_3)||\overline{\widetilde{v_\delta}(-n_4,-\tau_4)}||\widetilde{v_\delta}(n_5,\tau_5)|\}.$$

Let us now define the following functions:

$$F(x,t) := \sum_{n} \int d\tau \{ \frac{|c(n,\tau)|}{(1+|\tau+n^2|)^{1-b}} e^{inx+it\tau} \}. \tag{2.148}$$

$$G(x,t) := \sum_{n} \int d\tau \{(1+|n|)^s |\widetilde{v_\delta}(n,\tau)| e^{inx+it\tau} \}. \tag{2.149}$$

$$H(x,t) := \sum_{n} \int d\tau \{ |\widetilde{v_\delta}(n,\tau)| e^{inx+it\tau} \}. \tag{2.150}$$

Consequently, by using Parseval's identity, one obtains:

$$I_1 \lesssim \int\int F\bar{G}H\bar{H}H\bar{H}dxdt = |\int\int F\bar{G}H\bar{H}H\bar{H}dxdt|,$$

which by Hölder's inequality is:

$$\leq \|F\|_{L^4_{t,x}} \|G\|_{L^4_{t,x}} \|H\|^2_{L^4_{t,x}} \|H\|^2_{L^\infty_{t,x}}. \tag{2.151}$$

Recalling (2.22), and using the fact that $b = \frac{1}{2}+$, we have [7]:

$$\|F\|_{L^4_{t,x}} \lesssim \|F\|_{X^{0,\frac{3}{8}}} \leq \|F\|_{X^{0,1-b}} = \|c\|_{l^2_k L^2_\tau} = 1. \tag{2.152}$$

$$\|G\|_{L^4_{t,x}} \lesssim \|G\|_{X^{0,\frac{3}{8}}} = \|(1+|n|)^s |\widehat{v_\delta}(n,\tau)| (1+|\tau+n^2|)^{\frac{3}{8}}\|_{l^2_n L^2_\tau} =$$

$$= \|v_\delta\|_{X^{s,\frac{3}{8}}} \lesssim \|v\|_{X^{s,\frac{3}{8}}} \leq \|v\|_{X^{s,b}}. \tag{2.153}$$

The implied constant in the above inequality is independent of δ by (2.138). Also:

$$\|H\|_{L^4_{t,x}} \lesssim \|v_\delta\|_{X^{0,\frac{3}{8}}}$$

We interpolate between $X^{0,0}$ and $X^{0,b}$ for an appropriate $\theta \in (0,1)$ to deduce that this is:

$$\lesssim \|v_\delta\|^\theta_{X^{0,0}} \|v_\delta\|^{1-\theta}_{X^{0,b}}.$$

We estimate $\|v_\delta\|_{X^{0,0}}$ by:

$$\|v_\delta\|_{X^{0,0}} = \|v_\delta\|_{L^2_{x,t}}$$

which by the support properties of ψ_δ is:

$$= \|v_\delta \psi_\delta\|_{L^2_{x,t}} \leq \|\psi_\delta\|_{L^4_t} \|v_\delta\|_{L^4_t L^2_x}$$

$$\lesssim \delta^{\frac{1}{4}} \|v_\delta\|_{X^{0,\frac{1}{4}+}} \lesssim \delta^{\frac{1}{4}} \|v_\delta\|_{X^{0,b}}.$$

Here, we have used (2.21).

Hence:

$$\|H\|_{L^4_{t,x}} \lesssim (\delta^{\frac{1}{4}} \|v_\delta\|_{X^{0,b}})^\theta (\|v_\delta\|_{X^{0,b}})^{1-\theta} =$$

[7] In the following calculation, and later on, we crucially use the fact that one doesn't change the $X^{s,b}$ norm of a function when one takes absolute values in its Spacetime Fourier Transform.

$$= \delta^{\frac{\theta}{4}} \|v_\delta\|_{X^{0,b}} \lesssim \delta^{\frac{\theta}{4}+\frac{1-2b}{2}} \|v\|_{X^{0,b}}. \tag{2.154}$$

In the last step, we used (2.136).

Furthermore, by Sobolev embedding:

$$\|H\|_{L^\infty_{t,x}} \lesssim \|H\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} = \|v_\delta\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \leq \|v_\delta\|_{X^{1,b}} \lesssim \delta^{\frac{1-2b}{2}} \|v\|_{X^{1,b}}. \tag{2.155}$$

<u>We calculate θ :</u>

We know:

$$\frac{3}{8} = 0 \cdot \theta + b \cdot (1-\theta)$$

So:

$$\theta = \frac{b - \frac{3}{8}}{b} = \frac{1+8\epsilon}{4+8\epsilon}. \tag{2.156}$$

Combining (2.151) – (2.155), it follows that:

$$\||v_\delta|^4 v_\delta\|_{X^{s,b-1}} \lesssim \|v\|_{X^{s,b}} (\delta^{\frac{\theta}{4}+\frac{1-2b}{2}} \|v\|_{X^{0,b}})^2 (\delta^{\frac{1-2b}{2}} \|v\|_{X^{1,b}})^2 \leq$$

$$\leq \delta^{\,\theta_0 + 2(1-2b)} (\|v\|_{X^{1,b}})^4 \|v\|_{X^{s,b}}. \tag{2.157}$$

Here:

$$\theta_0 := \frac{\theta}{2} = \frac{1+8\epsilon}{8+16\epsilon}. \tag{2.158}$$

Hence, from (2.146) and (2.157),we obtain:

$$\|Lv\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} + c_1 \delta^{\,\theta_0+\frac{5}{2}(1-2b)} (\|v\|_{X^{1,b}})^4 \|v\|_{X^{s,b}}. \tag{2.159}$$

Here $c, c_1 > 0$ depend on s.

If we take c, c_1 possibly even smaller, and if we repeat the previous argument in the special case $s = 1$, it follows that:

$$\|Lv\|_{X^{1,b}} \leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^1} + c_1 \delta^{\,\theta_0+\frac{5}{2}(1-2b)} (\|v\|_{X^{1,b}})^5. \tag{2.160}$$

Now, we estimate $\|Lv - Lw\|_{X^{1,b}}$. In order to do this, we note that:

$|v|^4 v - |w|^4 w =$ Sum of quintic terms, each of which contains at least one factor of $v - w$ or $\overline{v-w}$. By the above proof, since the estimates (2.159) depended only on bounds on spacetime norms in x, t, we can put complex conjugates in the appropriate factors (so if $v - w$ comes with a conjugate, it doesn't matter). Furthermore, by the triangle inequality, we know: $\|v - w\|_{X^{1,b}} \leq \|v\|_{X^{1,b}} + \|w\|_{X^{1,b}}$. Thus, arguing as before, we can obtain, for some $c_2 > 0$:

$$\|Lv - Lw\|_{X^{1,b}} \leq c_2 \delta^{\,\theta_0 + \frac{5}{2}(1-2b)} (\|v\|_{X^{1,b}} + \|w\|_{X^{1,b}})^4 \|v - w\|_{X^{1,b}}. \tag{2.161}$$

Let

$$\Gamma := \{v : \|v\|_{X^{s,b}} \leq 2c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s}, \|v\|_{X^{1,b}} \leq 2c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^1}\}. \tag{2.162}$$

Let us give Γ the metric $d(v,w) := \|v - w\|_{X^{1,b}}$. Then, by Proposition 2.3.2, (Γ, d) is a Banach space.

From (2.159), we have for all $v \in \Gamma$

$$\|Lv\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} + c_1 \delta^{\theta_0 + \frac{5}{2}(1-2b)} (2c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^1})^4 \, 2c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} =$$

$$= c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} (1 + 32 c_1 c^4 \delta^{\,\theta_0 + \frac{9}{2}(1-2b)} \|\Phi\|^4_{H^1}). \tag{2.163}$$

Analogously, from (2.160) :

$$\|Lv\|_{X^{1,b}} \leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^1} (1 + 32 c_1 c^4 \delta^{\,\theta_0 + \frac{9}{2}(1-2b)} \|\Phi\|^4_{H^1}). \tag{2.164}$$

Finally, if $v, w \in \Gamma$, (2.161) implies that:

$$\|Lv - Lw\|_{X^{1,b}} \leq c_2\delta^{\,\theta_0+\frac{5}{2}(1-2b)}(4c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^1})^4\|v-w\|_{X^{1,b}}$$

$$\leq 256c_2c^4\delta^{\,\theta_0+\frac{9}{2}(1-2b)}\|\Phi\|_{H^1}^4\|v-w\|_{X^{1,b}}. \tag{2.165}$$

We recall that $\theta_0 = \frac{1+8\epsilon}{8+16\epsilon}, b = \frac{1}{2}+\epsilon$. We observe that for $\epsilon > 0$ sufficiently small, one has

$$\frac{1+8\epsilon}{8+16\epsilon} - 9\epsilon > 0 \tag{2.166}$$

From now, let us fix ϵ to satisfy the condition (2.166). In other words, we have:

$$\theta_0 + \frac{9}{2}(1-2b) > 0. \tag{2.167}$$

Hence, we can choose $\delta > 0$ sufficiently small such that:

$$32c_1c^4\delta^{\,\theta_0+\frac{9}{2}(1-2b)}\|\Phi\|_{H^1}^4 \leq 1. \tag{2.168}$$

$$256c_2c^4\delta^{\,\theta_0+\frac{9}{2}(1-2b)}\|\Phi\|_{H^1}^4 \leq \frac{1}{2}. \tag{2.169}$$

From (2.168), (2.169), the preceding bounds and the fact that (Γ, d) is a Banach Space, it follows that L *has a fixed point* $v \in \Gamma$.

By construction of L, for this v, we know:

- $v(t_0) = \Phi$.
- $iv_t + \Delta v = |v|^4 v$ for $t \in [t_0-\delta, t_0+\delta]$, and hence by uniqueness (which is proved by an application of Gronwall's inequality), it follows that:

$$v = u \text{ for } t \in [t_0-\delta, t_0+\delta].$$

- $\|v\|_{X^{s,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} = 2c\delta^{\frac{1-2b}{2}}\|u_0\|_{H^s}$.

It just remains to address the issue of choosing δ uniformly in t_0. However, from (2.168), (2.169), it follows that we just want δ to satisfy:

$$\delta^{\theta_0+\frac{9}{2}(1-2b)}\|\Phi\|_{H^1}^4 \lesssim 1. \tag{2.170}$$

By the fact that:

$$\|\Phi\|_{H^1} \lesssim_{Mass(u),Energy(u)} 1$$

it follows that we can choose

$$\delta \sim_{Mass(u),Energy(u)} 1$$

which is uniform in time, so the previous procedure can be iterated with fixed increment δ.

This proves (2.29) and (2.30). We now have to prove (2.31).

Let us recall that the function v that we have constructed satisfies:

$$\|v\|_{X^{1,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^1}. \tag{2.171}$$

$$\|v\|_{X^{s,b}} < \infty. \tag{2.172}$$

and

$$Lv = \chi_\delta(t)S(t)\Phi - i\chi_\delta(t)\int_0^t S(t-t')|v_\delta|^4 v_\delta(t')dt'.$$

We take $\mathcal{D}$'s in the previous equation, and since $\mathcal{D}$ acts only on the spatial variables (as a Fourier multiplier), we obtain:

$$\mathcal{D}v = \chi_\delta(t)S(t)\mathcal{D}\Phi - i\chi_\delta(t)\int_0^t S(t-t')\mathcal{D}(|v_\delta|^4 v_\delta(t'))dt'.$$

We know that:

$$\forall\, m,n \in \mathbb{Z},\ \theta(m+n) \lesssim_s \theta(m)+\theta(n).$$

From this *"Fractional Leibniz Rule"*, we deduce that for $n = n_1 - n_2 + n_3 - n_4 + n_5$, one has:

$$\theta(n) \lesssim_s max\{\theta(n_1), \theta(n_2), \theta(n_3), \theta(n_4), \theta(n_5)\}.$$

So, arguing analogously as earlier (c.f. (2.146)), we obtain:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq c_1 \delta^{\frac{1-2b}{2}} \|\mathcal{D}\Phi\|_{L^2} + c_2 \delta^{\frac{1-2b}{2}} \|\mathcal{D}(|v_\delta|^4 v_\delta)\|_{X^{0,b-1}} \leq$$

$$\leq c_1 \delta^{\frac{1-2b}{2}} \|\mathcal{D}\Phi\|_{L^2} + c_3 \delta^{\theta_0 + \frac{5(1-2b)}{2}} \|v\|^4_{X^{1,b}} \|\mathcal{D}v\|_{X^{0,b}}.$$

By using (2.171), we get:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq c_1 \delta^{\frac{1-2b}{2}} \|\mathcal{D}\Phi\|_{L^2} + c_4 \delta^{\theta_0 + \frac{9(1-2b)}{2}} \|\Phi\|^4_{H^1} \|\mathcal{D}v\|_{X^{0,b}}.$$

By using (2.167), we can choose $\delta > 0$ (possibly smaller than the one chosen before), such that:

$$c_4 \delta^{\theta_0 + \frac{9}{2}(1-2b)} \|\Phi\|^4_{H^1} \leq \frac{1}{2}. \tag{2.173}$$

Observe that then $\delta = \delta(s, Energy, Mass)$. Also, we note that choosing δ to be even smaller than the one chosen in the proof of (2.29),(2.30), yet still depending only on $(s, Energy, Mass)$ doesn't create problems with the estimates on $\|v\|_{X^{1,b}}, \|v\|_{X^{s,b}}$ we had earlier.

Note that:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq \|v\|_{X^{s,b}} < \infty.$$

where in the last inequality, we were using (2.172).

Hence:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq c_1\delta^{\frac{1-2b}{2}}\|\mathcal{D}\Phi\|_{L^2} + \frac{1}{2}\|\mathcal{D}v\|_{X^{0,b}}.$$

implies:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq 2c_1\|\mathcal{D}\Phi\|_{L^2}.$$

In other words, we obtain:

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \lesssim \|\mathcal{D}\Phi\|_{L^2}.$$

with the explicit constant depending only on $(s, Energy, Mass)$.

We may now conclude that (2.31) holds.

It remains to see the continuity of δ, C in the energy and mass. We recall from the construction of δ (c.f. (2.170),(2.173)) that we want, for some $\gamma > 0$:

$$\delta \lesssim \|\Phi\|_{H^1}^{-\gamma}.$$

Since $\|\Phi\|_{H^1}^2 \lesssim M(\Phi) + E(\Phi)$, we take:

$$\delta \sim (M(\Phi) + E(\Phi))^{-\frac{\gamma}{2}} \tag{2.174}$$

Such a δ depends continuously on the energy and mass. We notice that the C is obtained as a continuous function of δ, and the bounds on the H^1 norm of a solution, so it also depends continuously on energy and mass.

This proves Proposition 2.3.1 in the case $k = 2$.

If we are considering the general case $k \geq 2$, we have to modify the previous proof to consider the map:

$$Lv := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t)\int_0^t S(t-t')|v|^{2k}v(t')dt'.$$

Arguing as in (2.146), we deduce:

$$\|Lv\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} + c\delta^{\frac{1-2b}{2}}\||v_\delta|^{2k}v_\delta\|_{X^{s,b-1}}.$$

One then estimates the quantity $\||v_\delta|^{2k}v_\delta\|_{X^{s,b-1}}$ by duality.

The extra $k-2$ terms that are obtained here are estimated in $\|\cdot\|_{L^\infty_{t,x}}$ after an application of Hölder's inequality and we again use the fact that: $X^{\frac{1}{2}+,\frac{1}{2}+} \hookrightarrow L^\infty_{t,x}$. The proof then follows similarly as in the case $k=2$. We omit the details.

□

We now present the proof of Proposition 2.3.2, by which we can iterate our construction without changing the size of the increment:

Proof. (*of Proposition 2.3.2*)

The proof of this remarkable fact uses the special structure of the $X^{s,b}$ spaces. The main ingredient is the following fact, taken from [28]:

Theorem 1.2.5. *"Consider two Banach spaces $X \hookrightarrow Y$ and $1 < p,q \leq \infty$ and an open interval $I \subseteq \mathbb{R}$ (which can equal $\mathbb{R}$). Let $(f_n)_{n\geq 0}$ be a bounded sequence in $L^q(I,Y)$, and let $f : I \to Y$ be such that: $f_n(t) \rightharpoonup f(t)$ in Y as $n \to \infty$ for a.e. $t \in I$. If $(f_n)_{n\geq 0}$ is bounded in $L^p(I,X)$ and if X is reflexive, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \leq \liminf \|f_n\|_{L^p(I,X)}$."*

We now work on the Fourier transform side. For $\sigma \geq 0$, we define:

$$h_n^\sigma := \{(b_n)_{n\in\mathbb{Z}} : (\sum_n (1+|n|)^{2\sigma}|b_n|^2)^{\frac{1}{2}} < \infty\}$$

$$\|b\|_{h_\sigma^n} := (\sum_n (1+|n|)^{2\sigma}|b_n|^2)^{\frac{1}{2}}.$$

In this way, we get a Hilbert space, which is in particular a reflexive Banach space. The set $B := \{v : \|v\|_{X^{s,b}} \leq R\}$ is identified with the set:

$$E := \{\tilde{v} : \mathbb{Z} \times \mathbb{R} \to \mathbb{C} : \sum_n \int d\tau (1 + |\tau + n^2|)^{2b} (1 + |n|)^{2s} |\tilde{v}(n, \tau)|^2 \leq R^2\}.,$$

with the metric given by:

$$d(\tilde{v}, \tilde{w}) := (\sum_n \int d\tau (1 + |\tau + n^2|)^{2b} (1 + |n|)^{2s} |\tilde{v}(n, \tau) - \tilde{w}(n, \tau)|^2)^{\frac{1}{2}}.$$

We will now apply Theorem 1.2.5 from [28] with:

$$X = h_n^s, Y = h_n^1, p = q = 2, I = \mathbb{R}.$$

Let us now start with $(u_r)_{r \geq 0}$ a sequence in B such that: $u_r \to u$ as $r \to \infty$ in $X^{1,b}$ and we want to argue that $u \in B$.

Let us take:

$$f_r(n, \tau) := (1 + |\tau + n^2|)^b \widetilde{u_r}(n, \tau).$$

Then:

$$\|f_r\|_{L^2_\tau h^s_n} \leq R.$$

The claim we want to prove is:

$$\|f\|_{L^2_\tau h^s_n} \leq R \text{ where } f(n, \tau) := (1 + |\tau + n^2|)^b \widetilde{v}(n, \tau).$$

We know that:

$$\|u_r - u\|_{X^{1,b}} \to 0.$$

Thus:

$$\|\|f_r(\tau) - f(\tau)\|_{h^1_n}\|_{L^2_\tau} \to 0.$$

Hence:

$$\|f_r(\tau) - f(\tau)\|_{h^1_n} \to 0 \text{ in measure as a function of } \tau.$$

Thus, we can pass to a subsequence of $(f_r)_{r\geq 0}$ which we again call (f_r) such that:

$$\|f_r(\tau) - f(\tau)\|_{h^1_n} \to 0 \text{ pointwise almost everywhere as a function of } \tau.$$

In particular:

$$f_r(\tau) \to f(\tau) \text{ in } h^1_n = Y, \text{ for almost every } \tau.$$

So:

$$f_r(\tau) \rightharpoonup f(\tau) \text{ in } h^1_n = Y, \text{ for almost every } \tau.$$

Now, Theorem 1.2.5 from [28] implies that:

$$\|f\|_{L^2_\tau h^s_n} \leq \liminf \|f_r\|_{L^2_\tau h^s_n} \leq R. \tag{2.175}$$

□

We now prove the Approximation Lemma.

Proof. (*of Proposition 2.3.3*)

With notation as in the statement of the Proposition, we consider n sufficiently large so that:

$$M(\Phi_n) \sim M(\Phi), E(\Phi_n) \sim E(\Phi), \|\Phi_n\|_{H^s} \sim \|\Phi\|_{H^s}.$$

Let us denote $N(f) := |f|^{2k} f$.

With notation as in the proof of Proposition 2.3.1, we define:

$$Lv := \chi_\delta(t) S(t)\Phi - i\chi_\delta(t) \int_0^t S(t-t') N(v)(t') dt'.$$

$$L_n v^{(n)} := \chi_\delta(t) S(t)\Phi_n - i\chi_\delta(t) \int_0^t S(t-t') N(v^{(n)})(t') dt'.$$

From our earlier arguments, we can choose $\delta = \delta(s, E(\Phi), M(\Phi)) > 0$ sufficiently small so that L has a fixed point v that coincides with u for $t \in [0, \delta]$, and which satisfies:

$$\|v\|_{X^{s,b'}} \leq C(s, E(\Phi), M(\Phi)) \delta^{\frac{1-2b'}{2}} \|\Phi\|_{H^s}, b' = \frac{1}{2}+ \tag{2.176}$$

Let us fix $T > \delta$. By just iterating the local well-posedness bound, we get that for all $t \in [0, T]$:

$$\|u(t)\|_{H^s} \leq C(s, E(\Phi), M(\Phi)) \|\Phi\|_{H^s} e^{C_1(s,E(\Phi),M(\Phi))T} =: C_2. \tag{2.177}$$

Hence $C_2 = C_2(s, E(\Phi), M(\Phi), \|\Phi\|_{H^s}, T) > 0$.

We can repeat the same for L_n to obtain a fixed point $v^{(n)}$ which coincides with $u^{(n)}$ for $t \in [t_0, t_0 + \delta]$. The δ and C_2 will remain equivalent to the ones chosen earlier.

Then:

$$\|v - v^{(n)}\|_{X^{s,b'}} = \|Lv - L_n v^{(n)}\|_{X^{s,b'}} \leq$$

$$\|\chi_\delta(t) S(t)\Phi - \chi_\delta(t) S(t)\Phi_n\|_{X^{s,b'}} + \|\chi_\delta(t) \int_0^t S(t-t')(N(v) - N(v^{(n)}))(t') dt'\|_{X^{s,b'}} \leq$$

$$c\delta^{\frac{1-2b'}{2}}\|\Phi-\Phi_n\|_{H^s}+c\delta^{r_0}(P(\|v\|_{X^{s,b'}})+P(\|v^{(n)}\|_{X^{s,b'}}))\|v-v^{(n)}\|_{X^{s,b'}}.$$

Here, $c>0$ is a universal constant, P is a polynomial of fixed degree such that $P(0)=0$, and $r_0>0$ is fixed (independent of b').

Hence, by (2.176), it follows that:

$$\|v-v^{(n)}\|_{X^{s,b'}}\leq c\delta^{\frac{1-2b'}{2}}\|\Phi-\Phi_n\|_{H^s}+c\delta^{r_0}(P(\delta^{\frac{1-2b'}{2}}\|\Phi\|_{H^s})+P(\delta^{\frac{1-2b'}{2}}\|\Phi_n\|_{H^s}))\|v-v^{(n)}\|_{X^{s,b'}}$$

$$\leq c\delta^{\frac{1-2b'}{2}}\|\Phi-\Phi_n\|_{H^s}+\tilde{c}\delta^{r_0}P(\delta^{\frac{1-2b'}{2}}C_2)\|v-v^{(n)}\|_{X^{s,b'}}. \qquad (2.178)$$

The last inequality was obtained by combining (2.176) and (2.177).

We now choose δ even smaller such that:

$$\tilde{c}\delta^{r_0}P(\delta^{\frac{1-2b'}{2}}C_2)\leq\frac{1}{2}.$$

By choosing δ even smaller, the previous estimate (2.176), and all the subsequent estimates will remain otherwise unchanged. The new $\delta=\delta(s,E(\Phi),M(\Phi),C_2)>0$ now also depends on C_2.

We obtain:

$$\|v-v^{(n)}\|_{X^{s,b'}}\leq 2c\delta^{\frac{1-2b'}{2}}\|\Phi-\Phi_n\|_{H^s}.$$

By using (2.177), it follows that, this bound can be iterated on $\sim\frac{T}{\delta}$ time intervals, with the same δ. Namely, in the definition of L, L_n, we just have to consider $\chi_\delta(\cdot-r)$ for an appropriate time translation r, and instead of Φ,Φ_n as initial data, we consider $u(r),u^{(n)}(r)$ respectively.

Furthermore, let us use the fact that: $X^{s,b'}\hookrightarrow L^\infty_t H^s_x$ to deduce that, for the large enough n we are considering, one has:

$$\|u(t) - u^{(n)}(t)\|_{H^s} \leq C\|\Phi - \Phi_n\|_{H^s}.$$

Here, $C = C(s, E(\Phi), M(\Phi), \|\Phi\|_{H^s}, T) > 0$.

The claim now follows.

□

Remark 2.6.1. *An Approximation analogous to Proposition 2.3.3 is also holds for* (2.4), (2.5), *and* (2.6) *with the same proof.*

Chapter 3

Bounds on $\mathbb{R}$

3.1 Introduction

3.1.1 Statement of the main results

In this chapter, we will study the defocusing cubic nonlinear Schrödinger initial value problem on $\mathbb{R}$:

$$\begin{cases} iu_t + \Delta u = |u|^2 u,\, x \in \mathbb{R},\, t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}). \end{cases} \tag{3.1}$$

Furthermore, we will study the Hartree initial value problem on $\mathbb{R}$

$$\begin{cases} iu_t + \Delta u = (V * |u|^2)u, x \in \mathbb{R}, t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}). \end{cases} \tag{3.2}$$

The assumptions that we have on V are analogous to the ones we had on S^1:

(i) $V \in L^1(\mathbb{R})$.

(ii) $V \geq 0$.

(iii) V is even.

We recall from [46, 84] that (3.1) is completely integrable. Therefore, if $s = k$ is a positive integer, one can deduce, by using a fixed finite number of conserved quantities that there exists a function $B_k : H^k \to \mathbb{R}$ such that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^k} \leq B_k(\Phi). \tag{3.3}$$

From the preceding observation, it makes sense to consider only the case when s *is not an integer*. One notes that the uniform bounds for H^s norms when s is not an integer don't follow from the uniform bounds on the integer Sobolev norms if we are assuming only that $\Phi \in H^s(\mathbb{R})$.

The result that we prove for (3.1) is:

Theorem 3.1.1. *(Bound for the Cubic NLS) Suppose $s > 1$ is a real number. Let $\alpha := s - \lfloor s \rfloor$ denote the fractional part of s. Suppose $\Phi \in H^s(\mathbb{R})$, and let u denote the global solution to the corresponding problem (3.1). Then, there exists a continuous function $F_s : H^s \to \mathbb{R}$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq F_s(\Phi)(1 + |t|)^{\alpha+}.$$

Theorem 3.1.1 gives a solution to an open problem that was mentioned on the Dispersive Wiki Website [113].

Unlike the one-dimensional cubic NLS, the Hartree equation doesn't have infinitely many conserved quantities. The following quantities are conserved under the evolution of (3.2):

$$M(u(t)) = \int |u(x,t)|^2 dx \quad \textit{(Mass)}$$

and

$$E(u(t)) = \frac{1}{2}\int |\nabla u(x,t)|^2 dx + \frac{1}{4}\int (V * |u|^2)(x,t)|u(x,t)|^2 dx \quad \textit{(Energy)}$$

We hence deduce that $\|u(t)\|_{H^1}$ is uniformly bounded whenever u is a solution of

(3.2). The bound that we prove is:

Theorem 3.1.2. *(Bound for the Hartree equation) Let $s \geq 1$, and let u be the global solution of* (3.2)*. Then, there exists a function C_s, continuous on H^1 such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq C_s(\Phi)(1+|t|)^{\frac{1}{3}s+}\|\Phi\|_{H^s}. \tag{3.4}$$

Remark 3.1.3. *As in the previous chapter, we can see that the focusing-type analogues of Theorem 3.1.1 and Theorem 3.1.2 hold, if we suppose that the initial data is sufficiently small in L^2. Namely, if we take $\|\Phi\|_{L^2}$ sufficiently small, Theorem 3.1.1 holds for the focusing NLS on $\mathbb{R}$. The continuity of the higher conserved quantities is the same [46]. Furthermore, under the same smallness assumption, Theorem 3.1.2 still holds for* (3.2) *when the convolution potential is not necessarily non-negative, but is still real-valued.*

3.1.2 Previously known results

Let us note that the previously known techniques used to obtain polynomial bounds on $\|u(t)\|_{H^s}$ in [12, 27, 98, 99, 118] need either the assumption that s is a positive integer, or that Φ lies in a more regular space than H^s. The reason for this is that one wants to use an exact Leibniz rule for the operator D^s in order to cancel certain terms which can't be estimated in the appropriate $X^{s,b}$ space. Hence, the only bounds that one could previously obtain for $\|u(t)\|_{H^s}$, when u is a solution to (3.1) are exponential in time. This is clearly far from a sharp bound, since the equation is completely integrable.

Let us note that Theorem 3.1.2 would follow trivially if we knew that (3.2) scattered in H^s, since then all the Sobolev norms of solutions would be uniformly bounded in time. The currently known techniques to prove scattering don't seem to apply in this context though. Namely, the techniques from [49, 63] require for us to the have additional assumption that our solutions lie in weighted Sobolev spaces, and the obtained bounds depend on these weighted Sobolev norms. Hence we can't argue by density here. The methods from [56] require the initial data to belong to an appropriate subset of the Gevrey class. Finally, the techniques used in [85, 86] apply only in dimensions greater than or equal to 5.

Recently in [42], it was shown that the defocusing quintic NLS on $\mathbb{R}$:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, \; x \in \mathbb{R}, \; t \in \mathbb{R} \\ u|_{t=0} = \Phi \in L^2(\mathbb{R}). \end{cases} \tag{3.5}$$

exhibits scattering in L^2. Hence, if $\Phi \in H^s(\mathbb{R})$, one obtains uniform bounds on $\|u(t)\|_{H^s}$. More details on the *persistence of regularity* for scattering will be given in the following chapter.

3.1.3 Main ideas of the proofs

Main ideas of the proof of Theorem 3.1.1

The main idea of the proof of Theorem 3.1.1 is to look at the high and low-frequency part of the solution u as in [18], and to use the bound (3.3), which gives us uniform bounds on integral Sobolev norms of u. In particular, we let N be a parameter, which will be the threshold dividing the "low" and "high" frequencies, and we define Q to be the projection operator onto the high frequencies. From (3.3), i.e. from the uniform boundedness of the $H^{\lfloor s \rfloor}$ of a solution, we can derive that for all times t:

$$\|(I - Q)u(t)\|_{H^s}^2 \leq B. \tag{3.6}$$

Here $B = C(\Phi)N^{2\alpha}$, where $\alpha := s - \lfloor s \rfloor \in [0, 1)$ is the fractional part of s. We note that the exponent is then in $[0, 2)$ and is not a multiple of s. We use the estimate (3.6) to bound the low-frequency part of the solution.

One then has to bound $\|Qu(t)\|_{H^s}$. For $t_1 > 0$, we look at the quantity:

$$\|Qu(t_1)\|_{H^s}^2 - \|Qu(t_0)\|_{H^s}^2 = \int_{t_0}^{t_1} \frac{d}{dt}\|Qu(t)\|_{H^s}^2 dt.$$

Since we are working on the real line, we can use an appropriate dyadic decomposition and the *improved Strichartz estimate* (Proposition 3.2.2) to obtain a decay factor of $\frac{1}{N^{1-}}$ in the above integral in time. The exact bounds we obtain are the content of Proposition 3.3.4. At the end, we deduce that there exists an increment $\delta > 0$, and $C > 0$, both depending only on the initial data such that for all $t_0 \in \mathbb{R}$, one has:

$$\|Qu(t_0 + \delta)\|_{H^s}^2 \leq (1 + \frac{C}{N^{1-}})\|Qu(t_0)\|_{H^s}^2 + B_1. \tag{3.7}$$

Here, $B_1 \lesssim \frac{1}{N^{1-}} B$.

The idea now is to iterate (3.7) for times $t_0 = 0, \delta, \ldots, n\delta$, where $n \in \mathbb{N}$ is an integer such that $n \lesssim N^{1-}$.

Multiplying the obtained inequalities by appropriate powers of $1 + \frac{C}{N^{1-}}$, and telescoping, we show that:

$$\|Qu(n\delta)\|_{H^s}^2 \lesssim (1 + \frac{C}{N^{1-}})^n \|Qu(0)\|_{H^s}^2 + B \tag{3.8}$$

Since $n \lesssim N^{1-}$, we know:

$$(1 + \frac{1}{N^{1-}})^n = O(1).$$

Using the previous bound, (3.6) and (3.8), we can show that for all $t \in [0, n\delta]$:

$$\|u(t)\|_{H^s}^2 \lesssim C\|\Phi\|_{H^s}^2 + B.$$

Optimizing N in terms of the length of time interval $[0, T]$ on which we are considering the solution, and noting that then B becomes the leading term, Theorem 3.1.1 follows.

Main ideas of the proof of Theorem 3.1.2

The main argument is similar to the one given in Chapter 2. Given a parameter $N > 1$, we will use the method of an *upside down I-operator*, followed by the method of *higher modified energies* to define a quantity $E^2(u(t))$, which is linked to $\|u(t)\|_{H^s}^2$.

As before, our goal is to prove an iteration bound of the type:

$$E^2(u(t_0+\delta)) \leq (1+\frac{C}{N^\alpha})E^2(u(t_0)). \tag{3.9}$$

for all $t_0 \in \mathbb{R}$, with $\delta, \alpha > 0$, and the implied constant all independent of t_0.

Due to the presence of the decay factor $\frac{1}{N^\alpha}$, (3.9) can be iterated $\sim N^\alpha$ times to obtain that $E^2 \lesssim 1$ on a time interval of size $\sim N^\alpha$. One then uses the relation between $E^2(u(t))$ and $\|u(t)\|_{H^s}$ to get polynomial bounds for $\|u(t)\|_{H^s}$.

The bound (3.9) is proved in a similar way as the corresponding estimate in Chapter 2. In order to construct E^2, we need to consider the multiplier ψ which is defined by:

$$\psi := c\,\frac{((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2)\widehat{V}(\xi_3+\xi_4)}{\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2}$$

for some constant c when the denominator doesn't vanish, and $\psi := 0$ otherwise. Here, θ is an appropriately smoothed out and rescaled version of the operator D^s. For details, see (3.58), (3.70), and (3.71). The key is then to obtain pointwise bounds on such a ψ. This is done in Proposition 3.4.2

We observe that the bound we obtain in Theorem 3.1.2 is better than the corresponding bound in the periodic setting. This is a manifestation of stronger dispersion, which is present on the real line. In this chapter, we will prove that on $\mathbb{R}$, (3.9) holds for $\alpha = 3-$. We recall from Chapter 2 that the analogous estimate on S^1 holds for $\alpha = 2-$. Heuristically, the improvement is obtained by using the *improved Strichartz estimate*, i.e. Proposition 3.2.2, which holds on the real line.

Remark 3.1.4. *The techniques of proof of Theorem 3.1.2 apply to the derivative nonlinear Schrödinger equation:*

$$\begin{cases} iu_t + \Delta u = i\partial_x(|u|^2 u), \\ u(x,0) = \Phi(x), x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases} \tag{3.10}$$

The equation (3.10) occurs as a model for the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field [103]. In order to obtain global well-posedness in H^s, we need to have the smallness assumption:

$$\|\Phi\|_{L^2} < \sqrt{2\pi}, \tag{3.11}$$

From [68], we know that (3.10) is completely integrable. Hence, as for the cubic NLS, it makes sense to bound only the non-integral Sobolev norms of a solution.

Bound for the Derivative NLS. *For $s > 1$, not an integer, and $\Phi \in H^s(\mathbb{R})$, satisfying the smallness assumption (3.11), there exists $C(s, |\Phi\|_{H^1})$ such that the solution u of (3.10) satisfies:*

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{2s+}\|\Phi\|_{H^s}, \textit{ for all} t \in \mathbb{R}. \tag{3.12}$$

The proof of (3.12) *is quite involved. Unlike Theorem 3.1.1, we are not able to recover uniform bounds on the integral Sobolev norms of a solution. The techniques that we applied to the cubic NLS don't seem to work for the derivative NLS due to the derivative in the nonlinearity. A sketch of the proof of* (3.12) *is given in Appendix C of this chapter.*

3.2 Facts from harmonic analysis

On $\mathbb{R}$, we recall the following Strichartz estimate (c.f. [15, 106]).

$$\|f\|_{L^6_{t,x}} \lesssim \|f\|_{X^{0,\frac{1}{2}+}}. \tag{3.13}$$

Interpolating between (3.13) and $\|f\|_{L^2_{t,x}} = \|f\|_{X^{0,0}}$, it follows that:

$$\|f\|_{L^4_{t,x}} \lesssim \|f\|_{X^{0,\frac{3}{8}+}}. \tag{3.14}$$

From Sobolev embedding, we deduce that:

$$\|f\|_{L^\infty_t L^2_x} \lesssim \|f\|_{X^{0,\frac{1}{2}+}}. \tag{3.15}$$

and:

$$\|f\|_{L^\infty_t L^\infty_x} \lesssim \|f\|_{X^{\frac{1}{2}+,\frac{1}{2}+}}. \tag{3.16}$$

Interpolating between (3.13) and (3.15), we obtain:

$$\|f\|_{L^8_t L^4_x} \lesssim \|f\|_{X^{0,\frac{1}{2}+}}. \tag{3.17}$$

By an analogous proof as the localization estimate Lemma 2.2.1 in Chapter 2, we can deduce the following localization bound for $X^{s,b}$ spaces.

Lemma 3.2.1. *If $b \in (0, \frac{1}{2})$ and $s \in \mathbb{R}$, then, for $c < d$:*

$$\|\chi_{[c,d]}(t) f\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})} \lesssim \|f\|_{X^{s,b+}(\mathbb{R}\times\mathbb{R})} \tag{3.18}$$

where the implicit constant doesn't depend on u, c, d.

From [14, 31], we recall that on $\mathbb{R}$, the following *improved Strichartz estimate* holds:

Proposition 3.2.2. *(Improved Strichartz Estimate) Suppose $N_1 > 0$ and suppose that $f, g \in X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R})$ are such that for all $t \in \mathbb{R}$:*

$$supp\, \hat{f}(t) \subseteq \{|\xi| \sim N_1\},\ supp\, \hat{g}(t) \subseteq \{|\xi| \ll N_1\}.$$

Then, the following bound holds:

$$\|fg\|_{L^2_{t,x}} \lesssim \frac{1}{N_1^{\frac{1}{2}}} \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{0,\frac{1}{2}+}.$$

We observe the following consequence of Proposition 3.2.2:

Corollary 3.2.3. *For f, g as in Proposition 3.2.2, one has:*

$$\|fg\|_{L_t^{2+}L_x^2} \lesssim \frac{1}{N_1^{\frac{1}{2}-}} \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}}.$$

Let us prove Corollary 3.2.3.

Proof. Let f, g be as in the assumptions of the Lemma. We observe that by Hölder's inequality:

$$\|fg\|_{L_t^4 L_x^2} \leq \|f\|_{L_t^8 L_x^4} \|g\|_{L_t^8 L_x^4} \lesssim \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}}.$$

The last inequality follows from (3.17).

Given $\epsilon > 0$ small, we take:

$$\theta := \frac{2-\epsilon}{2+\epsilon} = 1- .$$

Then $\theta \in [0,1]$ satisfies:

$$\theta \cdot \frac{1}{2} + (1-\theta) \cdot \frac{1}{4} = \frac{1}{2+\epsilon}.$$

By using interpolation and Proposition 3.2.2, we deduce that:

$$\|fg\|_{L_t^{2+\epsilon}L_x^2} \leq (\|fg\|_{L^2_{t,x}})^{\theta} (\|fg\|_{L_t^4 L_x^2})^{1-\theta} \lesssim$$

$$(N_1{}^{-\frac{1}{2}} \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}})^{\theta} (\|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}})^{1-\theta} \lesssim N_1{}^{-\frac{\theta}{2}} \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}}.$$

Since $\theta = 1-$, Corollary 3.2.3 follows. □

In our analysis, we will have to work with $\chi = \chi_{[t_0,t_0+\delta]}(t)$, the characteristic function of the time interval $[t_0, t_0+\delta]$. It is difficult to deal with χ directly, since this function is not smooth, and since its Fourier transform doesn't have a sign. Instead, we will decompose χ as a sum of two functions which are easier to deal with. This goal will be achieved by using an appropriate approximation to the identity. We will use the following decomposition, which is originally found in [31]:

Given $\phi \in C_0^\infty(\mathbb{R})$, such that: $0 \leq \phi \leq 1$, $\int_{\mathbb{R}} \phi(t)\,dt = 1$, and $\lambda > 0$, we recall that the *rescaling* ϕ_λ of ϕ is defined by:

$$\phi_\lambda(t) := \frac{1}{\lambda}\,\phi(\frac{t}{\lambda}).$$

We observe that such a rescaling preserves the L^1 norm:

$$\|\phi_\lambda\|_{L^1_t} = \|\phi\|_{L^1_t}.$$

Having defined the rescaling, we write, for the scale $N_1 > 1$:

$$\chi(t) = a(t) + b(t), \text{ for } a := \chi * \phi_{N_1^{-1}}. \tag{3.19}$$

In Lemma 8.2. of [31], the authors note the following estimate:

$$\|a(t)f\|_{X^{0,\frac{1}{2}+}} \lesssim {N_1}^{0+}\|f\|_{X^{0,\frac{1}{2}+}}. \tag{3.20}$$

(The implied constant here is independent of N_1.)

On the other hand, for any $M \in (1, +\infty)$, one obtains:

$$\|b\|_{L^M_t} = \|\chi - \chi * \phi_{N_1{}^{-1}}\|_{L^M_t} \leq \|\chi\|_{L^M_t} + \|\chi * \phi_{N_1{}^{-1}}\|_{L^M_t}$$

which is by Young's inequality:

$$\leq \|\chi\|_{L^M_t} + \|\chi\|_{L^M_t}\|\phi_{N_1{}^{-1}}\|_{L^1_t} = 2\|\chi\|_{L^M_t} = C(M,\chi) = C(M,\Phi)$$

To explain the fact that $C(M,\chi) = C(M,\Phi)$, we note that χ is defined as the characteristic function of an interval of size δ, and δ, in turn, depends only on Φ.

If we now define:

$$b_1(t) := \int_{\mathbb{R}} |\hat{b}(\tau)| e^{it\tau} d\tau. \tag{3.21}$$

Since $M \in (1, \infty)$, we know by the Littlewood-Paley inequality [45] that:

$$\|b_1\|_{L^M_t} \lesssim \|b\|_{L^M_t}$$

Then, the previous bound on $\|b\|_{L^M_t}$ implies:

$$\|b_1\|_{L^M_t} \leq C(M, \Phi). \tag{3.22}$$

We will frequently use the following modification of Proposition 3.2.2

Proposition 3.2.4. *(Improved Strichartz Estimate with rough cut-off in time) Suppose $N_1 > 0$ and suppose $f, g \in X^{0,\frac{1}{2}+}(\mathbb{R} \times \mathbb{R})$ are such that for all $t \in \mathbb{R}$:*

$$supp\, \hat{f}(t) \subseteq \{|\xi| \sim N_1\},\ supp\, \hat{g}(t) \subseteq \{|\xi| \ll N_1\}.$$

Let f_1, g_1 be given by:

$$\widetilde{f_1} := |(\chi f)\widetilde{\ }|, \widetilde{g_1} := |\widetilde{g}\,|.$$

Then one has:

$$\|f_1 g_1\|_{L^2_{t,x}} \lesssim \frac{1}{N_1^{\frac{1}{2}-}} \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}} \tag{3.23}$$

The same bound holds if:

$$\widetilde{f_1} := |\widetilde{f}\,|,\ \widetilde{g_1} := |(\chi g)\widetilde{\ }|.$$

Proof. Let's consider the case when $\widetilde{f_1} = |(\chi f)\widetilde{\ }|, \widetilde{g_1} = |\widetilde{g}\,|$. With notation as earlier, let F_1, F_2 be given by:

$$\widetilde{F_1} := |(af)\widetilde{\ }|, \widetilde{F_2} := |(bf)\widetilde{\ }|.$$

Then, by the triangle inequality, one has:

$$\widetilde{f_1} \leq \widetilde{F_1} + \widetilde{F_2}.$$

Since $\widetilde{f_1}, \widetilde{g_1} \geq 0$, Plancherel's Theorem and duality imply that:

$$\|f_1 g_1\|_{L^2_{t,x}} \sim \sup_{\|c\|_{L^2_{\tau,\xi}}=1} \int_{\tau_1+\tau_2+\tau_3=0} \int_{\xi_1+\xi_2+\xi_3=0} \widetilde{f_1}(\xi_1,\tau_1)\widetilde{g_1}(\xi_2,\tau_2)|c(\xi_3,\tau_3)| d\xi_j d\tau_j$$

$$\leq \sup_{\|c\|_{L^2_{\tau,\xi}}=1} \int_{\tau_1+\tau_2+\tau_3=0} \int_{\xi_1+\xi_2+\xi_3=0} \widetilde{F_1}(\xi_1,\tau_1)\widetilde{g_1}(\xi_2,\tau_2)|c(\xi_3,\tau_3)| d\xi_j d\tau_j +$$

$$\sup_{\|c\|_{L^2_{\tau,\xi}}=1} \int_{\tau_1+\tau_2+\tau_3=0} \int_{\xi_1+\xi_2+\xi_3=0} \widetilde{F_2}(\xi_1,\tau_1)\widetilde{g_1}(\xi_2,\tau_2)|c(\xi_3,\tau_3)| d\xi_j d\tau_j$$

Since $\widetilde{F_1}, \widetilde{F_2}, \widetilde{v_1} \geq 0$, it follows that the latter expression is $\sim \|F_1 g_1\|_{L^2_{t,x}} + \|F_2 g_1\|_{L^2_{t,x}}$. Hence, it follows that:

$$\|f_1 g_1\|_{L^2_{t,x}} \lesssim \|F_1 g_1\|_{L^2_{t,x}} + \|F_2 g_1\|_{L^2_{t,x}}.$$

By Proposition 3.2.2, by the frequency assumptions on F_1 and v_1, and by the fact that taking absolute values in the spacetime Fourier transform doesn't change the $X^{s,b}$ norms, we know that:

$$\|F_1 g_1\|_{L^2_{t,x}} \lesssim \frac{1}{N_1^{\frac{1}{2}}} \|af\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}}$$

We now use (3.20) to deduce that this expression is:

$$\lesssim \frac{1}{N_1^{\frac{1}{2}}} (N^{0+} \|f\|_{X^{0,\frac{1}{2}+}}) \|g\|_{X^{0,\frac{1}{2}+}}$$

This expression is:

$$\lesssim \frac{1}{N_1^{\frac{1}{2}-}} \|f\|_{X^{0,\frac{1}{2}+}} \|g\|_{X^{0,\frac{1}{2}+}} \tag{3.24}$$

On the other hand, let us consider $c \in L^2_{\tau,\xi}$. With notation as before, one has:

$$|\int_{\tau_1+\tau_2=0}\int_{\xi_1+\xi_2=0}(F_2 g_1)\widetilde{\ }(\xi_1,\tau_1)c(\xi_2,\tau_2)d\xi_j d\tau_j\Big|$$
$$= |\int_{\tau_1+\tau_2+\tau_3=0}\int_{\xi_1+\xi_2+\xi_3=0}|(bf)\widetilde{\ }(\xi_1,\tau_1)|\widetilde{g_1}(\xi_2,\tau_2)c(\xi_3,\tau_3)d\xi_j d\tau_j\Big|$$

$$\leq \int_{\tau_0+\tau_1+\tau_2+\tau_3=0}\int_{\xi_1+\xi_2+\xi_3=0}|\widehat{b}(\tau_0)||\widetilde{f}(\xi_1,\tau_1)||\widetilde{g_1}(\xi_2,\tau_2)||c(\xi_3,\tau_3)|d\xi_j d\tau_j := I$$

We then define the functions $G_j, j=1,\ldots,3$ by:

$$\widetilde{G_1} := |\widetilde{f}|, \widetilde{G_2} := |\widetilde{g_1}|, \widetilde{G_3} := |c|$$

Recalling (3.21), and using Parseval's identity, it follows that:

$$I \lesssim \int_{\mathbb{R}\times\mathbb{R}} b_1(t)G_1(x,t)G_2(x,t)G_3(x,t)dxdt$$

We choose $M \in (1,\infty)$, and $2+$ such that: $\frac{1}{M}+\frac{1}{2+}=\frac{1}{2}$. By an $L^M_t, L^{2+}_t L^2_x, L^2_{t,x}$ Hölder inequality, we deduce that:

$$I \lesssim \|b_1\|_{L^M_t}\|G_1G_2\|_{L^{2+}_t L^2_x}\|G_3\|_{L^2_{t,x}}$$

We use (3.22), Corollary 3.2.3, and Plancherel's theorem to deduce that:

$$I \lesssim \frac{1}{N_1^{\frac{1}{2}-}}\|f\|_{X^{0,\frac{1}{2}+}}\|g\|_{X^{0,\frac{1}{2}+}}\|c\|_{L^2_{\tau,\xi}}.$$

By duality and by Plancherel's theorem, it follows that:

$$\|F_2 v_1\|_{L^2_{t,x}} \lesssim \frac{1}{N_1^{\frac{1}{2}-}}\|f\|_{X^{0,\frac{1}{2}+}}\|g\|_{X^{0,\frac{1}{2}+}} \tag{3.25}$$

The case when $\widetilde{f_1} := |\widetilde{f}|$, $\widetilde{g_1} := |(\chi g)\widetilde{\ }|$ is treated analogously. The Proposition now follows from (3.24) and (3.25).

□

Finally, let us recall the following Calculus fact, which is often referred to as the *Double Mean Value Theorem*:

Proposition 3.2.5. *Let $f \in C^2(\mathbb{R})$. Suppose that $x, \eta, \mu \in \mathbb{R}$ are such that $|\eta|, |\mu| \ll |x|$. Then, one has:*

$$|f(x+\eta+\mu) - f(x+\eta) - f(x+\mu) + f(x)| \lesssim |\eta||\mu||f''(x)|. \tag{3.26}$$

The proof of Proposition 3.2.5 follows from the standard Mean Value Theorem.

3.3 The cubic nonlinear Schrödinger equation

3.3.1 Basic facts about the equation

The equation (3.1) has conserved mass and energy:

$$M(u(t)) := \int_{\mathbb{R}} |u(x,t)|^2 dx \text{ (Mass)} \tag{3.27}$$

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}} |\nabla u(x,t)|^2 dx + \frac{1}{4}\int_{\mathbb{R}} |u(x,t)|^4 dx \text{ (Energy)} \tag{3.28}$$

The following local-in-time bound will be useful:

Proposition 3.3.1. *Suppose that u is a global solution of (3.1). Then, there exist $\delta = \delta(s, Energy, Mass), C = C(s, Energy, Mass)$ such that, for all $t_0 \in \mathbb{R}$, there exists $v \in X^{s,\frac{1}{2}+}$ satisfying the following properties:*

$$v|_{[t_0,t_0+\delta]} = u|_{[t_0,t_0+\delta]}, \tag{3.29}$$

$$\|v\|_{X^{s,\frac{1}{2}+}} \leq C\|u(t_0)\|_{H^s}, \tag{3.30}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C. \tag{3.31}$$

Furthermore, δ and C can be chosen to depend continuously on energy and mass.

The proof of Proposition 3.3.1 proceeds by an appropriate fixed-point method and is analogous to the proof of Proposition 2.3.1 in Chapter 2. Furthermore, from the proof, it follows that δ and C depend continuously on energy and mass. We refer to the mentioned proofs for details.

In the proof of the fact that F_s, as in the statement of Theorem 3.1.1, depends continuously on the initial data, w.r.t. the H^s topology, we will use the following:

Proposition 3.3.2. *(Continuity of conserved quantities) Suppose n is a positive integer. Let E_n denote the conserved quantity of (3.1), which, together with lower-order conserved quantities, we use to bound the H^n norm of a solution. Then E_n is continuous on H^n. Moreover, one can construct a function $B_n : H^n \to \mathbb{R}$ that satisfies (3.3) and is continuous on H^n.*

The proof of Proposition 3.3.2 is given in Appendix A of this chapter.

Although we are starting with initial data Φ, which we are only assuming belongs to H^s, and hence with solutions of (3.1), which we only know belong to H^s, our calculations will require us to work with solutions which have more regularity. Hence, we will have to approximate our solutions to (3.1) with smooth ones, and argue by density. The density argument is made precise by the following result:

Proposition 3.3.3. *Suppose u satisfies (3.1) with initial data $\Phi \in H^s$, and suppose each element of $(u^{(n)})$ satisfies (3.1) with initial data Φ_n, where $\Phi_n \in \mathcal{S}(\mathbb{R})$ and $\Phi_n \xrightarrow{H^s} \Phi$. Then, one has for all t:*

$$u^{(n)}(t) \xrightarrow{H^s} u(t).$$

The proof of Proposition 3.3.3 is analogous to the proof of Proposition 2.3.3 from Chapter 2. The proof is very similar, so it will be omitted.

Proposition 3.3.3 allows us to work with smooth solutions and pass to the limit in the end. Namely, we note that if we take initial data Φ_n as earlier, then, by persistence of regularity, $u^{(n)}(t)$ will belong to $H^{\infty}(\mathbb{R})$ for all t. If we knew that Theorem 3.1.1 were true for smooth solutions, we would obtain, for all $n \in \mathbb{N}$, and for all $t \in \mathbb{R}$:

$$\|u^{(n)}(t)\|_{H^s} \leq F_s(\Phi_n)(1+|t|)^{\alpha+}.$$

By letting $n \to \infty$, and using Proposition 3.3.2 and the continuity of F_s on H^s, it would follow that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq F_s(\Phi)(1+|t|)^{\alpha+}.$$

We may henceforth work with $\Phi \in \mathcal{S}(\mathbb{R})$, which implies that $u(t) \in H^{\infty}(\mathbb{R})$ for all t. The claimed result is then deduced from this special case by the approximation procedure given earlier. We will make the same assumption in our study of the Hartree equation.

3.3.2 An Iteration bound and Proof of Theorem 3.1.1

Let u denote the unique global solution to (3.1). From the previous arguments, we know that we can assume WLOG that for all $t \in \mathbb{R}$, $u(t) \in H^{\infty}(\mathbb{R})$. Our aim now is to use uniform bounds on $\|u(t)\|_{H^k}$ coming from (3.3) to deduce bounds on $\|u(t)\|_{H^s}$. The key is to perform a frequency decomposition, similarly as in [18].

Let $N > 1$ be a parameter which we will determine later. We define the operator Q by:

$$\widehat{Qf}(\xi) := \chi_{|\xi| \geq N} \hat{f}(\xi). \tag{3.32}$$

We write $s = k + \alpha$, for $k \in \mathbb{N}$, $\alpha \in (0,1)$. Using the definition (3.32) and (3.3), it follows that:

$$\|(I-Q)u(t)\|_{H^s} \lesssim N^{\alpha}\|(I-Q)u(t)\|_{H^k} \lesssim$$

$$\lesssim N^{\alpha}\|u(t)\|_{H^k} \leq N^{\alpha}B_k(\Phi). \tag{3.33}$$

We will use (3.33) to estimate the *low-frequency part* of the solution.

The key now is to estimate the *high-frequency part* of the solution. This is done by the following iteration bound:

Proposition 3.3.4. *Let $\delta = \delta(\Phi) > 0$ be as in Proposition 3.3.1. Then, there exists a continuous function $C : H^1 \to \mathbb{R}$ such that for all $t_0 \in \mathbb{R}$, one has:*

$$\|Qu(t_0+\delta)\|_{H^s}^2 - \|Qu(t_0)\|_{H^s}^2 \leq \frac{C(\Phi)}{N^{1-}}\|u(t_0)\|_{H^s}^2.$$

Before we prove Proposition 3.3.4, let us note how it implies Theorem 3.1.1.

Proof. (of Theorem 3.1.1 assuming Proposition 3.3.4)

Let us fix $t_0 \in \mathbb{R}$. It follows that:

$$\|Qu(t_0+\delta)\|_{H^s}^2 \leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)\|Qu(t_0)\|_{H^s}^2 + \frac{C(\Phi)}{N^{1-}}\|(I-Q)u(t_0)\|_{H^s}^2.$$

By (3.33), it follows that:

$$\frac{C(\Phi)}{N^{1-}}\|(I-Q)u(t_0)\|_{H^s}^2 \lesssim \frac{C(\Phi)}{N^{1-}}N^{2\alpha}B_k^2(\Phi) =: K(N,\Phi). \tag{3.34}$$

If we multiply K by an appropriate constant, we can write, for all $t_0 \in \mathbb{R}$:

$$\|Qu(t_0+\delta)\|_{H^s}^2 \leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)\|Qu(t_0)\|_{H^s}^2 + K(N,\Phi). \tag{3.35}$$

Given $n \in \mathbb{N}$, we take $t_0 = 0, \delta, 2\delta, \ldots, n\delta$ and apply (3.35) to deduce the inequalities:

$$
\begin{aligned}
\|Qu(\delta)\|_{H^s}^2 &\leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)\|Qu(0)\|_{H^s}^2 + K(N,\Phi)\\
\|Qu(2\delta)\|_{H^s}^2 &\leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)\|Qu(\delta)\|_{H^s}^2 + K(N,\Phi)\\
&\vdots\\
\|Qu((n-1)\delta)\|_{H^s}^2 &\leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)\|Qu((n-2)\delta)\|_{H^s}^2 + K(N,\Phi)\\
\|Qu(n\delta)\|_{H^s}^2 &\leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)\|Qu((n-1)\delta)\|_{H^s}^2 + K(N,\Phi)
\end{aligned}
$$

Let $\gamma := 1 + \frac{C(\Phi)}{N^{1-}}$. Let us multiply the first inequality by γ^{n-1}, the second inequality by $\gamma^{n-2}, \ldots$, and the $(n-1)$-st inequality by γ. We then sum to obtain:

$$
\|Qu(n\delta)\|_{H^s}^2 \leq \big(1+\frac{C(\Phi)}{N^{1-}}\big)^n\|Qu(0)\|_{H^s}^2 + K(N,\Phi)(1+\gamma+\cdots+\gamma^{n-1}). \tag{3.36}
$$

Let us now consider n such that $n \lesssim N^{1-}$. For such an n, we have:

$$
(1+\frac{C(\Phi)}{N^{1-}})^n = O(R_1(\Phi)). \tag{3.37}
$$

and hence:

$$
1+\gamma+\cdots+\gamma^{n-1} = \frac{\gamma^n-1}{\gamma-1} = \frac{\big(1+\frac{C(\Phi)}{N^{1-}}\big)^n-1}{\big(1+\frac{C(\Phi)}{N^{1-}}\big)-1} =
$$

$$
= \frac{\big(1+\frac{C(\Phi)}{N^{1-}}\big)^n-1}{\frac{C(\Phi)}{N^{1-}}} = O(N^{1-}R_2(\Phi)). \tag{3.38}
$$

We can take the functions $R_1, R_2 : H^1 \to \mathbb{R}$ to be continuous. If we then combine (3.34),(3.37),(3.38) with (3.36), it follows that:

$$
\|Qu(n\delta)\|_{H^s}^2 \lesssim R_1(\Phi)\|Q\Phi\|_{H^s}^2 + R_2(\Phi)N^{2\alpha}B_k^2(\Phi).
$$

Hence, by continuity properties of B_k coming from from Proposition 3.3.2, and by the construction of R_1, R_2, we can find a continuous function $R_3 : H^s \to \mathbb{R}$ such that for all $n \lesssim N^{1-}$:

$$\|Qu(n\delta)\|_{H^s} \leq R_3(\Phi)(1 + N^{\alpha}). \tag{3.39}$$

Combining (3.33) and (3.39), we deduce that there exists a continuous function $R_4 : H^s \to \mathbb{R}$, such that for all $n \lesssim N$, one has:

$$\|u(n\delta)\|_{H^s} \leq R_4(\Phi)(1 + N^{\alpha}).$$

Finally, by using appropriate local-in-time bounds on each of the n intervals of size δ, it follows that there exists a continuous function $R : H^s \to \mathbb{R}$ such that, for all $T \lesssim N^{1-}\delta$, one has:

$$\|u(T)\|_{H^s} \leq R(\Phi)(1 + N)^{\alpha}. \tag{3.40}$$

Let us now take:

$$T \sim N^{1-}\delta.$$

Then:

$$N \sim \left(\frac{T}{\delta}\right) + .$$

(This is the step in which we choose the parameter N.)

Consequently, since $\delta = \delta(\Phi) > 0$ is a continuous function on H^1, it follows that there exists a continuous function F_s on H^s such that for $T > 1$:

$$\|u(T)\|_{H^s} \leq F_s(\Phi)(1 + T)^{\alpha+}. \tag{3.41}$$

From local well-posedness, we get the same bound for times in $[0, 1]$. By time reversibility, we also get the bound for negative times. Theorem 3.1.1 now follows.

□

Let us now prove Proposition 3.3.4

Proof. We know that: $u_t = i\Delta u - i|u|^2 u$. Hence, we compute:

$$\frac{d}{dt}\|Qu(t)\|_{H^s}^2 = \frac{d}{dt}\langle D^s Qu(t), D^s Qu(t)\rangle = 2Re\langle D^s Qu, D^s Qu_t\rangle =$$

$$= 2Re\langle D^s Qu, D^s u_t\rangle = 2Re\langle D^s Qu, iD^s\Delta u\rangle - 2Re\langle D^s Qu, iD^s(|u|^2u)\rangle =$$

$$= -2\, Im\langle D^s Qu, D^s(|u|^2u)\rangle. \tag{3.42}$$

We note that in the third equality, we used Parseval's identity and the definition of Q to omit the operator Q in the second factor, and in the fifth equality, we argued similarly and used the fact that

$$\langle D^s Qu, D^s\Delta u\rangle = \langle D^s Qu, D^s\Delta Qu\rangle \in \mathbb{R}.$$

It is important to remark that this quantity is indeed finite since $u(t) \in H^\infty$. This is what allows us to differentiate in time and use the previous formulae.

Hence, if we fix $t_0 \in \mathbb{R}$, we obtain:

$$\|Qu(t_0+\delta)\|_{H^s}^2 - \|Qu(t_0)\|_{H^s}^2 = \int_{t_0}^{t_0+\delta} \frac{d}{dt}\|Qu(t)\|_{H^s}^2 =$$

$$= -\int_{t_0}^{t_0+\delta} 2Im\langle D^s Qu, D^s(|u|^2u)\rangle dt.$$

Thus, it suffices to estimate:

$$\left|\int_{t_0}^{t_0+\delta} \langle D^s Qu, D^s(|u|^2u)\rangle dt\,\right|.$$

Let v be the function we obtain by Proposition 3.3.1, if we are considering the time t_0 we fixed earlier. For the $\delta > 0$, which we obtain by Proposition 3.3.1, we denote:

$$\chi(t) := \chi_{[t_0,t_0+\delta]}(t).$$

Then:

$$\int_{t_0}^{t_0+\delta} \langle D^s Qu, D^s(|u|^2 u)\rangle dt = \int_{t_0}^{t_0+\delta} \langle D^s Qv, D^s(|v|^2 v)\rangle dt =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi(t) D^s Qv D^s(\bar{v} v \bar{v}) dx dt.$$

With notation as in Section 2 for dyadic integers N_1, N_2, N_3, N_4, we define:

$$I_{N_1,N_2,N_3,N_4} := \int_{\mathbb{R}} \int_{\mathbb{R}} \chi(t) D^s Qv_{N_1} D^s(\overline{v_{N_2}} v_{N_3} \overline{v_{N_4}}) dx dt.$$

By definition of Q and the fact that $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, |\xi_j| \sim N_j$, we deduce that I_{N_1,N_2,N_3,N_4} is zero unless the following conditions hold:

$$N_1 \gtrsim N. \tag{3.43}$$

$$max\,\{N_2, N_3, N_4\} \gtrsim N_1. \tag{3.44}$$

By Parseval's identity, the expression I_{N_1,N_2,N_3,N_4} is:

$$\sim \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} (\chi(t) D^s Qv_{N_1})\tilde{}(\xi_1,\tau_1) \langle \xi_2\rangle^s (\overline{v_{N_2}} v_{N_3} \overline{v_{N_4}})\tilde{}(\xi_2,\tau_2)\, d\xi_j d\tau_j =$$

$$= \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} (\chi(t) D^s Qv_{N_1})\tilde{}(\xi_1,\tau_1)$$
$$\langle \xi_2+\xi_3+\xi_4\rangle^s (\overline{v_{N_2}})\tilde{}(\xi_2,\tau_2)\, \widetilde{v_{N_3}}(\xi_3,\tau_3)\, (\overline{v_{N_4}})\tilde{}(\xi_4,\tau_4)\, d\xi_j d\tau_j$$

So, by the triangle inequality:

$$|I_{N_1,N_2,N_3,N_4}| \lesssim$$

$$\int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} |(\chi(t)D^s Q v_{N_1})^{\sim}(\xi_1,\tau_1)| \, \langle \xi_2+\xi_3+\xi_4 \rangle^s \, |(\overline{v_{N_2}})^{\sim}(\xi_2,\tau_2)|$$

$$|\widetilde{v_{N_3}}(\xi_3,\tau_3)| \, |(\overline{v_{N_4}})^{\sim}(\xi_4,\tau_4)| \, d\xi_j d\tau_j.$$

We now use a *"Fractional Leibniz Rule"*, i.e. we note that:

$$\langle \xi_2+\xi_3+\xi_4 \rangle^s \lesssim \langle \xi_2 \rangle^s + \langle \xi_3 \rangle^s + \langle \xi_4 \rangle^s.$$

Hence, by symmetry [1], it suffices to estimate:

$$J_{N_1,N_2,N_3,N_4} :=$$

$$\int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} |(\chi(t)D^s Q v_{N_1})^{\sim}(\xi_1,\tau_1)|$$

$$(\langle \xi_2 \rangle^s |(\overline{v_{N_2}})^{\sim}(\xi_2,\tau_2)|) \, |\widetilde{v_{N_3}}(\xi_3,\tau_3)| \, |(\overline{v_{N_4}})^{\sim}(\xi_4,\tau_4)| \, d\xi_j d\tau_j \sim$$

$$\sim \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} |(\chi(t)D^s Q v_{N_1})^{\sim}(\xi_1,\tau_1)|$$

$$|(\overline{D^s v_{N_2}})^{\sim}(\xi_2,\tau_2)| \, |\widetilde{v_{N_3}}(\xi_3,\tau_3)| \, |(\overline{v_{N_4}})^{\sim}(\xi_4,\tau_4)| \, d\xi_j d\tau_j.$$

Let us define:

$$F_1(x,t) := \int_{\mathbb{R}} \int_{\mathbb{R}} |(\chi(t)D^s Q v_{N_1})^{\sim}(\xi,\tau)| e^{i(x\xi+t\tau)} d\xi d\tau. \tag{3.45}$$

$$F_2(x,t) := \int_{\mathbb{R}} \int_{\mathbb{R}} |(D^s v_{N_2})^{\sim}(\xi,\tau)| e^{i(x\xi+t\tau)} d\xi d\tau. \tag{3.46}$$

$$F_j(x,t) := \int_{\mathbb{R}} \int_{\mathbb{R}} |\widetilde{v_{N_j}}(\xi,\tau)| e^{i(x\xi+t\tau)} d\xi d\tau, \text{ for } j=3,4. \tag{3.47}$$

[1] From the argument that follows, we see that the two other terms are estimated analogously. The fact that the D^s falls on a term with or without a complex conjugate doesn't matter in the argument.

Hence, by Parseval's identity, since all the $\widetilde{F_j}$ are real-valued, we obtain:

$$J_{N_1,N_2,N_3,N_4} \sim \int_{\mathbb{R}} \int_{\mathbb{R}} F_1 \overline{F_2} F_3 \overline{F_4} dx dt. \tag{3.48}$$

We consider the following Cases:

Case 1: $max\,\{N_2, N_3, N_4\} = N_3$ or $max\,\{N_2, N_3, N_4\} = N_4$.

Let us WLOG suppose that $max\,\{N_2, N_3, N_4\} = N_3$, since the case $max\,\{N_2, N_3, N_4\} = N_4$ is analogous. Here:

$|J_{N_1,N_2,N_3,N_4}| = J_{N_1,N_2,N_3,N_4}$, which is by (3.48) and by an $L^4_{t,x}, L^4_{t,x}, L^4_{t,x}, L^4_{t,x}$ Hölder's inequality:

$$\lesssim \|F_1\|_{L^4_{t,x}} \|\overline{F_2}\|_{L^4_{t,x}} \|F_3\|_{L^4_{t,x}} \|\overline{F_4}\|_{L^4_{t,x}} =$$

$$= \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^4_{t,x}} \|F_3\|_{L^4_{t,x}} \|F_4\|_{L^4_{t,x}},$$

which by using (3.14) is:

$$\lesssim \|F_1\|_{X^{0,\frac{3}{8}+}} \|F_2\|_{X^{0,\frac{3}{8}+}} \|F_3\|_{X^{0,\frac{3}{8}+}} \|F_4\|_{X^{0,\frac{3}{8}+}}$$

By definition of the functions F_j, and by the fact that taking absolute values in the spacetime Fourier transform doesn't change the $X^{s,b}$ norm, it follows that the previous expression is:

$$\sim \|\chi(t) D^s Q v_{N_1}\|_{X^{0,\frac{3}{8}+}} \|D^s v_{N_2}\|_{X^{0,\frac{3}{8}+}} \|v_{N_3}\|_{X^{0,\frac{3}{8}+}} \|v_{N_4}\|_{X^{0,\frac{3}{8}+}}$$

From Lemma 3.2.1 and the fact that $\frac{3}{8}+ < \frac{1}{2}$, this expression is:

$$\lesssim \|D^s Q v_{N_1}\|_{X^{0,\frac{3}{8}++}} \|D^s v_{N_2}\|_{X^{0,\frac{3}{8}+}} \|v_{N_3}\|_{X^{0,\frac{3}{8}+}} \|v_{N_4}\|_{X^{0,\frac{3}{8}+}} \lesssim$$

$$\lesssim \|D^s Q v_{N_1}\|_{X^{0,\frac{3}{8}++}} \|D^s v_{N_2}\|_{X^{0,\frac{3}{8}+}} \frac{1}{N_3} \|v_{N_3}\|_{X^{1,\frac{3}{8}+}} \|v_{N_4}\|_{X^{0,\frac{3}{8}+}} \lesssim$$

$$\lesssim \frac{1}{N_3} \|v\|^2_{X^{s,\frac{1}{2}+}} \|v\|^2_{X^{1,\frac{1}{2}+}}$$

From Proposition 3.3.1, we bound this by:

$$\leq \frac{C(\Phi)}{N_3} \|u(t_0)\|^2_{H^s}. \tag{3.49}$$

Let us observe that in this case, we have:

$$N_3 \geq N_2, N_4 \text{ and } N_3 \gtrsim N_1 \gtrsim N.$$

Hence, we have obtained a favorable decay factor of $\frac{1}{N_3}$.

Case 2: $max\{N_2, N_3, N_4\} = N_2$.

Subcase 1: $N_2 \gg N_3, N_4$.

Since $\sum \xi_j = 0$ and $|\xi_j| \sim N_j$, it follows that $N_1 \sim N_2$. Hence:

$$N_1 \sim N_2 \gtrsim N \text{ and } N_1 \sim N_2 \gg N_3, N_4. \tag{3.50}$$

In this subcase, we will have to argue a little bit harder. The main tools that we will use will be the *improved Strichartz estimate* Proposition 3.2.2, and its modification, Proposition 5.2.5.

We use (3.48) and an $L^2_{t,x}, L^2_{t,x}$ Hölder inequality to deduce that:

$$|J_{N_1,N_2,N_3,N_4}| \lesssim \|F_1 F_3\|_{L^2_{t,x}} \|F_2 F_4\|_{L^2_{t,x}}$$

By the assumption on the frequencies, (3.45), (3.46), (3.47), Proposition 5.2.5 and Proposition 3.2.2, this expression is:

$$\lesssim \Big(\frac{1}{N_1^{\frac{1}{2}-}}\|D^s Q v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|v_{N_3}\|_{X^{0,\frac{1}{2}+}}\Big)\Big(\frac{1}{N_2^{\frac{1}{2}}}\|D^s v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{0,\frac{1}{2}+}}\Big).$$

By (3.50), this expression is:

$$\lesssim \frac{1}{N_2^{1-}}\|v\|^2_{X^{s,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}}.$$

We now use Proposition 3.3.1 to deduce that in Subcase 1, one has:

$$|J_{N_1,N_2,N_3,N_4}| \leq \frac{C(\Phi)}{N_2^{1-}}\|u(t_0)\|^2_{H^s}. \tag{3.51}$$

By (3.50), we notice that in this Subcase $\frac{1}{N_2^{1-}}$ is again a favorable decay factor

Subcase 2: $N_2 \sim N_3 \gtrsim N_4$ or $N_2 \sim N_4 \gtrsim N_3$.

Let us consider WLOG the case when $N_2 \sim N_3 \gtrsim N_4$, since the case $N_2 \sim N_4 \gtrsim N_3$ is analogous. By the same argument as in Case 1, it follows that:

$$|J_{N_1,N_2,N_3,N_4|} \lesssim \frac{1}{N_3}\|v\|^2_{X^{s,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}}.$$

Since $N_3 \sim N_2$, it follows that:

$$|J_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_2}\|v\|^2_{X^{s,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}} \leq$$

$$\leq \frac{C(\Phi)}{N_2}\|u(t_0)\|^2_{H^s}. \tag{3.52}$$

In this Subcase, $\frac{1}{N_2}$ is an acceptable decay factor.

Combining (3.51) and (3.52), it follows that in Case 2, one has the bound:

$$|J_{N_1,N_2,N_3,N_4}| \leq \frac{C(\Phi)}{N_2^{1-}}\|u(t_0)\|^2_{H^s}. \tag{3.53}$$

We now combine (3.49), (3.53) and sum in the dyadic integers N_j, keeping in mind the assumptions (3.43), (3.44), and the assumptions of each case. It follows that:

$$|\sum_{N_j} J_{N_1,N_2,N_3,N_4}| \leq \frac{C(\Phi)}{N^{1-}} \|u(t_0)\|_{H^s}^2.$$

Hence, by construction of J_{N_1,N_2,N_3,N_4}, we deduce:

$$|\sum_{N_j} I_{N_1,N_2,N_3,N_4}| \leq \frac{C(\Phi)}{N^{1-}} \|u(t_0)\|_{H^s}^2.$$

The fact that $C(\Phi)$ depends continuously on Φ w.r.t the H^1 topology follows from Proposition 3.3.1, as well as the same continuous dependence of δ, energy, mass, and the uniform bound on the H^1 norm of u. Proposition 3.3.4 now follows. □

3.4 The Hartree equation

3.4.1 Definition of the $\mathcal{D}$ operator

As in the case of the cubic NLS, we will take $\Phi \in \mathcal{S}(\mathbb{R})$ in order to rigorously justify all of our calculations. The general claim follows by density and the Approximation Lemma, i.e. Proposition 3.3.3 applied to (3.2).

The same iteration argument that we used for the cubic equation doesn't work for (3.2), since the only conserved quantities that we have at our disposal are mass and energy. We now adapt to the non-periodic setting the *upside-down I-method* approach that we used on S^1 in Chapter 2.

We first define $\theta_0 : \mathbb{R} \rightarrow \mathbb{R}$ by:

$$\theta_0(\xi) := \begin{cases} |\xi|^s, \text{if } |\xi| \geq 2 \\ 1, \text{ if } |\xi| \leq 1. \end{cases} \tag{3.54}$$

We extend θ_0 for $1 \leq |\xi| \leq 2$ such that θ_0 is even, smooth on $\mathbb{R}$, and such that it is non-decreasing on $[0, +\infty)$. By construction, we then obtain:

$$|\theta_0'(\xi)| \lesssim \frac{|\theta_0(\xi)|}{|\xi|} \tag{3.55}$$

$$|\theta_0''(\xi)| \lesssim \frac{|\theta_0(\xi)|}{|\xi|^2} \tag{3.56}$$

$$\theta_0(x+y) \lesssim \theta_0(x) + \theta_0(y). \tag{3.57}$$

Suppose now that $N > 1$ is given. Then, we define:

$$\theta(\xi) := \theta_0(\frac{\xi}{N}).$$

Hence:

$$\theta(\xi) := \begin{cases} \left(\frac{|\xi|}{N}\right)^s, \text{if } |\xi| \geq 2N \\ 1, \text{ if } |\xi| \leq N. \end{cases} \tag{3.58}$$

From (3.55),(3.56), and (3.57), we obtain:

$$|\theta'(\xi)| \lesssim \frac{|\theta(\xi)|}{|\xi|} \tag{3.59}$$

$$|\theta''(\xi)| \lesssim \frac{|\theta(\xi)|}{|\xi|^2} \tag{3.60}$$

$$\theta(x+y) \lesssim \theta(x) + \theta(y). \tag{3.61}$$

Having defined θ, we define the $\mathcal{D}$-operator by:

$$\widehat{\mathcal{D}f}(\xi) := \theta(\xi)\widehat{f}(\xi). \tag{3.62}$$

One then has the bound:

$$\|\mathcal{D}f\|_{L^2} \lesssim \|f\|_{H^s} \lesssim N^s \|\mathcal{D}f\|_{L^2}. \tag{3.63}$$

Let u denote the global solution of (3.2). We then have the following result:

Proposition 3.4.1. *Given $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ satisfying the properties:*

$$v|_{[t_0,t_0+\delta]} = u|_{[t_0,t_0+\delta]}. \tag{3.64}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)). \tag{3.65}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{3.66}$$

Moreover, δ and C can be chosen to depend continuously on the energy and mass.

The proof of Proposition 3.4.1 is analogous to the proof of Propositions 2.3.1 and 2.4.1 from Chapter 2 and it will be omitted.

The point is that all the intermediate estimates that hold in the periodic setting carry over to the non-periodic setting. Since $V \in L^1(\mathbb{R})$, we know that $\widehat{V} \in L^\infty(\mathbb{R})$, so one can directly modify the proof for the cubic NLS to the Hartree equation as before.

3.4.2 An Iteration bound and proof of Theorem 3.1.2

As in the periodic case, let:

$$E^1(u(t)) := \|\mathcal{D}u(t)\|_{L^2}^2.$$

Then, arguing as in Chapter 2, we obtain, that for some $c \in \mathbb{R}$:

$$\frac{d}{dt}E^1(u(t)) = ci\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2)$$

$$\widehat{V}(\xi_3+\xi_4)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j. \tag{3.67}$$

Recalling the notation from the Introduction, as in Chapter 2, we consider the following *higher modified energy*

$$E^2(u) := E^1(u) + \lambda_4(M_4; u). \tag{3.68}$$

The quantity M_4 will be determined soon.

The modified energy E^2 is obtained by adding a "multilinear correction" to the modified energy E^1 considered earlier. In order to find $\frac{d}{dt}E^2(u)$, we need to find $\frac{d}{dt}\lambda_4(M_4; u)$. Thus, if we fix a multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4; u) =$$

$$= -i\lambda_4(M_4(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2); u)$$

$$-i\int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0}\big[M_4(\xi_{123},\xi_4,\xi_5,\xi_6)\widehat{V}(\xi_1+\xi_2)$$

$$-M_4(\xi_1,\xi_{234},\xi_5,\xi_6)\widehat{V}(\xi_2+\xi_3) + M_4(\xi_1,\xi_2,\xi_{345},\xi_6)\widehat{V}(\xi_3+\xi_4)$$

$$-M_4(\xi_1,\xi_2,\xi_3,\xi_{456})\widehat{V}(\xi_4+\xi_5)\big]\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)\widehat{u}(\xi_5)\widehat{\bar{u}}(\xi_6)d\xi_j \tag{3.69}$$

With the setup (3.67) and (3.69), we can use *higher modified energies* as in in the periodic setting. Namely, it follows that if we take:

$$M_4 := \Psi. \tag{3.70}$$

where Ψ is defined by:

$$\Psi : \Gamma_4 \to \mathbb{R}$$

$$\Psi := \begin{cases} c\frac{(\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2\widehat{V}(\xi_3+\xi_4)}{\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2}, \text{ if } \xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2 \neq 0 \\ 0, \text{ otherwise.} \end{cases} \tag{3.71}$$

for an appropriate real constant c. One then has:

$$\frac{d}{dt}E^2(u) = -i\lambda_6(M_6;u). \tag{3.72}$$

where:

$$M_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) := M_4(\xi_{123},\xi_4,\xi_5,\xi_6)\widehat{V}(\xi_1+\xi_2)$$

$$-M_4(\xi_1,\xi_{234},\xi_5,\xi_6)\widehat{V}(\xi_2+\xi_3) + M_4(\xi_1,\xi_2,\xi_{345},\xi_6)\widehat{V}(\xi_3+\xi_4)$$

$$-M_4(\xi_1,\xi_2,\xi_3,\xi_{456})\widehat{V}(\xi_4+\xi_5). \tag{3.73}$$

The key to continue our study of $E^2(u)$ is to deduce pointwise bounds on Ψ. We dyadically localize the frequencies as $|\xi_j| \sim N_j$. We then order the N_j^s in decreasing order to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. Let us show that the following result holds:

Proposition 3.4.2. *(Pointwise bound on the multiplier) Under the previous assump-*

tions, one has:

$$\text{If } N_2^* \gg N_3^*,\ \Psi = O(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)). \tag{3.74}$$

$$\text{If } N_2^* \sim N_3^*,\ \Psi = O(\frac{1}{(N_1^*)^3}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*). \tag{3.75}$$

In the proof of Proposition 3.4.2, the following bound will be useful:

Lemma 3.4.3. *Suppose that $|x| \geq |y|$. Then, one has:*

$$|(\theta(x))^2 - (\theta(y))^2| \lesssim (|x| - |y|)\frac{(\theta(x))^2}{|x|}.$$

We prove Proposition 3.4.2 and Lemma 3.4.3 in Appendix B of this chapter.

Using Proposition 3.4.2 and arguing as in Chapter 2, we deduce that, whenever u is a global solution of (3.2), one has:

$$E^2(u) \sim E^1(u). \tag{3.76}$$

Arguing as in Chapter 2, the key is to deduce the following bound:

Lemma 3.4.4. *For all $t_0 \in \mathbb{R}$, one has:*

$$|E^2(u(t_0+\delta)) - E^2(u(t_0))| \lesssim \frac{1}{N^{3-}}E^2(u(t_0)).$$

We see that Theorem 3.1.2 follows from Lemma 3.4.4:

Proof. (of Theorem 3.1.2 assuming Lemma 3.4.4)

By Lemma 3.4.4, there exists $C > 0$ such that for all $t_0 \in \mathbb{R}$, one has:

$$E^2(u(t_0+\delta)) \leq (1 + \frac{C}{N^{3-}})E^2(u(t_0)). \tag{3.77}$$

Using (3.77) iteratively, we obtain that [2] $\forall T > 1$:

[2]Strictly speaking, we are using (3.66) to deduce that we can get the bound for all such times, and not just those which are a multiple of δ.

$$E^2(u(T)) \leq (1 + \frac{C}{N^{3-}})^{\lceil \frac{T}{\delta} \rceil} E^2(\Phi).$$

Let us take:

$$T \sim N^{3-}. \tag{3.78}$$

For such a choice of T, one has:

$$E^2(u(T)) \lesssim E^2(\Phi). \tag{3.79}$$

Using (3.63), and (3.76), it follows that:

$$\|u(T)\|_{H^s} \lesssim N^s E^2(u(T)) \lesssim N^s E^2(\Phi) \lesssim N^s \|\Phi\|_{H^s}.$$

$$\lesssim T^{\frac{s}{3}+} \|\Phi\|_{H^s} \lesssim (1+T)^{\frac{s}{3}+} \|\Phi\|_{H^s}. \tag{3.80}$$

Since for times $t \in [0,1]$, we get the bound of Theorem 3.1.2 just by iterating the local well-posedness construction, the claim for these times follows immediately. Combining this observation, (3.80), recalling the approximation result, and using time-reversibility, Theorem 3.1.2 follows. □

We now prove Lemma 3.4.4.

Proof. Let us WLOG consider $t_0 = 0$. The general case follows analogously. By (3.72), we write:

$$E^2(u(\delta)) - E^2(u(0)) = \int_0^\delta \frac{d}{dt} E^2(u(t)) dt = -i \int_0^\delta \lambda_6(M_6; u) dt.$$

We recall (3.73), and we use symmetry to deduce that it suffices to bound:

$$\int_0^\delta \int_{\xi_1+\cdots+\xi_6=0} M_4(\xi_{123}, \xi_4, \xi_5, \xi_6) \widehat{V}(\xi_1+\xi_2) \widehat{u}(\xi_1) \widehat{\bar{u}}(\xi_2) \widehat{u}(\xi_3) \widehat{\bar{u}}(\xi_4) \widehat{u}(\xi_5) \widehat{\bar{u}}(\xi_6) d\xi_j dt.$$

Let v be as in Proposition 3.4.1, and let $\chi = \chi(t) = \chi_{[0,\delta]}(t)$. The above expression is then equal to:

$$\int_0^\delta \int_{\xi_1+\cdots+\xi_6=0} M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)\widehat{V}(\xi_1+\xi_2)\widehat{v}(\xi_1)\widehat{\bar{v}}(\xi_2)\widehat{v}(\xi_3)\widehat{\bar{v}}(\xi_4)\widehat{v}(\xi_5)(\chi\bar{v})\widehat{\ }(\xi_6)d\xi_j dt =$$

$$= \int_{\tau_1+\cdots+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} M_4(\xi_1, \xi_2, \xi_3, \xi_4)$$

$$((V * |v|^2)v)\widetilde{\ }(\xi_1, \tau_1)\widetilde{\bar{v}}(\xi_2, \tau_2)\widetilde{v}(\xi_3, \tau_3)(\chi\bar{v})\widetilde{\ }(\xi_4, \tau_4)d\xi_j d\tau_j.$$

Let $N_j, j = 1, \ldots 4$, be dyadic integers. We define:

$$I_{N_1,N_2,N_3,N_4} := \int_{\tau_1+\cdots+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} M_4(\xi_1, \xi_2, \xi_3, \xi_4)$$

$$((V * |v|^2)v)\widetilde{\ }_{N_1}(\xi_1, \tau_1)\widetilde{\bar{v}}_{N_2}(\xi_2, \tau_2)\widetilde{v}_{N_3}(\xi_3, \tau_3)(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_4, \tau_4)d\xi_j d\tau_j.$$

We want to bound I_{N_1,N_2,N_3,N_4}. Let us define by N_j^* the appropriate reordering of the N_j. We know:

$$N_1^* \sim N_2^*,\ N_1^* \gtrsim N. \tag{3.81}$$

We have to consider two Big Cases:

Big Case 1: $N_2^* \gg N_3^*$.

Big Case 2: $N_2^* \sim N_3^*$.

Big Case 1:

From Proposition 3.4.2, in this Big Case, we have the bound:

$$M_4 = O(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)). \tag{3.82}$$

We consider several Cases:

Case 1: $N_1^* \sim N_1$ (and hence $N_2^* \sim N_1$).

Let us assume WLOG that:

$$N_2^* \sim N_2, N_3^* \sim N_3, N_4^* \sim N_4.$$

The other cases are analogous.

By using (3.95), we deduce:

$$|I_{N_1,N_2,N_3,N_4}| \leq \int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0,|\xi_1+\xi_2+\xi_3|\sim N_1} \frac{1}{(N_1^*)^2}\theta(\xi_1+\xi_2+\xi_3)\theta(N_2^*)$$

$$|\widetilde{v}(\xi_1,\tau_1)||\widetilde{\bar{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)||\widetilde{\bar{v}}_{N_2}(\xi_4,\tau_4)||\widetilde{v}_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{}_{N_4}(\xi_6,\tau_6)|d\xi_j d\tau_j.$$

From (3.61), we know that:

$$\theta(\xi_1+\xi_2+\xi_3) \lesssim \theta(\xi_1)+\theta(\xi_2)+\theta(\xi_3).$$

By symmetry, we need to bound:

$$I^1_{N_1,N_2,N_3,N_4} := \int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0,|\xi_1+\xi_2+\xi_3|\sim N_1} \frac{1}{(N_1^*)^2}\theta(\xi_1)\theta(N_2^*)$$

$$|\widetilde{v}(\xi_1,\tau_1)||\widetilde{\bar{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)||\widetilde{\bar{v}}_{N_2}(\xi_4,\tau_4)||\widetilde{v}_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{}_{N_4}(\xi_6,\tau_6)|d\xi_j d\tau_j \lesssim$$

$$\int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0,|\xi_1+\xi_2+\xi_3|\sim N_1} \frac{1}{(N_1^*)^2}|(\mathcal{D}v)\widetilde{}(\xi_1,\tau_1)|$$

$$|\widetilde{\overline{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)||(\mathcal{D}\overline{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)||\widetilde{v}_{N_3}(\xi_5,\tau_5)||(\chi\overline{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)|d\xi_j d\tau_j.$$

Now, $|\xi_1+\xi_2+\xi_3| \sim N_1$, hence:

$$\max\{|\xi_1|,|\xi_2|,|\xi_3|\} \gtrsim N_1.$$

We have to consider several subcases:

Subcase 1: $|\xi_1| \gtrsim N_1$.

The contribution to $I^1_{N_1,N_2,N_3,N_4}$ in this subcase is:

$$\lesssim \int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0}\frac{1}{(N_1^*)^2}\big(|(\mathcal{D}v)\widetilde{\ }_{\gtrsim N_1}(\xi_1,\tau_1)||\widetilde{v}_{N_3}(\xi_5,\tau_5)|\big)$$

$$\big(|(\mathcal{D}\overline{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)||(\chi\overline{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)|\big)|\widetilde{\overline{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)|d\xi_j d\tau_j =$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(N_1^*)^2}F_1F_2\overline{F_3F_4F_5}F_6\,dxdt.$$

For the last equality, we used Parseval's identity for the functions F_j, which are chosen to satisfy:

$$\widetilde{F_1}=|(\mathcal{D}v)\widetilde{\ }_{\gtrsim N_1}|,\ \widetilde{F_2}=|\widetilde{v}_{N_3}|,\ \widetilde{F_3}=|(\mathcal{D}v)\widetilde{\ }_{N_2}|,\ \widetilde{F_4}=|(\chi v)\widetilde{\ }_{N_4}|,\ \widetilde{F_5}=|\widetilde{v}|,\ \widetilde{F_6}=|\widetilde{v}|.$$

We now use an $L^2_{t,x}, L^2_{t,x}, L^\infty_{t,x}, L^\infty_{t,x}$ Hölder inequality, Proposition 3.2.2, Proposition 5.2.5, and (3.16) to see that this expression is:

$$\lesssim \frac{1}{(N_1^*)^2}\Big(\frac{1}{N_1^{\frac{1}{2}}}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}\Big)\Big(\frac{1}{N_2^{\frac{1}{2}-}}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}\Big)\|v\|^2_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{3-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{3-}} E^2(u(0)). \tag{3.83}$$

Subcase 2: $|\xi_2| \gtrsim N_1$.

The contribution to $I^1_{N_1,N_2,N_3,N_4}$ in this subcase is:

$$\lesssim \int_{\tau_1+\cdots+\tau_6=0} \int_{\xi_1+\cdots+\xi_6=0} \frac{1}{(N_1^*)^2} |(\mathcal{D}v)\widetilde{\ }(\xi_1,\tau_1)||(\bar{v})\widetilde{\ }_{\gtrsim N_1}(\xi_2,\tau_2)|$$

$$\big(|(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)||(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)|\big)|\widetilde{v}(\xi_3,\tau_3)||\widetilde{v}_{N_3}(\xi_5,\tau_5)|d\xi_j d\tau_j$$

We argue similarly as in the previous Subcase, but we now use an $L^4_{t,x}, L^4_{t,x}, L^2_{t,x}, L^\infty_{t,x}, L^\infty_{t,x}$ Hölder inequality and Proposition 5.2.5 to deduce that the previous expression is:

$$\lesssim \frac{1}{(N_1^*)^2} \|\mathcal{D}v\|_{X^{0,\frac{3}{8}+}} \|v_{\gtrsim N_1}\|_{X^{0,\frac{3}{8}+}} \Big(\frac{1}{N_2^{\frac{1}{2}-}} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}}\Big) \|v\|^2_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{\frac{7}{2}-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^4_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{\frac{7}{2}-}} E^2(u(0)). \tag{3.84}$$

(We note that we used the fact that $\|v_{\gtrsim N_1}\|_{X^{0,\frac{3}{8}+}} \lesssim \frac{1}{N_1}\|v\|_{X^{1,\frac{1}{2}+}}$.)

Subcase 3: $|\xi_3| \gtrsim N_1$.

Subcase 3 is analogous to Subcase 2, and we get the same bound on the wanted contribution.

Case 2: $N_3^* \sim N_1$ or $N_4^* \sim N_1$. Let us WLOG consider the case $N_3^* \sim N_1$. (the case $N_4^* \sim N_1$ is analogous) Let us also WLOG suppose:

$$N_1^* \sim N_2,\ N_2^* \sim N_3,\ N_4^* \sim N_4.$$

Arguing similarly as earlier, we want to estimate:

$$\int_{\tau_1+\cdots+\tau_6=0} \int_{\xi_1+\cdots+\xi_6=0, |\xi_1+\xi_2+\xi_3| \sim N_1} \frac{1}{(N_1^*)^2} \theta(N_2)\theta(N_3)$$

$$|\widetilde{v}(\xi_1,\tau_1)||\widetilde{\overline{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)||\widetilde{\overline{v}}_{N_2}(\xi_4,\tau_4)||\widetilde{v}_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)| d\xi_j d\tau_j.$$

We write:

$$v = v_{\ll (N_1^*)^{\frac{1}{2}}} + v_{\gtrsim (N_1^*)^{\frac{1}{2}}}.$$

We consider the following subcases:

Subcase 1: $|\xi_1|, |\xi_2|, |\xi_3| \ll (N_1^*)^{\frac{1}{2}}$.

We have to estimate:

$$\int_{\tau_1+\cdots+\tau_6=0} \int_{\xi_1+\cdots+\xi_6=0, |\xi_1+\xi_2+\xi_3| \sim N_1} \frac{1}{(N_1^*)^2} \theta(N_2)\theta(N_3)$$

$$|\widetilde{v}_{\ll (N_1^*)^{\frac{1}{2}}}(\xi_1,\tau_1)||\widetilde{\overline{v}}_{\ll (N_1^*)^{\frac{1}{2}}}(\xi_2,\tau_2)||\widetilde{v}_{\ll (N_1^*)^{\frac{1}{2}}}(\xi_3,\tau_3)||\widetilde{\overline{v}}_{N_2}(\xi_4,\tau_4)||\widetilde{v}_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)| d\xi_j d\tau_j$$

$$\lesssim \int_{\tau_1+\cdots+\tau_6=0} \int_{\xi_1+\cdots+\xi_6=0, |\xi_1+\xi_2+\xi_3| \sim N_1} \frac{1}{(N_1^*)^2} (|\widetilde{v}_{\ll (N_1^*)^{\frac{1}{2}}}(\xi_1,\tau_1)||(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)|)$$

$$(|(\mathcal{D}v)\widetilde{\ }_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)|)|\widetilde{v}_{\ll (N_1^*)^{\frac{1}{2}}}(\xi_3,\tau_3)||\widetilde{v}_{\ll (N_1^*)^{\frac{1}{2}}}(\xi_2,\tau_2)| d\xi_j d\tau_j$$

We apply an $L^2_{t,x}, L^2_{t,x}, L^\infty_{t,x}, L^\infty_{t,x}$ Hölder inequality, Proposition 3.2.2, and Proposition 5.2.5 to deduce that the above expression is:

$$\lesssim \frac{1}{(N_1^*)^2}\big(\frac{1}{N_2^{\frac{1}{2}}}\|v\|_{X^{0,\frac{1}{2}+}}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\big)\big(\frac{1}{N_3^{\frac{1}{2}-}}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}\big)\|v\|^2_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{3-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{3-}}E^2(u(0)). \tag{3.85}$$

Subcase 2: $\max\{|\xi_1|,|\xi_2|,|\xi_3|\} \gtrsim (N_1^*)^{\frac{1}{2}}$.

We consider WLOG when $|\xi_1| \gtrsim (N_1^*)^{\frac{1}{2}}$. The other two cases are analogous. Hence, we have to estimate:

$$\int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0,|\xi_1+\xi_2+\xi_3|\sim N_1}\frac{1}{(N_1^*)^2}\theta(N_2)\theta(N_3)$$

$$|\widetilde{v}_{\gtrsim(N_1^*)^{\frac{1}{2}}}(\xi_1,\tau_1)||\widetilde{\bar{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)||\widetilde{\bar{v}}_{N_2}(\xi_4,\tau_4)||\widetilde{v}_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)|d\xi_j d\tau_j$$

$$\lesssim \int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0,|\xi_1+\xi_2+\xi_3|\sim N_1}\frac{1}{(N_1^*)^2}|\widetilde{v}_{\gtrsim(N_1^*)^{\frac{1}{2}}}(\xi_1,\tau_1)||(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)|$$

$$\big(|(\mathcal{D}v)\widetilde{\ }_{N_3}(\xi_5,\tau_5)||(\chi\bar{v})\widetilde{\ }_{N_4}(\xi_6,\tau_6)|\big)|\widetilde{\bar{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)|d\xi_j d\tau_j$$

We use an $L^4_{t,x}, L^4_{t,x}, L^2_{t,x}, L^\infty_{t,x}, L^\infty_{t,x}$ Hölder inequality, and Proposition 5.2.5 to deduce that this expression is:

$$\lesssim \frac{1}{(N_1^*)^2}\|v_{\gtrsim (N_1^*)^{\frac{1}{2}}}\|_{X^{0,\frac{3}{8}+}}\|\mathcal{D}v\|_{X^{0,\frac{3}{8}+}}\big(\frac{1}{(N_3)^{\frac{1}{2}-}}\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}\big)\|v\|^2_{X^{\frac{1}{2}+,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{3-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}}$$

Here we used the fact that $\|v_{\gtrsim (N_1^*)^{\frac{1}{2}}}\|_{X^{0,\frac{3}{8}+}} \lesssim \frac{1}{(N_1^*)^{\frac{1}{2}}}\|v\|_{X^{1,\frac{1}{2}+}}$.

Hence, the contribution from this Subcase is:

$$\lesssim \frac{1}{(N_1^*)^{3-}}E^2(u(0)). \tag{3.86}$$

Combining (3.83), (3.84), (3.85), (3.86), it follows that the contribution to I_{N_1,N_2,N_3,N_4} coming from Big Case 1 is:

$$O(\frac{1}{(N_1^*)^{3-}}E^2(u(0))). \tag{3.87}$$

Big Case 2: We recall that in this Big Case $N_2^* \sim N_3^*$.

From Proposition 3.4.2, we observe that in Big Case 2, one has:

$$M_4(\xi_1,\xi_2,\xi_3,\xi_4) = O(\frac{1}{(N_1^*)^3}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*) \tag{3.88}$$

In Big Case 2, we argue in the same way as we did for the Hartree equation on S^1 in Chapter 2. The same argument in this chapter implies that the contribution to I_{N_1,N_2,N_3,N_4} coming from Big Case 2 is:

$$O(\frac{1}{(N_1^*)^3}E^2(u(0))). \tag{3.89}$$

We refer the reader to the proof in Chapter 2 Let us note that in the periodic setting, we could only get a decay factor of $\frac{1}{(N_1^*)^2}$.

We use (3.87),(3.89),(3.81), and sum in the N_j^* to deduce Lemma 3.4.4 for $t_0 = 0$. By time-translation, the general claim follows. □

Remark 3.4.5. *If we use the method of proof of Theorem 3.1.1 for the Hartree equation (here we just use mass and energy as conserved quantities), we can obtain the bound* $\|u(t)\|_{H^s} \leq C(1+|t|)^{(s-1)+}$*, which is a weaker result than Theorem 3.1.2 when* s *is large.*

3.5 Appendix A: Auxiliary results for the cubic nonlinear Schrödinger equation

In Appendix A, we prove Proposition 3.3.2

Proof. By continuity of Energy and Mass on H^1, both of the claims clearly hold for $n = 1$. For higher n, we will need to work directly with the higher conserved quantities of (3.1). One can explicitly compute these quantities by means of a recursive formula. The formula that we use comes from [46, 84]. Let u be a solution of (3.1). Let us define a sequence of polynomials $(P_k)_{k \geq 1}$ by:

$$\begin{cases} P_1 := |u|^2, \\ P_{k+1} := -i\bar{u}\frac{\partial}{\partial x}(\frac{P_k}{\bar{u}}) + \sum_{l=1}^{k-1} P_l P_{k-l}, \text{ for } k \geq 1. \end{cases} \tag{3.90}$$

Then, for all $k \geq 1$, $\int P_k \, dx$ is a conserved quantity for (3.1).

For the details, we refer the reader to [46], more precisely to Page 53, where it is noted that formulas (4.19),(4.20),(4.34) in the textbook still remain valid for our equation. Let us now explicitly compute:

$$P_2 = -i\bar{u}\frac{\partial}{\partial x}u.$$

$$P_3 = -\bar{u}\frac{\partial^2}{\partial x^2}u + \frac{1}{2}|u|^4.$$

$$P_4 = i\bar{u}\frac{\partial^3}{\partial x^3}u - i|u|^2\bar{u}\frac{\partial}{\partial x}u.$$

The conserved quantity corresponding to P_1 is:

$$\int P_1\,dx = \int |u|^2 dx \;= Mass.$$

For the conserved quantities corresponding to P_2, P_3, we integrate by parts to obtain:

$$\int P_2\,dx = -\frac{i}{2}\int (\bar{u}\frac{\partial}{\partial x}u - u\frac{\partial}{\partial x}\bar{u})dx \;\sim Momentum.$$

$$\int P_3\,dx = \int \big|\frac{\partial}{\partial x}u\big|^2 dx + \frac{1}{2}\int |u|^4 dx \;\sim Energy.$$

So, we recover the well-known conserved quantities this way.

We argue by induction to deduce that:

$$P_n = c\bar{u}\frac{\partial^{n-1}}{\partial x^{n-1}}u + l.o.t.$$

Again, by induction, we obtain that each lower-order term contains in total at most $n-3$ derivatives. It follows that the conserved quantity we want to study is:

$$E_n(u) := \int P_{2n+1}\,dx = \pm c\int \big|\frac{\partial^n}{\partial x^n}u\big|^2 dx + l.o.t.$$

Here, each lower-order term is the integral of a polynomial in x-derivatives of $u, \bar{u}$ containing in total at most $2n-2$ derivatives. If we integrate by parts, we can arrange so that at most n derivatives fall on one factor, and that at most $n-2$ derivatives fall on all the other factors combined. By using Hölder's inequality [3] and by Sobolev embedding, there exists a polynomial $Q_n = Q_n(x)$ s.t.

$$E_n(u) \geq C(\|u\|^2_{\dot{H}^n} - Q_n(\|u\|_{H^{n-1}})\|u\|_{H^n}). \tag{3.91}$$

[3] we estimate the factor with the most derivatives, and an arbitrary other factor in L^2; the rest of the factors we estimate in L^∞

Similarly, if we also use multilinearity, it follows that there exists a polynomial R_n in (x, y) s.t.

$$|E_n(u) - E_n(v)| \lesssim (\|u\|_{H^n} + \|v\|_{H^n})\|u - v\|_{H^n} +$$

$$R_n(\|u\|_{H^{n-1}}, \|v\|_{H^{n-1}})\|u - v\|_{H^n}. \tag{3.92}$$

The fact that E_n is continuous on H^n follows immediately from (3.92). This proves the first part of the claim.

Furthermore, if we define:

$$\widetilde{E_n}(u) := E_n(u) + \|u\|_{L^2}^2,$$

then, by (3.91), it follows that:

$$\widetilde{E_n}(u) \geq C_n(\|u\|_{H^n}^2 - Q_n(\|u\|_{H^{n-1}})\|u\|_{H^n}).$$

This bound in turn implies:

$$\|u\|_{H^n} \leq \frac{1}{2}\Big(Q_n(\|u\|_{H^{n-1}}) + \sqrt{(Q_n(\|u\|_{H^{n-1}}))^2 + \frac{4}{C_n}\widetilde{E_n}(u)}\Big). \tag{3.93}$$

We finally define:

$$B_n(\Phi) := \frac{1}{2}\Big(Q_n(B_{n-1}(\Phi)) + \sqrt{(Q_n(B_{n-1}(\Phi)))^2 + \frac{4}{C_n}\widetilde{E_n}(\Phi)}\Big). \tag{3.94}$$

We combine the fact that E_n is continuous on H^n, conservation of mass, (3.93), and argue by induction to deduce the second part of the claim if we define B_n as in (3.94).

□

3.6 Appendix B: Auxiliary results for the Hartree equation

We first prove Proposition 3.4.2 assuming Lemma 3.4.3.

Proof. Let us first recall that:

$$\widehat{V} \in L^{\infty}. \tag{3.95}$$

As before, we consider $|\xi_j| \sim N_j$ for dyadic integers N_1, N_2, N_3, N_4. We order the N_j to obtain N_j^*, for $j = 1, \dots, 4$, s.t. $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. Let's recall the localization (3.81). By symmetry, let us also consider WLOG $N_1^* \sim N_1$.

We consider the following cases:

Case 1: $N_2^* \gg N_3^*$.

We must consider several subcases:

Subcase 1: $N_2^* \sim N_2$.

Since $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$, we obtain:

$$|\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2| = 2|(\xi_1 + \xi_2)(\xi_1 + \xi_4)|. \tag{3.96}$$

In this Subcase, this expression is:

$$\sim N_1^* |\xi_1 + \xi_2|.$$

By Lemma 3.4.3, we know:

$$|(\theta(\xi_1))^2 - (\theta(\xi_2))^2| \lesssim ||\xi_1| - |\xi_2|| \frac{(\theta(\xi_1))^2}{|\xi_1|} \lesssim |\xi_1 + \xi_2| \frac{\theta(N_1^*)\theta(N_2^*)}{N_1^*}.$$

Similarly, assuming WLOG that $|\xi_3| \geq |\xi_4|$, we use Lemma 3.4.3, and the fact that $\frac{(\theta(\xi_3))^2}{|\xi_3|} \lesssim \frac{(\theta(\xi_1))^2}{|\xi_1|}$, if $|\xi_3| \geq N$, and $(\theta(\xi_3))^2 - (\theta(\xi_4))^2 = 0$, if $|\xi_3| \leq N$ to deduce that:

$$|(\theta(\xi_3))^2 - (\theta(\xi_4))^2| \lesssim |\xi_3 + \xi_4| \frac{\theta(N_1^*)\theta(N_2^*)}{N_1^*} = |\xi_1 + \xi_2| \frac{\theta(N_1^*)\theta(N_2^*)}{N_1^*}$$

Combining the last three bounds and (3.95), we obtain:

$$\Psi = O(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)). \tag{3.97}$$

Subcase 2: $N_2^* \sim N_3$.

In this subcase, we don't expect any cancelation in neither the numerator nor the denominator. So, we just estimate the numerator as $O(\theta(N_1^*)\theta(N_2^*))$, and we estimate the denominator as $\sim (N_1^*)^2$. Consequently:

$$\Psi = O(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)). \tag{3.98}$$

Case 2: $N_2^* \sim N_3^*$.

As before, we consider two subcases:

Subcase 1: $N_3^* \gg N_4^*$.

It suffices to WLOG consider when $N_2^* \sim N_2, N_3^* \sim N_3, N_4^* \sim N_4$.

We have:

$$|\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2| = 2|(\xi_1 + \xi_2)(\xi_1 + \xi_4)| \sim N_1^* |\xi_1 + \xi_2|.$$

We argue now as in Subcase 1 of Case 1 to obtain:

$$\Psi = O\big(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big).$$

Since $N_3^* \sim N_1^*$, in this subcase, we obtain:

$$\Psi = O\big(\frac{1}{(N_1^*)^3}\theta(N_1^*)\theta(N_2^*)N_3^*\big). \tag{3.99}$$

Subcase 2: $N_1^* \sim N_2^* \sim N_3^* \sim N_4^*$.

We know:

$$|\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2| \sim |(\xi_1+\xi_2)(\xi_1+\xi_4)|.$$

We must consider several sub-subcases.

Sub-subcase 1: $|\xi_1+\xi_2| \ll 1,\ |\xi_1+\xi_4| \ll 1.$

Since $\xi_1+\xi_2+\xi_3+\xi_4 = 0$, we get:

$$\xi_3 + (\xi_1+\xi_2) = -\xi_4$$

$$\xi_3 + (\xi_1+\xi_4) = -\xi_2$$

$$\xi_3 + (\xi_1+\xi_2) + (\xi_1+\xi_4) = \xi_1.$$

From the previous identities, the Double Mean Value Theorem (3.26), and (3.60), we obtain that:

$$|(\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2| =$$

$$= |(\theta(\xi_3+(\xi_1+\xi_2)+(\xi_1+\xi_4)))^2 - (\theta(\xi_3+(\xi_1+\xi_4)))^2 + (\theta(\xi_3+(\xi_1+\xi_2)))^2 - (\theta(\xi_4))^2|$$

$$\lesssim |\xi_1+\xi_2||\xi_1+\xi_4||(\theta^2)''(\xi_3)| \lesssim |\xi_1+\xi_2||\xi_1+\xi_4|\,\frac{(\theta(\xi_3))^2}{|\xi_3|^2}$$

$$\lesssim |\xi_1+\xi_2||\xi_1+\xi_4|\frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}$$

So, in this sub-subcase, we obtain that:

$$\Psi = O(\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)) =$$

$$= O(\frac{1}{(N_1^*)^4}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*) \tag{3.100}$$

Sub-subcase 2: $|\xi_1 + \xi_4| \gtrsim 1$.

Here:

$$|\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2| = 2|\xi_1 + \xi_2||\xi_1 + \xi_4| \gtrsim |\xi_1 + \xi_2| = |\xi_3 + \xi_4|.$$

Hence, by Lemma 3.4.3:

$$\Psi = O\big(\frac{|(\theta(\xi_1))^2 - (\theta(\xi_2))^2}{|\xi_1 + \xi_2|} + \frac{|(\theta(\xi_3))^2 - (\theta(\xi_4))^2|}{|\xi_3 + \xi_4|}\big) =$$

$$= O(\frac{1}{|\xi_1|}(\theta(\xi_1))^2) = O(\frac{1}{N_1^*}\theta(N_1^*)\theta(N_2^*)) =$$

$$O(\frac{1}{(N_1^*)^3}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*) \tag{3.101}$$

Sub-subcase 3: $|\xi_1 + \xi_2| \gtrsim 1$.

We group the terms in the numerator as:

$$\big((\theta(\xi_1))^2 - (\theta(\xi_4))^2\big) + \big((\theta(\xi_3))^2 - (\theta(\xi_2))^2\big)$$

Then, we argue as in the previous sub-subcase to obtain:

$$O\big(\frac{1}{(N_1^*)^3}\theta(N_1^*)\theta(N_2^*)N_3^*N_4^*\big) \tag{3.102}$$

□

Let us now prove Lemma 3.4.3.

Proof. We have to consider five cases:

1. $N \leq |y|$, $2N \leq |x|$.

2. $N \leq |y| \leq |x| \leq 2N$.

3. $|y| \leq |x| \leq N$.

4. $|y| \leq N$, $2N \leq |x|$.

5. $|y| \leq N \leq |x| \leq 2N$.

We consider each case separately:

1. $|(\theta(x)^2 - (\theta(y))^2| \leq (|x| - |y|) \sup_{[|y|,|x|]} |(\theta^2)'(z)| \leq (|x| - |y|) \sup_{[N,|x|]} |(\theta^2)'(z)|$

   By using (3.59), this expression is:

$$\lesssim (|x| - |y|) \sup_{[N,|x|]} \big(\frac{(\theta(z))^2}{|z|}\big) \lesssim (|x| - |y|) \sup_{[2N,|x|]} \big(\frac{(\theta(z))^2}{|z|}\big) =$$

$$= (|x| - |y|) \sup_{[2N,|x|]} \frac{|z|^{2s-1}}{N^{2s}} = (|x| - |y|) \frac{|x|^{2s-1}}{N^{2s}} = (|x| - |y|) \frac{(\theta(x))^2}{|x|}.$$

2. $|(\theta(x))^2 - (\theta(y))^2| \leq (|x| - |y|) \sup_{[|y|,|x|]} |(\theta^2)'(z)| \lesssim (|x| - |y|) \sup_{[|y|,|x|]} \big(\frac{(\theta(z))^2}{|z|}\big)$

   For $z \in [|y|, |x|]$, one has:

$$\frac{(\theta(z))^2}{|z|} \sim \frac{(\theta(N))^2}{|N|} \sim \frac{(\theta(x))^2}{|x|}.$$

   Hence, we get the wanted bound in this case.

3. In this case: $(\theta(x))^2 - (\theta(y))^2 = 0$.

4. $|(\theta(x))^2 - (\theta(y))^2| = |\frac{|x|^{2s}}{N^{2s}} - (\theta(N))^2|$, and we argue as in the first case.

5. $|(\theta(x))^2 - (\theta(y))^2| = |(\theta(x))^2 - (\theta(N))^2|$, and we argue as in the second case.

Lemma 3.4.3 now follows.

$\square$

3.7 Appendix C: The derivative nonlinear Schrödinger equation

In this Appendix, we give a brief sketch of the proof of (3.12).

We don't consider derivative nonlinear Schrödinger equation directly. Rather, we argue as in [61, 64, 65], and we apply to (3.10) the following gauge transform:

$$\mathcal{G}f(x) := e^{-i\int_{-\infty}^{x}|f(y)|^2 dy} f(x). \tag{3.103}$$

For u a solution of (3.10),we take $w := \mathcal{G}u$. Then, it can be shown that w solves:

$$\begin{cases} iw_t + \Delta w = -iw^2\bar{w}_x - \frac{1}{2}|w|^4 w \\ w(x,0) = w_0(x) = \mathcal{G}\Phi(x), x \in \mathbb{R}, t \in \mathbb{R}. \end{cases} \tag{3.104}$$

The equation (3.104) has as a corresponding Hamiltonian:

$$E(f) := \int \partial_x f \partial_x \bar{f} dx - \frac{1}{2} Im \int f \bar{f} f \partial_x \bar{f} dx. \tag{3.105}$$

Although the problem is not defocusing a priori, in [31],[32], it is noted that the smallness condition (3.11) guarantees that the energy $E(w(t))$ is positive and that it gives us a priori bounds on $\|w(t)\|_{H^1}$.

It can be shown that the gauge transform satisfies the following boundedness property:

Gauge transform bound. *For $s \geq 1$, there exists a polynomial $P_s = P_s(x)$ such that:*

$$\|\mathcal{G}f\|_{H^s} \leq P_s(\|f\|_{H^1})\|f\|_{H^s},$$

$$\|\mathcal{G}^{-1}f\|_{H^s} \leq P_s(\|f\|_{H^1})\|f\|_{H^s}.$$

From the bi-continuity of gauge transform, and the uniform bounds on $\|w(t)\|_{H^1}$, it suffices to prove for solutions of (3.104) the bounds that we want to hold for solutions of the derivative NLS.

One can show that a local-in-time estimate, analogous to Proposition 3.4.1, holds for (3.104). The key is to use the following:

Trilinear Estimate. *Let $s \geq 1, b \in (\frac{1}{2}, \frac{5}{8}], b' > \frac{1}{2}$, then for $v_1, v_2, v_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, the following estimate holds:*

$$\|v_1 v_2 \overline{(v_3)}_x\|_{X^{s,b-1}} \lesssim \|v_1\|_{X^{1,b'}} \|v_2\|_{X^{1,b'}} \|v_3\|_{X^{s,b'}}$$

$$+\|v_1\|_{X^{1,b'}} \|v_2\|_{X^{s,b'}} \|v_3\|_{X^{1,b'}} + \|v_1\|_{X^{s,b'}} \|v_2\|_{X^{1,b'}} \|v_3\|_{X^{1,b'}}. \tag{3.106}$$

This estimate is the analogue of Proposition 2.4. in [104], where the identical statement is proved in the context of low regularities. The proof for $s \geq 1$ is similar, with minor modifications.

We now argue as in Theorem 3.1.2, by using the technique of *higher modified energies*. We define E^1 as before. We consider the higher modified energy E^2 given by:

$$E^2(w) := E^1(w) + \lambda_4(M_4; w). \tag{3.107}$$

Using the equation (3.104), it follows that a good choice for the multiplier M_4 on the set Γ_4 is:

$$M_4 \sim \frac{(\theta(\xi_1))^2 \xi_3 + (\theta(\xi_2))^2 \xi_4 + (\theta(\xi_3))^2 \xi_1 + (\theta(\xi_4))^2 \xi_2}{\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2} \tag{3.108}$$

We define the ordered dyadic localizations N_j^* as before. With this notation, one can show the following:

Multiplier bound. *On Γ_4, one has the pointwise bound:*

$$|M_4| \lesssim \frac{1}{N_1^*} \theta(N_1^*) \theta(N_2^*). \tag{3.109}$$

By construction of M_4, we obtain:

$$\frac{d}{dt}E^2(w(t)) = \frac{d}{dt}E^1(w) + \frac{d}{dt}\lambda_4(M_4; w)$$

$$= \lambda_6(\sigma_6; w) + \lambda_6(M_6; w) + \lambda_8(M_8; w). \tag{3.110}$$

Here:

$$\sigma_6 = (\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2 + (\theta(\xi_5))^2 - (\theta(\xi_6))^2. \tag{3.111}$$

$$M_6 \sim M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)\xi_2 + M_4(\xi_1, \xi_{234}, \xi_5, \xi_6)\xi_3$$

$$+ M_4(\xi_1, \xi_2, \xi_{345}, \xi_6)\xi_4 + M_4(\xi_1, \xi_2, \xi_3, \xi_{456})\xi_5. \tag{3.112}$$

$$M_8 \sim M_4(\xi_{12345}, \xi_6, \xi_7, \xi_8) - M_4(\xi_1, \xi_{23456}, \xi_7, \xi_8)$$

$$+ M_4(\xi_1, \xi_2, \xi_{34567}, \xi_8) - M_4(\xi_1, \xi_2, \xi_3, \xi_{45678}). \tag{3.113}$$

Using (3.110) and (3.109), we can argue similarly as in the proof of Theorem 3.1.2 to deduce that:

$$|E^2(w(t_0 + \delta)) - E^2(w(t_0))| \lesssim \frac{1}{N^{\frac{1}{2}-}} E^2(w(t_0)). \tag{3.114}$$

The bound for the derivative NLS follows from (3.114). $\square$

Chapter 4

Bounds on $\mathbb{T}^2$ and $\mathbb{R}^2$

4.1 Introduction

In this chapter, we study the 2D Hartree initial value problem:

$$\begin{cases} iu_t + \Delta u = (V * |u|^2)u, \; x \in \mathbb{T}^2 \text{ or } x \in \mathbb{R}^2, \; t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{T}^2), \text{ or } \Phi \in H^s(\mathbb{R}^2), \; s > 1. \end{cases} \tag{4.1}$$

The assumptions that we have on V are the following:

(i) $V \in L^1(\mathbb{T}^2)$, or $V \in L^1(\mathbb{R}^2)$, respectively.

(ii) $V \geq 0$.

(iii) V is even.

The equation (4.1) has the following conserved quantities:

$$M(u(t)) := \int |u(x,t)|^2 dx, \; \textit{(Mass)}$$

$$E(u(t)) := \frac{1}{2}\int |\nabla u(x,t)|^2 dx + \frac{1}{4}\int (V * |u|^2)(x,t)|u(x,t)|^2 dx, \; \textit{(Energy)}.$$

The region of integration is either $\mathbb{T}^2$ or $\mathbb{R}^2$, depending whether we are considering the periodic or the non-periodic setting. As in the one-dimensional case, the fact that mass is conserved follows from the fact that V is real-valued. The fact that energy is conserved follows from integration by parts, by using the fact that V is even [28].

By using the two conservation laws, and by arguing as in [52], we can deduce global existence of (4.1) in H^1 and a priori bounds on the H^1 norm of a solution, in the non-periodic setting. By persistence of regularity, we obtain global existence in H^s, for $s > 1$. Hence, it makes sense to analyze the behavior of $\|u(t)\|_{H^s}$. A similar argument holds in the periodic setting, whereas here, we need to use periodic variants of Strichartz estimates [9].

4.1.1 Statement of the main results

The first result that we prove is:

Theorem 4.1.1. *(Bound for the Hartree equation on $\mathbb{T}^2$) Let u be the global solution of* (4.1) *on $\mathbb{T}^2$. Then, there exists a function C_s, continuous on $H^1(\mathbb{T}^2)$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C_s(\Phi)(1+|t|)^{s+}\|\Phi\|_{H^s(\mathbb{T}^2)}. \tag{4.2}$$

Similarly, in the non-periodic setting one has:

Theorem 4.1.2. *(Bound for the Hartree equation on $\mathbb{R}^2$) Let u be the global solution of* (4.1) *on $\mathbb{R}^2$. Then, there exists a function C_s, continuous on $H^1(\mathbb{R}^2)$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{R}^2)} \leq C_s(\Phi)(1+|t|)^{\frac{4}{7}s+}\|\Phi\|_{H^s(\mathbb{R}^2)}. \tag{4.3}$$

Heuristically, we expect to get a better bound in the non-periodic setting, due to the presence of stronger dispersion.

In the non-periodic setting, let us formally take $V = \delta$. Then, (4.1) becomes:

$$\begin{cases} iu_t + \Delta u = |u|^2 u, x \in \mathbb{T}^2 \text{ or } x \in \mathbb{R}^2, t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{T}^2) \text{ or } \Phi \in H^s(\mathbb{R}^2),\ s > 1. \end{cases} \tag{4.4}$$

The Cauchy problem (4.4) is also known to be globally well-posed in H^s [51]. We will see that the proof of Theorem 4.1.2 holds when we formally take $V = \delta$. Hence, we also deduce the following:

Corollary 4.1.3. *(Bound for the Cubic NLS on $\mathbb{R}^2$) Let u be the global solution of (4.4) on $\mathbb{R}^2$. Then, there exists a function C_s, continuous on $H^1(\mathbb{R}^2)$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{R}^2)} \leq C_s(\Phi)(1 + |t|)^{\frac{4}{7}s+} \|\Phi\|_{H^s(\mathbb{R}^2)}. \tag{4.5}$$

Corollary 4.1.3 improves the previously known bound $\|u(t)\|_{H^s} \lesssim (1+|t|)^{\frac{2}{3}s+} \|\Phi\|_{H^s}$, for all $s \in \mathbb{N}$. The latter bound was proved in [30]. However, after the submission of our paper, Dodson [44] proved that the equation (4.4) scatters in $L^2(\mathbb{R}^2)$. From this fact, one can deduce uniform bounds on $\|u(t)\|_{H^s}$.

We can also take $V = \delta$ in the periodic setting.

Corollary 4.1.4. *(Bound for the Cubic NLS on $\mathbb{T}^2$) Let u be the global solution of (4.4) on $\mathbb{T}^2$. Then, there exists a function C_s, continuous on $H^1(\mathbb{T}^2)$ such that for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C_s(\Phi)(1 + |t|)^{s+} \|\Phi\|_{H^s(\mathbb{T}^2)}. \tag{4.6}$$

We note that essentially the same bound was proved for $s \in \mathbb{N}$ in [118].

4.1.2 Previously known results

We note that the growth of high Sobolev norms for the cubic NLS on $\mathbb{T}^2$ was previously studied in [118]. The presented methods can be applied to (4.1). The bounds are essentially the same as we obtain in Theorem 4.1.1. However, the approach from [118] applies only to the case when s is a positive integer, whereas our method works for all real $s \geq 1$.

If we knew that (4.1) scattered in H^s, we would immediately obtain uniform bounds on $\|u(t)\|_{H^s}$. However, in the periodic setting, no scattering results have ever been proved, and one doesn't expect them to hold due to limited dispersion. In the non-periodic setting, there are several known scattering results [49, 55, 54, 56, 63, 85], but none of them are strong enough to imply scattering in H^s for (4.1) on $\mathbb{R}^2$. For a detailed explanation, we refer the reader to Remark 4.4.6. It is not known whether the methods presented in [44] apply to the (4.1).

Let us finally mention that the problem of Sobolev norm growth for the cubic NLS on $\mathbb{T}^2$ was also recently studied in [39], but in the sense of bounding the growth from below. In this paper, the authors exhibit the existence of smooth solutions of the cubic defocusing nonlinear Schrödinger equation on $\mathbb{T}^2$, whose H^s norm is arbitrarily small at time zero, and is arbitrarily large at some large finite time. One should note that behavior at infinity is still an open problem. If one starts from a specific initial data containing only five frequencies, an analysis of which Fourier modes become excited has recently been studied in [25] by different methods.

4.1.3 Main ideas of the proofs

As in the previous chapters, the main idea is to define $\mathcal{D}$ to be an *upside-down I-operator*. Similarly as before, we will use *higher modified energies*, i.e. quantities obtained from $\|\mathcal{D}u(t)\|_{L^2}^2$ by adding an appropriate multilinear correction. In this way, we will obtain $E^2(u(t)) \sim \|\mathcal{D}u(t)\|_{L^2}^2$, which is even more slowly varying. Due to more a more complicated resonance phenomenon in two dimensions, the construction of E^2 is going to be more involved than it was in one dimension. In the periodic setting, E^2 is constructed in Subsection 4.3.3. In the non-periodic setting, E^2 is constructed in Subsection 4.4.3.

We prove Theorem 4.1.1 and Theorem 4.1.2 for initial data Φ, which we assume lies only in $H^s(\mathbb{T}^2)$ and $H^s(\mathbb{R}^2)$, respectively. We don't assume any further regularity on the initial data. However, in the course of the proof, we work with Φ which is smooth, in order to make our formal calculations rigorous. The fact that we can do this follows from an appropriate Approximation Lemma (Proposition 4.3.2 and Proposition 4.4.2).

4.2 Facts from harmonic analysis

4.2.1 Estimates on $\mathbb{T}^2$

By Sobolev embedding on $\mathbb{T}^2$, we know that, for all $2 \leq q < \infty$, one has:

$$\|u\|_{L^q} \lesssim \|u\|_{H^1} \tag{4.7}$$

From [58], we know that on $\mathbb{T}^2$:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}} \tag{4.8}$$

(A similar local-in-time estimate was earlier noted in [9].)

By Plancherel's Theorem, one has:

$$\|u\|_{L^2_{t,x}} \sim \|u\|_{X^{0,0}} \tag{4.9}$$

From Sobolev embedding, it follows that:

$$\|u\|_{L^\infty_{t,x}} \lesssim \|u\|_{X^{1+,\frac{1}{2}+}} \tag{4.10}$$

If we take the $\frac{1}{2}+$ in (4.8) to be very close to $\frac{1}{2}$, we can interpolate between (4.8) and (4.9) to deduce:

$$\|u\|_{L^{4-}_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}-}} \tag{4.11}$$

Similarly, we can interpolate between (4.8) and (4.10) to obtain:

$$\|u\|_{L^{4+}_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}} \tag{4.12}$$

Lemma 4.2.1. *Let $c < d$ be real numbers, and let us denote by $\chi = \chi(t) = \chi_{[c,d]}(t)$. One then has, for all $s \in \mathbb{R}$, and for all $b < \frac{1}{2}$:*

$$\|\chi u\|_{X^{s,b}(\mathbb{R}\times\Lambda)} \lesssim \|u\|_{X^{s,b+}(\mathbb{R}\times\Lambda)}.$$

Here, Λ denotes either $\mathbb{T}^2$ or $\mathbb{R}^2$.

The proof of Lemma 4.2.1 is the same as the proof of Lemma 2.2.1 (see also [24, 36]). From the proof, we note that the implied constant is independent of c and d. We omit the details.

We can interpolate between (4.9) and (4.10) to deduce that, for $M \gg 1$, one has:

$$\|u\|_{L^M_{t,x}} \lesssim \|u\|_{X^{1,\frac{1}{2}+}} \tag{4.13}$$

Furthermore, from Sobolev embedding in time, we know that:

$$\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{0,\frac{1}{2}+}} \tag{4.14}$$

We can interpolate between (4.9) and (4.14) to obtain:

$$\|u\|_{L^4_t L^2_x} \lesssim \|u\|_{X^{0,\frac{1}{4}+}} \tag{4.15}$$

An additional estimate we will use is:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{4}+}} \tag{4.16}$$

The estimate (4.16) is a consequence of the following:

Lemma 4.2.2. *Suppose that Q is a ball in $\mathbb{Z}^2$ of radius N, and center n_0. Suppose that u satisfies $supp\, \widehat{u} \subseteq Q$. Then, one has:*

$$\|u\|_{L^4_{t,x}} \lesssim N^{\frac{1}{2}} \|u\|_{X^{0,\frac{1}{4}+}}. \tag{4.17}$$

Lemma 5.2.6 is proved in [15] by using the Hausdorff-Young inequality and Hölder's inequality. We omit the details.

To deduce (4.16), we write $u = \sum_N u_N$. By the triangle inequality and Lemma 5.2.6, we obtain:

$$\|u\|_{L^4_{t,x}} \leq \sum_N \|u_N\|_{L^4_{t,x}} \lesssim \sum_N N^{\frac{1}{2}} \|u_N\|_{X^{0,\frac{1}{4}+}}$$

$$\lesssim \sum_N \frac{1}{N^{0+}} \|u_N\|_{X^{\frac{1}{2}+,\frac{1}{4}+}} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{4}+}}$$

We can now interpolate between (4.8) and (4.16) to deduce:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{s_1,b_1}} \tag{4.18}$$

whenever $\frac{1}{4} < b_1 < \frac{1}{2}+, s_1 > 1 - 2b_1$.

By using an appropriate transformation, as in Lemma 2.4 in [58], we see that (4.18) implies:

Lemma 4.2.3. *Suppose that u is as in the assumptions of Lemma 5.2.6, and suppose that* $b_1, s_1 \in \mathbb{R}$ *satisfy* $\frac{1}{4} < b_1 < \frac{1}{2}+, s_1 > 1 - 2b_1$. *Then, one has:*

$$\|u\|_{L^4_{t,x}} \lesssim N^{s_1} \|u\|_{X^{0,b_1}}. \tag{4.19}$$

4.2.2 Estimates on $\mathbb{R}^2$

We note that all the mentioned estimates in the periodic setting carry over to the non-periodic setting. However, there are some estimates which hold only in the non-periodic setting, which express the fact that the dispersion phenomenon is stronger on $\mathbb{R}^2$ than on $\mathbb{T}^2$. Such estimates allow us to get a better bound in Theorem 4.1.2 than the one we obtained in Theorem 4.1.1.

The first modification is that, on the plane, (4.8) is improved to:

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}. \tag{4.20}$$

Consequently, one can improve (4.11) to:

$$\|u\|_{L^{4-}_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{2}-}}. \tag{4.21}$$

On the plane, we will use the following estimate:

$$\|u\|_{L^{2+}_{t,x}} \lesssim \|u\|_{X^{0+,0+}}. \tag{4.22}$$

(4.22) follows from (4.20), the fact that $\|u\|_{L^2_{t,x}} \sim \|u\|_{X^{0,0}}$, and interpolation.

Furthermore, a key fact is the following result, which was first noted by Bourgain in [14]:

Proposition 4.2.4. *(Improved Strichartz Estimate) Suppose that N_1, N_2 are dyadic integers such that $N_1 \gg N_2$, and suppose that $u, v \in X^{0,\frac{1}{2}+}(\mathbb{R}^2 \times \mathbb{R})$ satisfy, for all t: supp $\widehat{u}(t) \subseteq \{|\xi| \sim N_1\}$, and supp $\widehat{v}(t) \subseteq \{|\xi| \sim N_2\}$. Then, one has:*

$$\|uv\|_{L^2_{t,x}} \lesssim \frac{N_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}}. \tag{4.23}$$

An alternative proof (in the 1D case) is given in [31].

Let us note the following corollary of Proposition 4.2.4.

Corollary 4.2.5. *Let $u, v \in X^{0,\frac{1}{2}+}(\mathbb{R}^2 \times \mathbb{R})$ be as in the assumptions of Proposition 4.2.4. Then one has:*

$$\|uv\|_{L^{2+}_t L^2_x} \lesssim \frac{N_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}-}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}}. \tag{4.24}$$

Proof. We observe that:

$$\|uv\|_{L^\infty_t L^2_x} \leq \|u\|_{L^\infty_t L^4_x} \|v\|_{L^\infty_t L^4_x} \lesssim N_1^{\frac{1}{2}} \|u\|_{L^\infty_t L^2_x} N_2^{\frac{1}{2}} \|v\|_{L^\infty_t L^2_x}.$$

$$\lesssim N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}}. \tag{4.25}$$

In order to deduce this bound, we used Bernstein's inequality, and the non-periodic analogue of (4.14).

For completeness, we recall Bernstein's inequality [106]. Namely, if $1 \leq p \leq q \leq \infty$, and if $f \in L^p(\mathbb{R}^2)$ satisfies $supp\,\widehat{f} \subseteq \{|\xi| \sim N\}$, then one has:

$$\|f\|_{L^q_x} \lesssim N^{\frac{2}{p}-\frac{2}{q}} \|f\|_{L^p_x}. \tag{4.26}$$

We interpolate between (4.23) and (4.25) and the Corollary follows.

□

As in the previous chapter, we will have to work with the Fourier transform of $\chi = \chi_{[t_0,t_0+\delta]}(t)$, the characteristic function of the time interval $[t_0, t_0+\delta]$. We treat this issue as before. Let us briefly recall that, given $\phi \in C_0^\infty(\mathbb{R})$ with $0 \leq \phi \leq 1$, $\int_{\mathbb{R}} \phi(t)\,dt = 1$, and $\lambda > 0$, we define: $\phi_\lambda(t) := \frac{1}{\lambda}\,\phi(\frac{t}{\lambda})$. Given a scale $N > 1$, we write:

$$\chi(t) = a(t) + b(t), \text{ for } a := \chi * \phi_{N^{-1}}. \tag{4.27}$$

We recall Lemma 8.2. of [31], by which one has the estimate:

$$\|a(t)f\|_{X^{0,\frac{1}{2}+}} \lesssim N^{0+} \|f\|_{X^{0,\frac{1}{2}+}}. \tag{4.28}$$

(The implied constant here is independent of N.)

On the other hand, for any $M \in (1,+\infty)$, one obtains by Young's Inequality:

$$\|b\|_{L^M_t} \leq C(M, \Phi).$$

If we now define:

$$b_1(t) := \int_{\mathbb{R}} |\hat{b}(\tau)| e^{it\tau} d\tau. \tag{4.29}$$

One also has, as before:

$$\|b_1\|_{L^M_t} \leq C(M,\Phi). \tag{4.30}$$

We will frequently use the following consequence of Proposition 4.2.4

Proposition 4.2.6. *(Improved Strichartz Estimate with rough cut-off in time) Let $u,v \in X^{0,\frac{1}{2}+}(\mathbb{R}^2 \times \mathbb{R})$ satisfy the assumptions of Proposition 4.2.4. Suppose that $N_1 \gtrsim N$. Let u_1, v_1 be given by:*

$$\widetilde{u_1} := |(\chi u)\widetilde{\ }|, \widetilde{v_1} := |\widetilde{v}\,|.$$

Then one has:

$$\|u_1 v_1\|_{L^2_{t,x}} \lesssim \frac{N_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}-}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \tag{4.31}$$

The same bound holds if

$$\widetilde{u_1} := |\widetilde{u}\,|, \widetilde{v_1} := |(\chi v)\widetilde{\ }|.$$

Proposition 5.2.5 follows from Proposition 4.2.4, Corollary 4.2.5, the decomposition (4.27), and the estimates associated to this decomposition. We omit the details of the proof. An analogous statement is proved in one dimension in [96]. The only difference is that on $\mathbb{R}^2$, the coefficient on the right-hand side of (4.23) is $\frac{N_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}$, instead of $\frac{1}{N_1^{\frac{1}{2}}}$, and hence we obtain the coefficient $\frac{N_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}-}}$ on the right-hand side of (4.31).

We also must consider estimates on the product uv, when u and v are localized in dyadic annuli as before, but when we no longer assume that $N_1 \gg N_2$.

By using Hölder's inequality and (4.20), it follows that:

$$\|uv\|_{L^2_{t,x}} \leq \|u\|_{L^4_{t,x}} \|v\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \tag{4.32}$$

We note that (4.25) still holds. We now interpolate between (4.25) and (4.32) to deduce:

$$\|uv\|_{L^{2+}_t L^2_x} \lesssim N_1^{0+} N_2^{0+} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \tag{4.33}$$

An additional form of a bilinear Strichartz Estimate that we will have to use will be the following bound, which was first observed in [38]:

Proposition 4.2.7. *(Angular Improved Strichartz Estimate) Let $0 < N_1 \leq N_2$ be dyadic integers, and suppose $\theta_0 \in (0,1)$. Suppose $v_j \in X^{0,\frac{1}{2}+}, j = 1,2$ satisfy: supp$\widehat{v_j} \subseteq \{|\xi| \sim N_j\}$. Then the function F defined by:*

$$F(t,x) := \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it(\tau_1+\tau_2)+i\langle x,\xi_1+\xi_2\rangle} \chi_{|\cos\angle(\xi_1,\xi_2)|\leq\theta_0} \widetilde{v_1}(\xi_1,\tau_1)\widetilde{v_2}(\xi_2,\tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2$$

obeys the bound:

$$\|F\|_{L^2_{t,x}} \lesssim \theta_0^{\frac{1}{2}} \|v_1\|_{X^{0,\frac{1}{2}+}} \|v_2\|_{X^{0,\frac{1}{2}+}} \tag{4.34}$$

For the proof of Proposition 4.2.7, we refer the reader to the proof of Lemma 8.2. in [38].

We record the 2D version of the *Double Mean Value Theorem*:

Proposition 4.2.8. *Let $f \in C^2(\mathbb{R})$. Suppose that $x, \eta, \mu \in \mathbb{R}^2$ are such that: $|\eta|, |\mu| \ll |x|$. Then, one has:*

$$|f(x+\eta+\mu) - f(x+\eta) - f(x+\mu) + f(x)| \lesssim |\eta||\mu|\|\nabla^2 f(x)\|. \tag{4.35}$$

The proof of Proposition 4.2.8 follows from the standard Mean Value Theorem.

4.3 The Hartree equation on $\mathbb{T}^2$

4.3.1 Definition of the $\mathcal{D}$-operator

As in our previous work [97, 96], we want to define an *upside-down I operator*. We start by defining an appropriate multiplier:

Suppose $N > 1$ is given. Let $\theta : \mathbb{Z}^2 \to \mathbb{R}$ be given by:

$$\theta(n) := \begin{cases} \left(\frac{|n|}{N}\right)^s, \text{if } |n| \geq N \\ 1, \text{ if } |n| \leq N \end{cases} \tag{4.36}$$

Then, if $f : \mathbb{T}^2 \to \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(n) := \theta(n)\hat{f}(n). \tag{4.37}$$

We observe that:

$$\|\mathcal{D}f\|_{L^2} \lesssim_s \|f\|_{H^s} \lesssim_s N^s \|\mathcal{D}f\|_{L^2}. \tag{4.38}$$

Our goal is to then estimate $\|\mathcal{D}u(t)\|_{L^2}$, from which we can estimate $\|u(t)\|_{H^s}$ by (4.38). In order to do this, we first need to have good local-in-time bounds.

4.3.2 Local-in-time bounds

Let u denote the global solution to (4.1) on $\mathbb{T}^2$. One then has:

Proposition 4.3.1. *(Local-in-time bounds for the Hartree equation on $\mathbb{T}^2$) There exist $\delta = \delta(s, E(\Phi), M(\Phi)), C = C(s, E(\Phi), M(\Phi)) > 0$, which are continuous in energy and mass, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{C}$ such that:*

$$v|_{[t_0, t_0+\delta]} = u|_{[t_0, t_0+\delta]}. \tag{4.39}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{4.40}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{4.41}$$

Proposition 4.3.1 is similar to local-in-time bounds we had to prove in Chapters

2 and 3. Since we are working in two dimensions, the proof is going to be a little different. Our proof of Proposition 4.3.1 is similar to the proof of Theorem 2.7. in Chapter V of [15]. For completeness, we present it in the Appendix of this chapter.

As before, Proposition 4.3.1 implies the following:

Proposition 4.3.2. *(Approximation Lemma for the Hartree equation on $\mathbb{T}^2$)*

If Φ satisfies:

$$\begin{cases} iu_t + \Delta u = (V * |u|^2)u, \\ u(x,0) = \Phi(x). \end{cases} \tag{4.42}$$

and if the sequence $(u^{(n)})$ satisfies:

$$\begin{cases} iu_t^{(n)} + \Delta u^{(n)} = (V * |u^{(n)}|^2)u^{(n)}, \\ u^{(n)}(x,0) = \Phi_n(x). \end{cases} \tag{4.43}$$

where $\Phi_n \in C^\infty(\mathbb{T}^2)$ and $\Phi_n \xrightarrow{H^s} \Phi$, then, one has for all t:

$$u^{(n)}(t) \xrightarrow{H^s} u(t).$$

The mentioned approximation Lemma allows us to work with smooth solutions and pass to the limit in the end. Namely, we note that if we take initial data Φ_n as earlier, then $u^{(n)}(t)$ will belong to $H^\infty(\mathbb{T}^2)$ for all t. This allows us to rigorously justify all of our calculations. Now, we want to argue by density. For this, we first need to know that energy and mass are continuous on H^1 The fact that mass is continuous on H^1 is obvious. To see that energy is continuous on H^1, let $1 = \frac{1}{1+} + \frac{1}{M}$. Then, by Hölder's inequality, Young's inequality, and (4.7), we obtain:

$$|\int (V * (u_1 u_2)) u_3 u_4 dx| \leq \|V * (u_1 u_2)\|_{L^{1+}_x} \|u_3 u_4\|_{L^M_x}$$

$$\leq \|V\|_{L^1_x} \|u_1\|_{L^{2+}_x} \|u_2\|_{L^{2+}_x} \|u_3\|_{L^{2M}_x} \|u_4\|_{L^{2M}_x}$$

$$\lesssim \|u_1\|_{H^1}\|u_2\|_{H^1}\|u_3\|_{H^1}\|u_4\|_{H^1}. \tag{4.44}$$

Continuity of energy on H^1 follows from (4.44).

Now, by continuity of mass,energy, and the H^s norm on H^s, it follows that:

$$M(\Phi_n) \to M(\Phi),\ E(\Phi_n) \to E(\Phi),\ \|\Phi_n\|_{H^s} \to \|\Phi\|_{H^s}.$$

Suppose that we knew that Theorem 4.1.1 were true in the case of smooth solutions. Then, for all $t \in \mathbb{R}$, it would follow that:

$$\|u^{(n)}(t)\|_{H^s} \leq C(s,k,E(\Phi_n),M(\Phi_n))(1+|t|)^{2s+}\|\Phi_n\|_{H^s},$$

The claim for u would now follow by applying the continuity properties of C and the Approximation Lemma. So, from now on, we can work with $\Phi \in C^\infty(\mathbb{T}^2)$.

4.3.3 A higher modified energy and an iteration bound

As in [97, 96], we let:

$$E^1(u(t)) := \|\mathcal{D}u(t)\|_{L^2}^2.$$

Arguing as in [97, 96], we obtain that for some $c \in \mathbb{R}$, one has:

$$\frac{d}{dt}E^1(u(t)) = ic \sum_{n_1+n_2+n_3+n_4=0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - ((\theta(n_4))^2\big)$$

$$\widehat{V}(n_3+n_4)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) \tag{4.45}$$

As in the previous works, we consider the *higher modified energy*:

$$E^2(u) := E^1(u) + \lambda_4(M_4;u) \tag{4.46}$$

The quantity M_4 will be determined soon.

The modified energy E^2 is obtained by adding a "multilinear correction" to the modified energy E^1 we considered earlier. In order to find $\frac{d}{dt}E^2(u)$, we need to find $\frac{d}{dt}\lambda_4(M_4;u)$. If we fix a multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4;u) =$$

$$-i\lambda_4(M_4(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2);u)$$

$$-i \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \big[M_4(n_{123},n_4,n_5,n_6)\widehat{V}(n_1+n_2)$$

$$-M_4(n_1,n_{234},n_5,n_6)\widehat{V}(n_2+n_3) + M_4(n_1,n_2,n_{345},n_6)\widehat{V}(n_3+n_4)$$

$$-M_4(n_1,n_2,n_3,n_{456})\widehat{V}(n_4+n_5)\big]\widehat{u}(n_1)\widehat{\overline{u}}(n_2)\widehat{u}(n_3)\widehat{\overline{u}}(n_4)\widehat{u}(n_5)\widehat{\overline{u}}(n_6) \tag{4.47}$$

We can compute that for $(n_1,n_2,n_3,n_4) \in \Gamma_4$, one has:

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 2n_{12}\cdot n_{14} \tag{4.48}$$

We notice that the numerator vanishes not only when $n_{12} = n_{14} = 0$, but also when n_{12} and n_{14} are orthogonal. Hence, on Γ_4, it is possible for $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2$ to vanish, but for $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2$ to be non-zero. Consequently, unlike in our previous work on the 1D Hartree equation in Chapters 2 and 3, we can't cancel the whole quadrilinear term in (4.45). We remedy this by canceling the *non-resonant part* of the quadrilinear term. A similar technique was used in [38]. More precisely, given $\beta_0 \ll 1$, which we determine later, we decompose:

$$\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.$$

Here, the set Ω_{nr} of *non-resonant* frequencies is defined by:

$$\Omega_{nr} := \{(n_1,n_2,n_3,n_4) \in \Gamma_4; n_{12}, n_{14} \neq 0, |cos\angle(n_{12},n_{14})| > \beta_0\} \tag{4.49}$$

and the set Ω_r of *resonant* frequencies Ω_r is defined to be its complement in Γ_4.

We now define the multiplier M_4 by:

$$M_4(n_1,n_2,n_3,n_4) := \begin{cases} c\frac{((\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2)}{|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2}\widehat{V}(n_3+n_4)\,, \text{if } (n_1,n_2,n_3,n_4) \in \Omega_{nr} \\ 0, \text{ if } (n_1,n_2,n_3,n_4) \in \Omega_r \end{cases} \tag{4.50}$$

Let us now define the multiplier M_6 on Γ_6 by:

$$M_6(n_1,n_2,n_3,n_4,n_5,n_6) := M_4(n_{123},n_4,n_5,n_6)\widehat{V}(n_1+n_2) - M_4(n_1,n_{234},n_5,n_6)\widehat{V}(n_2+n_3) +$$

$$+M_4(n_1,n_2,n_{345},n_6)\widehat{V}(n_3+n_4) - M_4(n_1,n_2,n_3,n_{456})\widehat{V}(n_4+n_5) \tag{4.51}$$

We now use (4.45) and (4.47), and the construction of M_4 and M_6 to deduce that [1]:

$$\frac{d}{dt}E^2(u) =$$

$$\sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})| \leq \beta_0} \big((\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2\big)\widehat{V}(n_3+n_4)$$

$$\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)+$$

$$+ \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_6(n_1,n_2,n_3,n_4,n_5,n_6)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6)$$

$$=: I + II \tag{4.52}$$

[1]Since $(\theta(n_1))^2-(\theta(n_2))^2+(\theta(n_3))^2-(\theta(n_4))^2 = 0$ whenever $n_{12}=0$ or $n_{14}=0$, the terms where $n_{12}=0$ or $n_{14}=0$ don't contribute to the first sum. We henceforth don't have to worry about defining the quantity $\cos(0,\cdot)$

Before we proceed, we need to prove pointwise bounds on the multiplier M_4. In order to do this, let $(n_1, n_2, n_3, n_4) \in \Gamma_4$ be given. We dyadically localize the frequencies, i.e, we find dyadic integers N_j s.t. $|n_j| \sim N_j$. We then order the N_j's to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. We slightly abuse notation by writing $\theta(N_j^*)$ for $\theta(N_j^*, 0)$.

Lemma 4.3.3. *With notation as above, the following bound holds:*

$$M_4 = O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{4.53}$$

Proof. By construction of the set Ω_{nr}, and by the fact that $|\widehat{V}| \lesssim 1$, we note that:

$$|M_4| \lesssim \frac{|(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2|}{|n_{12}||n_{14}|\beta_0} \tag{4.54}$$

Let us assume, without loss of generality, that:

$$|n_1| \geq |n_2|, |n_3|, |n_4|, \text{ and } |n_{12}| \geq |n_{14}|. \tag{4.55}$$

We now have to consider three cases:

Case 1: $|n_1| \sim |n_{12}| \sim |n_{14}|$

In this Case, one has:

$$M_4 = O\big(\frac{1}{\beta_0}\frac{(\theta(n_1))^2}{|n_1|^2}\big) = O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big)$$

Case 2: $|n_1| \sim |n_{12}| \gg |n_{14}|$

We use the *Mean Value Theorem*, and monotonicity properties of the function $\frac{(\theta(n))^2}{|n|}$ to deduce:

$$(\theta(n_1))^2 - (\theta(n_4))^2 = (\theta(n_1))^2 - (\theta(n_1 - n_{14}))^2 = O\big(|n_{14}|\frac{(\theta(n_1))^2}{|n_1|}\big) \tag{4.56}$$

$$(\theta(n_2))^2 - (\theta(n_3))^2 = (\theta(n_3 + n_{14}))^2 - (\theta(n_3))^2 =$$

$$O\big(|n_{14}| \sup_{N \leq |z| \lesssim |n_1|} \frac{(\theta(z))^2}{|z|}\big) = O\big(|n_{14}| \frac{(\theta(n_1))^2}{|n_1|}\big). \tag{4.57}$$

Using (4.54), (4.56), (4.57), and the fact that $|n_{12}| \sim |n_1|$, it follows that:

$$M_4 = O\big(\frac{(\theta(n_1))^2}{|n_1|^2 \beta_0}\big) = O\big(\frac{1}{\beta_0} \frac{1}{(N_1^*)^2} \theta(N_1^*)\theta(N_2^*)\big).$$

Case 3: $|n_1| \gg |n_{12}|, |n_{14}|$

We write:

$$(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = (\theta(n_1))^2 - (\theta(n_1 - n_{12}))^2 + (\theta(n_1 - n_{12} - n_{14}))^2 - (\theta(n_1 - n_{14}))^2$$

By using the *Double Mean-Value Theorem* (4.35), it follows that this expression is $O\big(\frac{(\theta(n_1))^2}{|n_1|^2} |n_{12}||n_{14}|\big)$.

Consequently:

$$M_4 = O\big(\frac{1}{\beta_0} \frac{1}{(N_1^*)^2} \theta(N_1^*)\theta(N_2^*)\big).$$

The Lemma now follows.

$\square$

Let us choose:

$$\beta_0 \sim \frac{1}{N} \tag{4.58}$$

The reason why we choose such a β_0 will become clear later. For details, see Remark 4.3.6.

Hence Lemma 4.3.3 implies:

$$M_4 = O\big(\frac{N}{(N_1^*)^2} \theta(N_1^*)\theta(N_2^*)\big). \tag{4.59}$$

The bound from (4.59) allows us to deduce the equivalence of E^1 and E^2. We have the following bound:

Proposition 4.3.4. *One has that:*

$$E^1(u) \sim E^2(u). \tag{4.60}$$

Here, the constant is independent of N as long as N is sufficiently large.

Proof. We estimate $E^2(u) - E^1(u) = \lambda_4(M_4; u)$. By construction, one has:

$$|\lambda_4(M_4; u)| \lesssim \sum_{n_1+n_2+n_3+n_4=0} |M_4(n_1, n_2, n_3, n_4)||\widehat{u}(n_1)||\widehat{\bar{u}}(n_2)||\widehat{u}(n_3)||\widehat{\bar{u}}(n_4)|$$

Let us dyadically localize the n_j, i.e., we find N_j dyadic integers such that $|n_j| \sim N_j$. We consider the case when $N_1 \geq N_2 \geq N_3 \geq N_4$. The other cases are analogous. We know that the nonzero contributions occur when:

$$N_1 \sim N_2 \gtrsim N \tag{4.61}$$

Let us denote the corresponding contribution to $\lambda_4(M_4; u)$ by I_{N_1,N_2,N_3,N_4}. We use Parseval's identity and (4.59) to deduce that:

$$|I_{N_1,N_2,N_3,N_4}| \lesssim \sum_{n_1+n_2+n_3+n_4=0, |n_j| \sim N_j} \frac{N}{N_1^2} |\widehat{\mathcal{D}u}_{N_1}(n_1)||\widehat{\mathcal{D}\bar{u}}_{N_2}(n_2)||\widehat{u}_{N_3}(n_3)||\widehat{\bar{u}}_{N_4}(n_4)|$$

Let us define $F_j : j = 1, \ldots, 4$ by:

$$\widehat{F_1} := |\widehat{\mathcal{D}u_{N_1}}|, \widehat{F_2} := |\widehat{\mathcal{D}u_{N_2}}|, \widehat{F_3} := |\widehat{u}_{N_3}|, \widehat{F_4} := |\widehat{u}_{N_4}|.$$

By Parseval's identity, one has:

$$|I_{N_1,N_2,N_3,N_4}| \lesssim \frac{N}{N_1^2} \int_{\mathbb{T}^2} F_1 \overline{F_2} F_3 \overline{F_4} dx$$

which by an $L^2_x, L^2_x, L^\infty_x, L^\infty_x$ Hölder's inequality is:

$$\lesssim \frac{N}{N_1^2}\|F_1\|_{L^2_x}\|F_2\|_{L^2_x}\|F_3\|_{L^\infty_x}\|F_4\|_{L^\infty_x}$$

Furthermore, we use Sobolev embedding, and the fact that taking absolute values in the Fourier transform doesn't change Sobolev norms to deduce that this expression is:

$$\lesssim \frac{N}{N_1^2}\|F_1\|_{L^2_x}\|F_2\|_{L^2_x}\|F_3\|_{H^{1+}_x}\|F_4\|_{H^{1+}_x} \lesssim \frac{N}{N_1^2}\|\mathcal{D}u_{N_1}\|_{L^2_x}\|\mathcal{D}u_{N_2}\|_{L^2_x}\|u_{N_3}\|_{H^{1+}_x}\|u_{N_4}\|_{H^{1+}_x} \lesssim$$

$$\lesssim \frac{N}{N_1^{2-}}\|\mathcal{D}u\|^2_{L^2_x}\|u\|^2_{H^1_x} \lesssim \frac{N}{N_1^{2-}}E^1(u)$$

Here, we used the fact that $\|u\|_{H^1_x} \lesssim 1$.

We now recall (4.61) and sum in the N_j to deduce that:

$$|E^2(u) - E^1(u)| = |\lambda_4(M_4; u)| \lesssim \frac{1}{N^{1-}}E^1(u).$$

The claim now follows.

□

Let $\delta > 0, v$ be as in Proposition 4.3.1. For $t_0 \in \mathbb{R}$, we are interested in estimating:

$$E^2(u(t_0+\delta)) - E^2(u(t_0)) = \int_{t_0}^{t_0+\delta} \frac{d}{dt}E^2(u(t))dt = \int_{t_0}^{t_0+\delta} \frac{d}{dt}E^2(v(t))dt$$

The iteration bound that we will show is:

Lemma 4.3.5. *For all $t_0 \in \mathbb{R}$, one has:*

$$\left|E^2(u(t_0+\delta)) - E^2(u(t_0))\right| \lesssim \frac{1}{N^{1-}}E^2(u(t_0)).$$

Arguing as in Chapter 2, Theorem 4.1.1 will follow from Lemma 4.3.5. Let us prove Lemma 4.3.5.

Proof. (of Lemma 4.3.5)

Let us WLOG consider $t_0 = 0$. The general claim will follow by time translation, and the fact that all of the implied constants are uniform in time. Let v be the function constructed in Proposition 4.3.1, corresponding to $t_0 = 0$.

By (4.52), and with notation as in this equation, we need to estimate:

$$\int_0^\delta \Big(\sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})| \leq \beta_0}$$

$$\big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big) \widehat{V}(n_3+n_4)\widehat{v}(n_1)\widehat{\bar{v}}(n_2)\widehat{v}(n_3)\widehat{\bar{v}}(n_4) +$$

$$+ \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_6(n_1,n_2,n_3,n_4,n_5,n_6)\widehat{v}(n_1)\widehat{\bar{v}}(n_2)\widehat{v}(n_3)\widehat{\bar{v}}(n_4)\widehat{v}(n_5)\widehat{\bar{v}}(n_6)\Big) dt =$$

$$= \int_0^\delta I dt + \int_0^\delta II dt =: A + B.$$

We now have to estimate A and B separately. Throughout our calculations, let us denote by $\chi = \chi(t) = \chi_{[0,\delta]}(t)$.

Estimate of A (Quadrilinear Terms)

By symmetry, we can consider WLOG the contribution when:

$$|n_1| \geq |n_2|, |n_3|, |n_4|, \text{and } |n_2| \geq |n_4|.$$

We note that when all $|n_j| \leq N$, one has: $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2 = 0$. Hence, we need to consider the contribution in which one has:

$$|n_1| > N, |cos\angle(n_{12}, n_{14})| \leq \beta_0.$$

We dyadically localize the frequencies: $|n_j| \sim N_j; j = 1, \ldots, 4$. We order the N_j to obtain $N_j^* \geq N_2^* \geq N_3^* \geq N_4^*$. Since $n_1 + n_2 + n_3 + n_4 = 0$, we know that:

$$N_1^* \sim N_2^* \gtrsim N. \tag{4.62}$$

Let us note that $N_1 \sim N_2$. Namely, if it were the case that: $N_1 \gg N_2$, then, one would also have: $N_1 \gg N_4$, and the vectors n_{12} and n_{14} would form a very small angle. Hence, $cos\angle(n_{12}, n_{14})$ would be close to 1, which would be a contradiction to the assumption that $|cos\angle(n_{12}, n_{14})| \leq \beta_0$. Consequently:

$$N_1 \sim N_2 \sim N_1^* \gtrsim N. \tag{4.63}$$

We denote the corresponding contribution to A by A_{N_1,N_2,N_3,N_4}. In other words:

$$A_{N_1,N_2,N_3,N_4} :=$$
$$\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})| \leq \beta_0} \big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)\widehat{V}(n_3+n_4)$$
$$\widehat{v}_{N_1}(n_1)\widehat{\overline{v}}_{N_2}(n_2)\widehat{v}_{N_3}(n_3)\widehat{\overline{v}}_{N_4}(n_4)dt.$$

Arguing analogously as in the proof of Lemma 4.3.3, it follows that for the n_j that occur in the above sum, one has:

$$\big((\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2\big)\widehat{V}(n_3+n_4) = O\big(|n_{12}||n_{14}|\frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}\big) \tag{4.64}$$

By (4.63), it follows that $|n_3|, |n_4| \lesssim N_3^*$. Consequently:

$$|n_{12}| = |n_{34}| \leq |n_3| + |n_4| \lesssim N_3^*.$$

One also knows that:

$$|n_{14}| \leq |n_1| + |n_4| \lesssim N_1^*.$$

Substituting the last two inequalities into the multiplier bound (4.64), and using Parseval's identity in time, it follows that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \sum_{n_1+n_2+n_3+n_4=0, |cos\angle(n_{12},n_{14})|\leq \beta_0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} N_3^* N_1^* \frac{\theta(N_1^*)\theta(N_2^*)}{(N_1^*)^2}$$

$$|\widetilde{v}_{N_1}(n_1,\tau_1)||\widetilde{\overline{v}}_{N_2}(n_2,\tau_2)||\widetilde{v}_{N_3}(n_3,\tau_3)||(\chi\bar{v})\widetilde{\ }_{N_4}(n_4,\tau_4)|d\tau_j$$

$$\lesssim \frac{1}{N_1^*} \sum_{n_1+n_2+n_3+n_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} |(\mathcal{D}v)\widetilde{\ }_{N_1}(n_1,\tau_1)||(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(n_2,\tau_2)||(\nabla v)\widetilde{\ }_{N_3}(n_3,\tau_3)||(\chi\bar{v})\widetilde{\ }_{N_4}(n_4,\tau_4)|d\tau_j$$

Let us define $F_j; j = 1, \ldots, 4$ by:

$$\widetilde{F_1} := |(\mathcal{D}v)\widetilde{\ }_{N_1}|, \widetilde{F_2} := |(\mathcal{D}v)\widetilde{\ }_{N_2}|, \widetilde{F_3} := |(\nabla v)\widetilde{\ }_{N_3}|, \widetilde{F_4} := |(\chi v)\widetilde{\ }_{N_4}|$$

Consequently, by Parseval's identity:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*} \int_{\mathbb{R}} \int_{\mathbb{T}^2} F_1 \overline{F_2} F_3 \overline{F_4} dx dt$$

By using an $L^4_{t,x}, L^4_{t,x}, L^{4+}_{t,x}, L^{4-}_{t,x}$ Hölder inequality, the corresponding term is:

$$\lesssim \frac{1}{N_1^*} \|F_1\|_{L^4_{t,x}} \|F_2\|_{L^4_{t,x}} \|F_3\|_{L^{4+}_{t,x}} \|F_4\|_{L^{4-}_{t,x}}.$$

By using (4.8), (4.12), (4.11), and the fact that taking absolute values in the spacetime Fourier transforms doesn't change the $X^{s,b}$ norm, it follows that this term is:

$$\lesssim \frac{1}{N_1^*} \|\mathcal{D}v_{N_1}\|_{X^{0+,\frac{1}{2}+}} \|\mathcal{D}v_{N_2}\|_{X^{0+,\frac{1}{2}+}} \|v_{N_3}\|_{X^{1+,\frac{1}{2}+}} \|(\chi v)_{N_4}\|_{X^{0+,\frac{1}{2}-}}$$

By using frequency localization and Lemma 4.2.1), this expression is:

$$\lesssim \frac{1}{(N_1^*)^{1-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^2_{X^{1,\frac{1}{2}+}} \lesssim \frac{1}{(N_1^*)^{1-}} E^1(\Phi).$$

In the last inequality, we used Proposition 4.3.1. By using the previous inequality, and by recalling (4.60), it follows that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{(N_1^*)^{1-}} E^2(\Phi) \tag{4.65}$$

Using (4.65), summing in the N_j, and using (4.62) to deduce that:

$$|A| \lesssim \frac{1}{N^{1-}} E^2(\Phi). \tag{4.66}$$

Estimate of B (Sextilinear Terms)

Let us consider just the first term in B coming from the summand $M_4(n_{123}, n_4, n_5, n_6)$ in the definition of M_6. The other terms are bounded analogously. In other words, we want to estimate:

$$B^{(1)} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} M_4(n_{123}, n_4, n_5, n_6)\widehat{(v\bar{v}v)}(n_1+n_2+n_3)\widehat{\bar{v}}(n_4)\widehat{v}(n_5)\widehat{\bar{v}}(n_6)dt$$

We now dyadically localize n_{123}, n_4, n_5, n_6, i.e., we find $N_j; j = 1, \ldots, 4$ such that:

$$|n_{123}| \sim N_1, |n_4| \sim N_2, |n_5| \sim N_3, |n_6| \sim N_4.$$

Let us define:

$$B^{(1)}_{N_1,N_2,N_3,N_4} := \int_0^\delta \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0}$$

$$M_4(n_{123}, n_4, n_5, n_6)\widehat{(v\bar{v}v)}_{N_1}(n_1 + n_2 + n_3)\widehat{\bar{v}}_{N_2}(n_4)\widehat{v}_{N_3}(n_5)\widehat{\bar{v}}_{N_4}(n_6)dt.$$

We now order the N_j to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. As before, we know the following localization bound:

$$N_1^* \sim N_2^* \gtrsim N. \tag{4.67}$$

In order to obtain a bound on the wanted term, we have to consider two cases.

Case 1: $N_1 = N_1^*$ or $N_1 = N_2^*$.

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$

Case 1:

It suffices to consider the case when $N_1 = N_1^*, N_2 = N_2^*, N_3 = N_3^*, N_4 = N_4^*$. The other cases are analogous. We use (4.59) and Parseval's identity to obtain that:

$$|B^{(1)}_{N_1,N_2,N_3,N_4}| \lesssim$$

$$\sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} \theta(N_1^*)\theta(N_2^*)|(v\bar{v}v)\widetilde{\ }_{N_1}(n_1+n_2+n_3, \tau_1+\tau_2+\tau_3)|$$

$$|\widetilde{\bar{v}}_{N_2}(n_4,\tau_4)||(\chi v)\widetilde{\ }_{N_3}(n_5,\tau_5)||\widetilde{\bar{v}}_{N_4}(n_6,\tau_6)|d\tau_j.$$

Since $|(v\bar{v}v)\widetilde{\ }_{N_1}| \leq |(v\bar{v}v)\widetilde{\ }|$, and since $\theta(N_1^*) \sim \theta(n_1+n_2+n_3) \lesssim \theta(n_1)+\theta(n_2)+\theta(n_3)$, by symmetry, it follows that we just have to bound:

$$K_{N_1,N_2,N_3,N_4} :=$$

$$\sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} \theta(n_1)|\widetilde{v}(n_1,\tau_1)||\widetilde{\bar{v}}(n_2,\tau_2)||\widetilde{v}(n_3,\tau_3)|$$

$$\theta(N_2)|\widetilde{\bar{v}}_{N_2}(n_4,\tau_4)||(\chi v)\widetilde{\ }_{N_3}(n_5,\tau_5)||\widetilde{\bar{v}}_{N_4}(n_4,\tau_4)|d\tau_j \lesssim$$

$$\sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} |(\mathcal{D}v)\widetilde{\ }(n_1,\tau_1)||\widetilde{\bar{v}}(n_2,\tau_2)||\widetilde{v}(n_3,\tau_3)|$$

$$|(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(n_4,\tau_4)||(\chi v)\widetilde{\ }_{N_3}(n_5,\tau_5)||\widetilde{\bar{v}}_{N_4}(n_4,\tau_4)|d\tau_j.$$

Let us define the functions $F_j; j = 1,\ldots,6$ by:

$$\widetilde{F_1} := |(\mathcal{D}v)\widetilde{\ }|, \widetilde{F_2} = \widetilde{F_3} := |\widetilde{v}|, \widetilde{F_4} := |(\mathcal{D}v)\widetilde{\ }_{N_2}|, \widetilde{F_5} := |(\chi v)\widetilde{\ }_{N_3}|, \widetilde{F_1} := |\widetilde{v}_{N_4}|$$

For $M \gg 1$, we use an $L^2_{t,x}, L^M_{t,x}, L^M_{t,x}, L^{4+}_{t,x}, L^{4-}_{t,x}, L^M_{t,x}$ Hölder inequality to deduce that:

$$K_{N_1,N_2,N_3,N_4} \lesssim \frac{N}{(N_1^*)^2} \|F_1\|_{L^2_{t,x}} \|F_2\|_{L^M_{t,x}} \|F_3\|_{L^M_{t,x}} \|F_4\|_{L^{4+}_{t,x}} \|F_5\|_{L^{4-}_{t,x}} \|F_6\|_{L^M_{t,x}}.$$

By using (4.13), (4.12), (4.11), and the fact that taking absolute values in the spacetime Fourier transform leaves the $X^{s,b}$ norm invariant, it follows that the previous expression is:

$$\lesssim \frac{N}{(N_1^*)^2} \|\mathcal{D}v\|_{X^{0,0}} \|v\|_{X^{1,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}} \|\mathcal{D}v_{N_2}\|_{X^{0+,\frac{1}{2}-}} \|\chi v_{N_3}\|_{X^{0+,\frac{1}{2}-}} \|v_{N_4}\|_{X^{1,\frac{1}{2}+}}$$

We use frequency localization and Lemma 4.2.1 to deduce that this is:

$$\lesssim \frac{N}{(N_1^*)^2} \|\mathcal{D}v\|_{X^{0,0}} \|v\|^2_{X^{1,\frac{1}{2}+}} (N_2^{0+} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}}) \|v_{N_3}\|_{X^{0+,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}}.$$

$$\lesssim \frac{N}{(N_1^*)^{2-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^4_{X^{1,\frac{1}{2}+}} \lesssim \frac{N}{(N_1^*)^{2-}} E^1(\Phi). \tag{4.68}$$

In the last inequality, we used Proposition 4.3.1.

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$.

Let us assume that:

$$N_3 \gtrsim N_2 \gtrsim N_1 \gtrsim N_4.$$

The other cases are dealt with similarly.

Arguing similarly as in Case 1, it follows that:

$$|B^{(1)}_{N_1,N_2,N_3,N_4}| \lesssim$$

$$\sum_{n_1+\cdots+n_6=0} \int_{\tau_1+\cdots+\tau_6=0} \frac{N}{(N_1^*)^2} |\widetilde{v}(n_1,\tau_1)| |\widetilde{\overline{v}}(n_2,\tau_2)| |\widetilde{v}(n_3,\tau_3)|$$

$$|(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(n_4,\tau_4)||(\chi\mathcal{D}v)\widetilde{\ }_{N_3}(n_5,\tau_5)||\widetilde{\widetilde{v}}_{N_4}(n_6,\tau_6)|d\tau_j$$

We now use an $L^M_{t,x}, L^M_{t,x}, L^M_{t,x}, L^{4+}_{t,x}, L^{4-}_{t,x}, L^2_{t,x}$ Hölder inequality and argue as earlier to see that this term is:

$$\lesssim \frac{N}{(N_1^*)^2}\|v\|^3_{X^{1,\frac{1}{2}+}}\|\mathcal{D}v_{N_2}\|_{X^{0+,\frac{1}{2}+}}\|(\chi\mathcal{D}v)_{N_3}\|_{X^{0+,\frac{1}{2}-}}\|v_{N_4}\|_{X^{0,0}}$$

$$\lesssim \frac{N}{(N_1^*)^{2-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^4_{X^{1,\frac{1}{2}+}} \lesssim \frac{N}{(N_1^*)^{2-}}E^1(\Phi) \tag{4.69}$$

From (4.68), (4.69), and (4.60), it follows that:

$$|B_{N_1,N_2,N_3,N_4}| \lesssim \frac{N}{(N_1^*)^{2-}}E^2(\Phi) \tag{4.70}$$

We now use (4.70), sum in the N_j, and recall (4.67) to deduce that:

$$|B| \lesssim \frac{1}{N^{1-}}E^2(\Phi) \tag{4.71}$$

The Lemma now follows from (4.66) and (4.71).

□

4.3.4 Further remarks on the equation

Remark 4.3.6. *The quantity* β_0 *was chosen as in* (4.58) *in order to get the same decay factor in the quantities* A *and* B*. We note that the quantity* β_0 *only occurred in the bound for* B*, whereas in the bound for* A*, we only used the fact that the terms corresponding to the largest two frequencies in the multiplier* $(\theta(n_1))^2 - (\theta(n_2))^2 + (\theta(n_3))^2 - (\theta(n_4))^2$ *appear with an opposite sign. As we will see, in the non-periodic setting, the quantity* β_0 *will occur both in the bound for* A *and in the bound for* B*. For details, see* (4.104) *and* (4.112)*.*

Remark 4.3.7. *Let us observe that, when* s *is an integer, or when* Φ *is smooth, essentially the same bound as in Theorem 4.1.1 can be proved by using the techniques of [118]. The approach is more complicated due to the presence of the convolution* *potential, but the proof for the cubic NLS can be shown to work for the Hartree equation too. The reason why one uses the fact that s is an integer is because one wants to use exact formulae for the (Fractional) Leibniz Rule for D^s. By using an exact Leibniz Rule, one sees that certain terms which are difficult to estimate are in fact equal to zero. We omit the details here.*

Remark 4.3.8. *The equation* (4.1) *on* $\mathbb{T}^2$ *has non-trivial solutions which have all Sobolev norms uniformly bounded in time. Similarly as on* S^1*, given* $\alpha \in \mathbb{C}$ *and* $n \in \mathbb{Z}^2$*, the function:*

$$u(x,t) := \alpha e^{-i\widehat{V}(0)|\alpha|^2 t} e^{i(\langle n,x\rangle - |n|^2 t)}$$

is a solution to (4.1) *on* $\mathbb{T}^2$ *with initial data* $\Phi = \alpha e^{i\langle n,x\rangle}$*. A similar construction was used in [21] to prove instability properties in Sobolev spaces of negative index.*

4.4 The Hartree equation on $\mathbb{R}^2$

4.4.1 Definition of the $\mathcal{D}$-operator

Let us now consider (4.1) on $\mathbb{R}^2$. The proof of Theorem 4.1.2 will be based on the adaptation of the previous techniques to the non-periodic setting. We start by defining an appropriate *upside-down I-operator*.

Let $N > 1$ be given. Similarly as in the periodic setting, we define $\theta : \mathbb{R}^2 \rightarrow \mathbb{R}$ to be given by:

$$\theta(\xi) := \begin{cases} \left(\frac{|\xi|}{N}\right)^s, \text{if } |\xi| \geq 2N \\ 1, \text{if } |\xi| \leq N \end{cases} \tag{4.72}$$

We then extend θ to all of $\mathbb{R}^2$ so that it is radial and smooth. Arguing similarly as in the 1D setting in Chapter 3, it follows that, for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, one has:

$$\|\nabla\theta(\xi)\| \lesssim \frac{\theta(\xi)}{|\xi|} \tag{4.73}$$

$$\|\nabla^2\theta(\xi)\| \lesssim \frac{\theta(\xi)}{|\xi|^2} \tag{4.74}$$

Then, if $f : \mathbb{R}^2 \to \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(\xi) := \theta(\xi)\hat{f}(\xi). \tag{4.75}$$

We also observe that:

$$\|\mathcal{D}f\|_{L^2} \lesssim_s \|f\|_{H^s} \lesssim_s N^s \|\mathcal{D}f\|_{L^2}. \tag{4.76}$$

4.4.2 Local-in-time bounds

Let u denote the global solution of (4.1) on $\mathbb{R}^2$. As in the periodic setting, our goal is to estimate $\|\mathcal{D}u(t)\|_{L^2}$.

We start by noting:

Proposition 4.4.1. *(Local-in-time bounds for the Hartree equation on $\mathbb{R}^2$) There exist $\delta = \delta(s, E(\Phi), M(\Phi)), C = C(s, E(\Phi), M(\Phi)) > 0$, which are continuous in energy and mass, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ such that:*

$$v|_{[t_0, t_0+\delta]} = u|_{[t_0, t_0+\delta]}. \tag{4.77}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{4.78}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{4.79}$$

Furthermore, we have:

Lemma 4.4.2. *If u satisfies:*

$$\begin{cases} iu_t + \Delta u = (V * |u|^2)u, \\ u(x,0) = \Phi(x). \end{cases} \tag{4.80}$$

and if the sequence $(u^{(n)})$ satisfies:

$$\begin{cases} iu_t^{(n)} + \Delta u^{(n)} = (V * |u^{(n)}|^2)u^{(n)}, \\ u^{(n)}(x,0) = \Phi_n(x). \end{cases} \tag{4.81}$$

where $\Phi_n \in C^\infty(\mathbb{R}^2)$ and $\Phi_n \xrightarrow{H^s} \Phi$, then, one has for all t:

$$u^{(n)}(t) \xrightarrow{H^s} u(t).$$

The proofs of Propositions 4.4.1 and 4.4.2 are analogous to the proofs of Propositions 4.3.1 and 4.3.2. The main point is that all the auxiliary estimates still hold in the non-periodic setting. As before, we can assume WLOG that $\Phi \in \mathcal{S}(\mathbb{R}^2)$. We omit the details.

4.4.3 A higher modified energy and an iteration bound

As in the periodic setting, we will apply the method of *higher modified energies*. We will see that we can obtain better estimates in the non-periodic setting due to the fact that we can apply the *improved Strichartz estimate* (Proposition 4.2.4), and the *angular improved Strichartz estimate* (Proposition 4.2.7).

We start by defining:

$$E^1(u(t)) := \|\mathcal{D}u(t)\|_{L^2}^2.$$

As before, we obtain that for some $c \in \mathbb{R}$, one has:

$$\frac{d}{dt}E^1(u(t) = ic\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \big((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2\big)$$

$$\widehat{V}(\xi_3+\xi_4)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j \tag{4.82}$$

As in the previous works, we consider the *higher modified energy*:

$$E^2(u) := E^1(u) + \lambda_4(M_4; u) \tag{4.83}$$

The quantity M_4 will be determined soon.

For a fixed multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4; u) =$$

$$-i\lambda_4(M_4(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2); u)$$

$$-i\sum_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} \big[M_4(\xi_{123},\xi_4,\xi_5,\xi_6)\widehat{V}(\xi_1+\xi_2)$$

$$-M_4(\xi_1,\xi_{234},\xi_5,\xi_6)\widehat{V}(\xi_2+\xi_3) + M_4(\xi_1,\xi_2,\xi_{345},\xi_6)\widehat{V}(\xi_3+\xi_4)$$

$$-M_4(\xi_1,\xi_2,\xi_3,\xi_{456})\widehat{V}(\xi_4+\xi_5)\big]\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)\widehat{u}(\xi_5)\widehat{\bar{u}}(\xi_6) \tag{4.84}$$

As in the periodic setting, we can compute that for $(\xi_1,\xi_2,\xi_3,\xi_4) \in \Gamma_4$, one has:

$$|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 = 2\xi_{12}\cdot\xi_{14} \tag{4.85}$$

As before, we decompose:

$$\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.$$

Here, the set Ω_{nr} of *non-resonant* frequencies is defined by:

$$\Omega_{nr} := \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4; \xi_{12}, \xi_{14} \neq 0, |cos\angle(\xi_{12}, \xi_{14})| > \beta_0\} \tag{4.86}$$

and the set Ω_r of *resonant* frequencies Ω_r is defined to be its complement in Γ_4.

We now define the multiplier M_4 by:

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) := \begin{cases} c\frac{((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2)}{|\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2}\widehat{V}(\xi_3+\xi_4)\,, \text{if } (\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_{nr} \\ 0, \text{ if } (\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega_r \end{cases} \tag{4.87}$$

Let us now define the multiplier M_6 on Γ_6 by:

$$M_6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) := M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)\widehat{V}(\xi_1+\xi_2) - M_4(\xi_1, \xi_{234}, \xi_5, \xi_6)\widehat{V}(\xi_2+\xi_3) + \tag{4.88}$$

$$+M_4(\xi_1, \xi_2, \xi_{345}, \xi_6)\widehat{V}(\xi_3+\xi_4) - M_4(\xi_1, \xi_2, \xi_3, \xi_{456})\widehat{V}(\xi_4+\xi_5)$$

We now use (4.82) and (4.84), and the construction of M_4 to deduce that [2]:

$$\frac{d}{dt}E^2(u) =$$

$$\int_{\xi_1+\xi_2+\xi_3+\xi_4=0, |cos\angle(\xi_{12},\xi_{14})|\leq\beta_0} \big((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2\big)\widehat{V}(\xi_3+\xi_4)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j +$$

$$+\int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} M_6(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)\widehat{u}(\xi_5)\widehat{\bar{u}}(\xi_6)$$

$$=: I + II \tag{4.89}$$

As before, we need to prove pointwise bounds on the multiplier M_4. Given $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4$, we dyadically localize the frequencies, i.e, we find dyadic integers N_j s.t. $|\xi_j| \sim N_j$. We then order the N_j's to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. We

[2] As in the periodic setting, we recall that $(\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2 = 0$, whenever $\xi_{12} = 0$ or $\xi_{14} = 0$, hence the corresponding terms again don't contribute to the quadrilinear term. Therefore, we don't have to worry about defining the quantity $\cos(0, \cdot)$.

again abuse notation by writing $\theta(N_j^*)$ for $\theta(N_j^*,0)$. One then has:

Lemma 4.4.3. *With notation as above, the following bound holds:*

$$M_4 = O\big(\frac{1}{\beta_0}\frac{1}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{4.90}$$

The proof of Lemma 4.4.3 is analogous to the proof of Lemma 4.3.3 and it will be omitted.

In the non-periodic setting, we will see that we can choose a larger β_0 from which we can get a better bound. Let us choose:

$$\beta_0 \sim \frac{1}{N^{\alpha}}. \tag{4.91}$$

Here, we take $\alpha \in (0,1)$. We determine α precisely later (see (4.116)). For now, we notice:

$$\beta_0 \geq \frac{1}{N}. \tag{4.92}$$

We observe that Lemma 4.4.3 and (4.92) imply:

$$M_4 = O\big(\frac{N}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*)\big). \tag{4.93}$$

The bound from (4.93) allows us to deduce the equivalence of E^1 and E^2. We have the following bound:

Proposition 4.4.4. *One has that:*

$$E^1(u) \sim E^2(u). \tag{4.94}$$

Here, the constant is independent of N as long as N is sufficiently large.

The proof of Proposition 4.4.4 is analogous to the proof of Proposition 4.3.4. We omit the details.

Let $\delta > 0, v$ be as in Proposition 4.4.1. For $t_0 \in \mathbb{R}$, we are interested in estimating:

$$E^2(u(t_0+\delta)) - E^2(u(t_0)) = \int_{t_0}^{t_0+\delta} \frac{d}{dt}E^2(u(t))dt = \int_{t_0}^{t_0+\delta} \frac{d}{dt}E^2(v(t))dt$$

The iteration bound that we will show is:

Lemma 4.4.5. *For all $t_0 \in \mathbb{R}$, one has:*

$$\big|E^2(u(t_0+\delta)) - E^2(u(t_0))\big| \lesssim \frac{1}{N^{\frac{7}{4}-}} E^2(u(t_0)).$$

Arguing as in the case of (4.1) on $\mathbb{T}^2$, Theorem 4.1.2 will follow from Lemma 4.4.5.

We now prove Lemma 4.4.5

Proof. It suffices to consider the case when $t_0 = 0$. As on $\mathbb{T}^2$, we compute that $E^2(u(\delta)) - E^2(u(0))$ equals:

$$\int_0^\delta \Big(\int_{\xi_1+\xi_2+\xi_3+\xi_4=0, |cos\angle(\xi_{12},\xi_{14})|\leq\beta_0} \big((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2\big)\widehat{V}(\xi_3+\xi_4)\widehat{v}(\xi_1)$$

$$\widehat{\overline{v}}(\xi_2)\widehat{v}(\xi_3)\widehat{\overline{v}}(\xi_4)d\xi_j +$$

$$+\int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} M_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\widehat{v}(\xi_1)\widehat{\overline{v}}(\xi_2)\widehat{v}(\xi_3)\widehat{\overline{v}}(\xi_4)\widehat{v}(\xi_5)\widehat{\overline{v}}(\xi_6)d\xi_j\Big)dt =$$

$$= \int_0^\delta I dt + \int_0^\delta II dt =: A + B \tag{4.95}$$

We now have to estimate A and B separately.

Estimate of A (Quadrilinear Terms)

By symmetry, we can consider WLOG the contribution when:

$$|\xi_1| \geq |\xi_2|, |\xi_3|, |\xi_4|, \text{and } |\xi_2| \geq |\xi_4|.$$

Hence, we are considering the contribution in which one has:

$$|\xi_1| > N, |cos\angle(\xi_{12},\xi_{14})| \leq \beta_0.$$

We dyadically localize the frequencies: $|\xi_j| \sim N_j; j = 1, \ldots, 4$. We order the N_j to obtain $N_j^* \geq N_2^* \geq N_3^* \geq N_4^*$. As in the periodic setting, we have:

$$N_1 \sim N_2 \sim N_1^* \gtrsim N \tag{4.96}$$

We denote the corresponding contribution to A by A_{N_1,N_2,N_3,N_4}. In other words:

$$A_{N_1,N_2,N_3,N_4} :=$$
$$\int_0^\delta \int_{\xi_1+\xi_2+\xi_3+\xi_4=0, |cos\angle(\xi_{12},\xi_{14})|\leq\beta_0} \big((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2\big)\widehat{V}(\xi_3+\xi_4)$$
$$\widehat{v}_{N_1}(\xi_1)\widehat{\overline{v}}_{N_2}(\xi_2)\widehat{v}_{N_3}(\xi_3)\widehat{\overline{v}}_{N_4}(\xi_4)d\xi_j dt$$

As in the periodic setting, we have:

$$\big((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2\big)\widehat{V}(\xi_3+\xi_4) = O\big(\frac{N_3^*}{N_1^*}\theta(N_1^*)\theta(N_2^*)\big) \tag{4.97}$$

Using Parseval's identity in time, it follows that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \int_{\xi_1+\xi_2+\xi_3+\xi_4=0, |cos\angle(\xi_{12},\xi_{14})|\leq\beta_0} \frac{N_3^*}{N_1^*}\theta(N_1^*)\theta(N_2^*)$$
$$|(\chi v)\widetilde{\ }_{N_1}(\xi_1,\tau_1)||\widetilde{\overline{v}}_{N_2}(\xi_2,\tau_2)||\widetilde{v}_{N_3}(\xi_3,\tau_3)||\widetilde{\overline{v}}_{N_4}(\xi_4,\tau_4)|d\xi_j d\tau_j$$

We now consider two subcases:

Subcase 1: $N_4 \sim N_1$

We observe that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*}\int_{\tau_1+\tau_2+\tau_3+\tau_4=0}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}|(\mathcal{D}v)\widetilde{\ }_{N_1}(\xi_1,\tau_1)||(\chi\mathcal{D}\overline{v})\widetilde{\ }_{N_2}(\xi_2,\tau_2)||$$

$$(\nabla v)\widetilde{\ }_{N_3}(\xi_3,\tau_3)||\widetilde{\overline{v}}_{N_4}(\xi_4,\tau_4)|d\xi_j d\tau_j$$

Let us define $F_j; j=1,\ldots,4$ by:

$$\widetilde{F_1} := |(\mathcal{D}v)\widetilde{\ }_{N_1}|, \widetilde{F_2} := |(\chi\mathcal{D}v)\widetilde{\ }_{N_2}|, \widetilde{F_3} := |(\nabla v)\widetilde{\ }_{N_3}|, \widetilde{F_4} := |\widetilde{v}_{N_4}| \tag{4.98}$$

Consequently, by Parseval's identity:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*}\int_{\mathbb{R}}\int_{\mathbb{R}^2} F_1\overline{F_2}F_3\overline{F_4}dxdt$$

We use an $L^{4+}_{t,x}, L^{4-}_{t,x}, L^4_{t,x}, L^4_{t,x}$ Hölder inequality, and argue as earlier to deduce that, in this subcase:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*}\|(\mathcal{D}v)_{N_1}\|_{X^{0+,\frac{1}{2}+}}\|(\chi\mathcal{D}v)_{N_2}\|_{X^{0,\frac{1}{2}-}}\|(\nabla v)_{N_3}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{0,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{(N_1^*)^{1-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|_{X^{1,\frac{1}{2}+}}\big(\frac{1}{N_4}\|v\|_{X^{1,\frac{1}{2}+}}\big)$$

$$\lesssim \frac{1}{(N_1^*)^{2-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}} \lesssim \frac{1}{(N_1^*)^{2-}}E^1(\Phi) \tag{4.99}$$

In the last step, we used Proposition 4.4.1.

Subcase 2: $N_1 \gg N_4$

In this subcase, we need to consider two sub-subcases. Let $\gamma \in (0,1)$ be fixed. We will determine γ later. (in equation (4.114))

Sub-subcase 1: $N_3 \lesssim N_1^\gamma$

Let the functions $F_j; j=1,\ldots,4$ be defined as in (4.98). We use an $L^2_{t,x}, L^2_{t,x}$ Hölder inequality, and we argue as before to deduce that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*}\|F_1F_3\|_{L^2_{t,x}}\|F_2F_4\|_{L^2_{t,x}}$$

We use Proposition 4.2.4 and Proposition 5.2.5 to deduce that this expression is:

$$\lesssim \frac{1}{N_1^*}\Big(\frac{N_3^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}\|\mathcal{D}v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|\nabla v_{N_3}\|_{X^{0,\frac{1}{2}+}}\Big)\Big(\frac{N_4^{\frac{1}{2}}}{N_2^{\frac{1}{2}-}}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{0,\frac{1}{2}+}}\Big)$$

$$\lesssim \frac{1}{(N_1^*)^{2-}}N_1^{\frac{\gamma}{2}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|_{X^{1,\frac{1}{2}+}}\|v\|_{X^{\frac{1}{2},\frac{1}{2}+}} \lesssim \frac{1}{(N_1^*)^{2-\frac{\gamma}{2}-}}E^1(\Phi) \tag{4.100}$$

Sub-subcase 2: $N_3 \gtrsim N_1^{\gamma}$

In this sub-subcase, we have to work a little bit harder. The crucial estimate will be Proposition 4.2.7. We suppose that $(\xi_1,\xi_2,\xi_3,\xi_4)$ is a frequency configuration occurring in the integral defining A_{N_1,N_2,N_3,N_4}. We argue as in [38]. We note the elementary trigonometry fact that in this frequency regime, one has: $\angle(\xi_1,\xi_{14}) = O\big(\frac{N_4}{N_1}\big), \angle(\xi_3,\xi_{34}) = O\big(\frac{N_4}{N_3}\big)$. Furthermore, one can use Lipschitz properties of the cosine function to deduce that:

$$|cos\angle(\xi_1,\xi_3)| \lesssim \beta_0 + \frac{N_4}{N_3} \tag{4.101}$$

We now define:

$$F(x,t) := \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it(\tau_1+\tau_2)+i\langle x,\xi_1+\xi_2\rangle}\chi_{|\cos\angle(\xi_1,\xi_2)|\leq\beta_0+\frac{N_4}{N_3}}\widetilde{F_1}(\xi_1,\tau_1)\widetilde{F_3}(\xi_2,\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2$$

We now use an $L^2_{t,x}, L^2_{t,x}$ Hölder inequality, and recall (4.98) to deduce that one now has:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1^*}\|F\|_{L^2_{t,x}}\|F_2F_4\|_{L^2_{t,x}}$$

which by using Proposition 4.2.7 and Proposition 5.2.5 is:

$$\lesssim \frac{1}{N_1^*}\big(\beta_0 + \frac{N_4}{N_3}\big)^{\frac{1}{2}}\|F_1\|_{X^{0,\frac{1}{2}+}}\|F_3\|_{X^{0,\frac{1}{2}+}}\big(\frac{N_4^{\frac{1}{2}}}{N_2^{\frac{1}{2}-}}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{0,\frac{1}{2}+}}\big)$$

$$\lesssim \frac{\beta_0^{\frac{1}{2}}}{(N_1^*)^{\frac{3}{2}-}}\|\mathcal{D}v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_3}\|_{X^{1,\frac{1}{2}+}}\|v_{N_4}\|_{X^{\frac{1}{2},\frac{1}{2}+}}$$

$$+\frac{1}{(N_1^*)^{\frac{3}{2}+\frac{\gamma}{2}-}}\|\mathcal{D}v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}\|v_{N_3}\|_{X^{1,\frac{1}{2}+}}\|v_{N_4}\|_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \big(\frac{\beta_0^{\frac{1}{2}}}{(N_1^*)^{\frac{3}{2}-}} + \frac{1}{(N_1^*)^{\frac{3}{2}+\frac{\gamma}{2}-}}\big)E^1(\Phi) \tag{4.102}$$

We combine (4.99), (4.100), and (4.102) to deduce that:

$$|A_{N_1,N_2,N_3,N_4}| \lesssim \big(\frac{\beta_0^{\frac{1}{2}}}{(N_1^*)^{\frac{3}{2}-}} + \frac{1}{(N_1^*)^{\frac{3}{2}+\frac{\gamma}{2}-}} + \frac{1}{(N_1^*)^{2-\frac{\gamma}{2}-}}\big)E^1(\Phi) \tag{4.103}$$

We then sum in the N_j, use (4.96), and Proposition 4.4.4 to deduce that:

$$|A| \lesssim \big(\frac{\beta_0^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{\frac{3}{2}+\frac{\gamma}{2}-}} + \frac{1}{N^{2-\frac{\gamma}{2}-}}\big)E^2(\Phi) \tag{4.104}$$

Estimate of B (Sextilinear Terms)

Let us consider just the first term in B coming from the summand $M_4(\xi_{123},\xi_4,\xi_5,\xi_6)$ in the definition of M_6. The other terms are bounded analogously. In other words, we want to estimate:

$$B^{(1)} := \int_0^\delta \int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} M_4(\xi_{123},\xi_4,\xi_5,\xi_6)\widehat{(v\bar{v}v)}(\xi_1+\xi_2+\xi_3)\widehat{\bar{v}}(\xi_4)\widehat{v}(\xi_5)\widehat{\bar{v}}(\xi_6)d\xi_j dt$$

The bounds that we will prove for $B^{(1)}$ will also hold for B, with different constants.

We now dyadically localize $\xi_{123},\xi_4,\xi_5,\xi_6$, i.e., we find $N_j; j=1,\ldots,4$ such that:

$$|\xi_{123}| \sim N_1, |\xi_4| \sim N_2, |\xi_5| \sim N_3, |\xi_6| \sim N_4.$$

Let us define:

$$B^{(1)}_{N_1,N_2,N_3,N_4} := \int_0^\delta \int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} M_4(\xi_{123},\xi_4,\xi_5,\xi_6)$$
$$\widehat{(v\bar{v}v)}_{N_1}(\xi_1+\xi_2+\xi_3)\widehat{\bar{v}}_{N_2}(\xi_4)\widehat{v}_{N_3}(\xi_5)\widehat{\bar{v}}_{N_4}(\xi_6)d\xi_j dt$$

We now order the N_j to obtain: $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. As before, we know the following localization bound:

$$N_1^* \sim N_2^* \gtrsim N \tag{4.105}$$

In order to obtain a bound on the wanted term, we have to consider two cases.

Case 1: $N_1 = N_1^*$ or $N_1 = N_2^*$.

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$

Case 1: As in the periodic case, we consider the case when:

$$N_1 = N_1^*, N_2 = N_2^*, N_3 = N_3^*, N_4 = N_4^*.$$

The other cases are analogous.

We use Parseval's identity together with the *Fractional Leibniz Rule for $\mathcal{D}$*, and argue as in the periodic case to deduce that it suffices to bound the quantity:

$$K_{N_1,N_2,N_3,N_4} :=$$
$$\int_{\tau_1+\cdots+\tau_6=0}\int_{\xi_1+\cdots+\xi_6=0} \frac{1}{\beta_0(N_1^*)^2}|(\mathcal{D}v)\widetilde{\ }(\xi_1,\tau_1)||\widetilde{\bar{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)|$$
$$|(\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)||(\chi v)\widetilde{\ }_{N_3}(\xi_5,\tau_5)||\widetilde{\bar{v}}_{N_4}(\xi_4,\tau_4)|d\xi_j d\tau_j$$

We must consider several subcases:

Subcase 1: $N_1 \gg N_3$

Let us define the functions $F_j; j = 1,\ldots,6$ by:

$$\widetilde{F_1} := |(\mathcal{D}v)\widetilde{\ }|, \widetilde{F_2} = \widetilde{F_3} := |\widetilde{v}|, \widetilde{F_4} := |(\mathcal{D}v)\widetilde{\ }_{N_2}|, \widetilde{F_5} := |(\chi v)\widetilde{\ }_{N_3}|, \widetilde{F_6} := |\widetilde{v}_{N_4}| \quad (4.106)$$

We first use an $L^2_{t,x}, L^M_{t,x}, L^M_{t,x}, L^2_{t,x}, L^{4+}_{t,x}$ Hölder inequality to deduce that:

$$K_{N_1,N_2,N_3,N_4} \lesssim \frac{1}{\beta_0(N_1^*)^2} \|F_4F_5\|_{L^2_{t,x}} \|F_2\|_{L^M_{t,x}} \|F_3\|_{L^M_{t,x}} \|F_1\|_{L^4_{t,x}} \|F_6\|_{L^{4+}_{t,x}}$$

By Proposition 5.2.5, (4.13), (4.20), (4.12) adapted to the non-periodic setting, by the fact that taking absolute values in the spacetime Fourier transform, and since $N_1 \sim N_2$, it follows that this expression is:

$$\lesssim \frac{1}{\beta_0(N_1^*)^2} \big(\frac{N_3^{\frac{1}{2}}}{N_1^{\frac{1}{2}-}} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \|v_{N_3}\|_{X^{0,\frac{1}{2}+}}\big) \|v\|_{X^{1,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \|v_{N_4}\|_{X^{0+,\frac{1}{2}+}}$$

We use localization in frequency to deduce that this is:

$$\lesssim \frac{1}{\beta_0(N_1^*)^{\frac{5}{2}-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^4_{X^{1,\frac{1}{2}+}}$$

which by Proposition 4.4.1 is:

$$\lesssim \frac{1}{\beta_0(N_1^*)^{\frac{5}{2}-}} E^1(\Phi). \quad (4.107)$$

Subcase 2: $N_3 \sim N_1$

We use an $L^4_{t,x}, L^M_{t,x}, L^M_{t,x}, L^4_{t,x}, L^{4-}_{t,x}, L^{4+}_{t,x}$ Hölder inequality, and we argue as in the periodic case to deduce that:

$$K_{N_1,N_2,N_3,N_4} \lesssim \frac{1}{\beta_0(N_1^*)^2} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}} \|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \|\chi v_{N_3}\|_{X^{0,\frac{1}{2}-}} \|v_{N_4}\|_{X^{0+,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{\beta_0(N_1^*)^2} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^2_{X^{1,\frac{1}{2}+}} \big(\frac{1}{N_3} \|v\|_{X^{1,\frac{1}{2}+}}\big) \|v\|_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{\beta_0 (N_1^*)^3} E^1(\Phi). \tag{4.108}$$

Case 2: $N_1 = N_3^*$ or $N_1 = N_4^*$.

We assume as in the periodic case that $N_1 = N_3^*$. Let's also suppose that $N_3 = N_1^*, N_2 = N_2^*$. The other contributions are bounded analogously. Arguing as in the periodic case, we have to bound:

$$L_{N_1,N_2,N_3,N_4} := \int_{\tau_1+\cdots+\tau_6=0} \int_{\xi_1+\cdots+\xi_6=0} \frac{1}{\beta_0 (N_1^*)^2}$$

$$|\widetilde{v}(\xi_1,\tau_1)||\widetilde{\bar{v}}(\xi_2,\tau_2)||\widetilde{v}(\xi_3,\tau_3)||(\chi\mathcal{D}\bar{v})\widetilde{\ }_{N_2}(\xi_4,\tau_4)||(\mathcal{D}v)\widetilde{\ }_{N_3}(\xi_5,\tau_5)||\widetilde{\bar{v}}_{N_4}(\xi_6,\tau_6)|d\xi_j d\tau_j$$

We consider two subcases:

Subcase 1: $N_1^* \gg N_4$

We know that: $N_2 \gg N_4$

Let us estimate L_{N_1,N_2,N_3,N_4}. We define $F_j, j = 1, \ldots, 4$ by:

$$\widetilde{F_1} := |\widetilde{v}|, \widetilde{F_2} := |(\chi\mathcal{D}v)\widetilde{\ }_{N_2}|, \widetilde{F_3} := |(\mathcal{D}v)\widetilde{\ }_{N_3}|, \widetilde{F_4} := |\widetilde{v}_{N_4}|.$$

We use an $L^M_{t,x}, L^M_{t,x}, L^M_{t,x}, L^{2+}_{t,x}, L^2_{t,x}$ Hölder inequality, (4.13) adapted to the non-periodic setting, Proposition 5.2.5, and (4.22) to deduce that:

$$L_{N_1,N_2,N_3,N_4} \lesssim \frac{1}{\beta_0 (N_1^*)^2} \|F_1\|^3_{L^M_{t,x}} \|F_2 F_4\|_{L^2_{t,x}} \|F_3\|_{L^{2+}_{t,x}}$$

$$\lesssim \frac{1}{\beta_0 (N_1^*)^2} \|v\|^3_{X^{1,\frac{1}{2}+}} \Big(\frac{N_4^{\frac{1}{2}}}{N_2^{\frac{1}{2}-}} \|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}} \|v_{N_4}\|_{X^{0,\frac{1}{2}+}} \Big) \|\mathcal{D}v_{N_3}\|_{X^{0+,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{\beta_0 (N_1^*)^{\frac{5}{2}-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^3_{X^{1,\frac{1}{2}+}} \|v_{N_4}\|_{X^{\frac{1}{2},\frac{1}{2}+}}$$

$$\lesssim \frac{1}{\beta_0 (N_1^*)^{\frac{5}{2}-}} \|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}} \|v\|^4_{X^{1,\frac{1}{2}+}} \lesssim \frac{1}{\beta_0 (N_1^*)^{\frac{5}{2}-}} E^1(\Phi) \tag{4.109}$$

For the last inequality, we used Proposition 4.4.1.

Subcase 2: $N_4 \sim N_1^*$

We argue similarly as in Subcase 2 of Case 1 to deduce that:

$$L_{N_1,N_2,N_3,N_4} \lesssim \frac{1}{\beta_0 (N_1^*)^3} E^1(\Phi) \tag{4.110}$$

We use (4.107), (4.108), (4.109), and (4.110) to deduce that:

$$|B^{(1)}_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{\beta_0 (N_1^*)^{\frac{5}{2}-}} E^1(\Phi) \tag{4.111}$$

We sum in N_j. Using (4.105) and (4.111), it follows that:

$$|B^{(1)}| \lesssim \frac{1}{\beta_0 N^{\frac{5}{2}-}} E^1(\Phi)$$

By Proposition 4.4.4, and by construction of $B^{(1)}$, we deduce that:

$$|B| \lesssim \frac{1}{\beta_0 N^{\frac{5}{2}-}} E^2(\Phi) \tag{4.112}$$

4.4.4 Choice of the optimal parameters

By (4.95), (4.104), and (4.112), it follows that:

$$|E^2(u(\delta)) - E^2(u(0))| \lesssim \big(\frac{\beta_0^{\frac{1}{2}}}{N^{\frac{3}{2}-}} + \frac{1}{N^{\frac{3}{2}+\frac{\gamma}{2}-}} + \frac{1}{N^{2-\frac{\gamma}{2}-}} + \frac{1}{\beta_0 N^{\frac{5}{2}-}} \big) E^2(\Phi) \tag{4.113}$$

We now choose γ s.t. $\frac{3}{2} + \frac{\gamma}{2} = 2 - \frac{\gamma}{2}$. Hence, we choose:

$$\gamma := \frac{1}{2} \tag{4.114}$$

One then has that:

$$\frac{3}{2} + \frac{\gamma}{2} = 2 - \frac{\gamma}{2} = \frac{7}{4} \tag{4.115}$$

Let us now choose β_0. We recall that by (4.91), one has: $\beta_0 \sim \frac{1}{N^\alpha}, \alpha \in (0,1)$. In order to have $\frac{\beta_0^{\frac{1}{2}}}{N^{\frac{3}{2}-}} \lesssim \frac{1}{N^{\frac{7}{4}-}}$, we should take: $\alpha \geq \frac{1}{2}$.

In order to have $\frac{1}{\beta_0 N^{\frac{5}{2}-}} \lesssim \frac{1}{N^{\frac{7}{4}-}}$, we should take: $\alpha \leq \frac{3}{4}$.

Consequently, we take:

$$\alpha \in [\frac{1}{2}, \frac{3}{4}] \tag{4.116}$$

From the preceding, we may conclude that:

$$|E^2(u(\delta)) - E^2(u(0))| \lesssim \frac{1}{N^{\frac{7}{4}-}} E^2(u(0)) \tag{4.117}$$

Lemma 4.4.5 now follows.

□

4.4.5 Remarks on the scattering result of Dodson

Let us briefly explain why the L^2-scattering result of Dodson [44] for the defocusing cubic NLS on $(\mathbb{R}^2)$ (4.4) can be used to deduce scattering in H^s of the same equation, assuming that the initial data Φ lies in H^s. In other words, we want to justify the *persistence of regularity phenomenon* for scattering. We note that a similar argument is given in [37].

Let u be a global solution to (4.4). In [44], it is shown that whenever $\Phi \in L^2$, u satisfies the spacetime bound:

$$\|u\|_{L^4_{t,x}(\mathbb{R}^2 \times \mathbb{R})} < \infty. \tag{4.118}$$

It can be seen that (4.118) implies scattering in L^2. Given $s > 1$, and assuming that $\Phi \in H^s$, we are interested in obtaining:

$$\|D^s u\|_{L^4_{t,x}(\mathbb{R}^2 \times \mathbb{R})} < \infty. \tag{4.119}$$

In order to prove (4.119), we start with $T \in \mathbb{R}$ and we observe that for all $t \in \mathbb{R}$, one has:

$$u(t) = S(t-T)u(T) - i\int_T^t S(t-\tau)(|u|^2u)(\tau)d\tau. \tag{4.120}$$

Taking D^s on both sides, it follows that:

$$D^su(t) = S(t-T)D^su(T) - i\int_T^t S(t-\tau)D^s(|u|^2u)(\tau)d\tau.$$

We suppose that I is an closed interval in $\mathbb{R}$ whose left endpoint is T and whose right endpoint can be $+\infty$. By Strichartz estimates, we deduce:

$$\|D^su\|_{L^4_{t,x}(I\times\mathbb{R}^2)} \lesssim \|D^su(T)\|_{L^2_x(\mathbb{R}^2)} + \|D^s(|u|^2u)\|_{L^{\frac{4}{3}}_{t,x}(I\times\mathbb{R}^2)}.$$

By using the Fractional Leibniz Rule and Hölder's inequality, this implies:

$$\|D^su\|_{L^4_{t,x}(I\times\mathbb{R}^2)} \lesssim \|D^su(T)\|_{L^2_x(\mathbb{R}^2)} + \|D^su\|_{L^4_{t,x}(I\times\mathbb{R}^2)}\|u\|^2_{L^4_{t,x}(I\times\mathbb{R}^2)}. \tag{4.121}$$

Given $\epsilon > 0$, by (4.118), we can make the interval I small enough so that:

$$\|u\|_{L^4_{t,x}(I\times\mathbb{R}^2)} \leq \epsilon. \tag{4.122}$$

Choosing ϵ small enough, (4.121), and (4.122) imply:

$$\|D^su\|_{L^4_{t,x}(I\times\mathbb{R}^2)} \lesssim \|D^su(T)\|_{L^2_x(\mathbb{R}^2)} = \|u(T)\|_{H^s_x(\mathbb{R}^2)} \tag{4.123}$$

We now cover $\mathbb{R}$ by such intervals I, with a small modification when we take the left endpoint of the interval to be $-\infty$. The bound (4.119) now follows.

Let us now observe why (4.119) implies scattering in H^s. Namely, given $\delta > 0$ small, we can find $T(\delta) > 0$ such that:

$$\|D^su\|_{L^4_{t,x}([T(\delta),+\infty)\times\mathbb{R}^2)} \leq \delta \tag{4.124}$$

We use (4.120), Strichartz estimates and we argue as before to obtain that for all $t \geq T(\delta)$, one has:

$$\|D^s u(t)-S(t-T(\delta))D^s u(T(\delta))\|_{L^\infty_t L^2_x([T(\delta),+\infty)\times\mathbb{R}^2)} \lesssim \|D^s u\|_{L^4_{t,x}([T(\delta),+\infty)\times\mathbb{R}^2)}\|u\|^2_{L^4_{t,x}([T(\delta),+\infty)\times\mathbb{R}^2)}.$$

Using (4.118) and (4.124), it follows that, for all $t \geq T(\delta)$:

$$\|D^s u(t) - S(t-T(\delta))D^s u(T(\delta))\|_{L^\infty_t L^2_x([T(\delta),+\infty)\times\mathbb{R}^2)} \lesssim \delta \tag{4.125}$$

We now let $\delta_k := 2^{-k} \to 0$, and we choose $T(\delta_k)$ as above such that $T(\delta_k) \to +\infty$. Using (4.125) and the unitarity of $S(t)$ on L^2, it follows that $(S(-T(\delta_k)u(T(\delta_k)))$ is Cauchy in H^s. By completeness, there exists $u_+ \in H^s$ such that $S(-T(\delta_k))u(T(\delta_k)) \xrightarrow{H^s} u_+$. By using (4.125) again, we note that:

$$S(-t)u(t) \xrightarrow{H^s} u_+, \text{ as}\, t \to +\infty.$$

By unitarity, it follows that, for the obtained $u_+ \in H^s$, one has:

$$\|u(t) - S(t)u_+\|_{H^s_x(\mathbb{R}^2)} \to 0, \text{ as}\, t \to +\infty. \tag{4.126}$$

An analogous argument shows that there exists $u_- \in H^s$ such that:

$$\|u(t) - S(t)u_-\|_{H^s_x(\mathbb{R}^2)} \to 0, \text{ as}\, t \to -\infty. \tag{4.127}$$

Hence, the H^s scattering result for the cubic NLS (4.4) follows, thus implying uniform bounds on $\|u(t)\|_{H^s}$ whenever $\Phi \in H^s$.

4.4.6 Further remarks on the equation

Remark 4.4.6. *Let us observe that Theorem 4.1.2 would follow immediately if we knew that the equation* (4.1) *on* $\mathbb{R}^2$ *scattered in* H^s. *To the best of our knowledge, this result isn't available, and it can't be deduced from currently known techniques used to prove scattering. Some scattering results for the Hartree equation were previously* *studied in [55, 54, 56]. In [55, 54], the asymptotic completeness step was proved by using techniques from [87], which work in dimensions $n \geq 3$. In [56], the one and two-dimensional equations are studied. In this case, scattering results are deduced for a subset of solutions with initial data which belongs to a Gevrey class.*

Further scattering results for the Hartree equation are noted in [49, 63]. In these papers, one assumes that the initial data lies in an appropriate weighted Sobolev space. The implied bounds depend on the corresponding weighted Sobolev norms of the initial data. Hence, uniform bounds on appropriate Sobolev norms of solutions whose initial data doesn't lie in the weighted Sobolev spaces can't be deduced by density. Also, the techniques used to prove [85] and similar results are restricted to dimensions $n \geq 5$.

Let us finally note that the techniques used to prove scattering for the defocusing cubic NLS on $\mathbb{R}^2$ in [44] rely on the construction of a Morawetz functional. It is not clear if this construction can be modified to (4.1)*. This would be an interesting problem to examine. We expect scattering for* (4.1) *to be harder than for the defocusing NLS since the nonlinearity is non-local.*

4.5 Appendix: Proof of Proposition 4.3.1

Proof. The proof is based on a fixed-point argument. Let us WLOG look at $t_0 = 0$. With notation as in Chapter 2, we consider:

$$Lw := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t)\int_0^t S(t-t')(V*|w_\delta|^2)w_\delta(t')dt' \tag{4.128}$$

Let $c > 0$ be the constant[3] such that $\|\chi_\delta S(t)\Phi\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s}$. Such a constant exists by using arguments from [71]. We then define:

$$B := \{w; \|w\|_{X^{1,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^1}, \|w\|_{X^{s,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s}\}$$

Arguing as in Chapter 2, B is complete w.r.t $\|\cdot\|_{X^{1,b}}$. For $w \in B$, we obtain:

[3] This time localization estimate, and all the other similar estimates that we had to use in previous chapters carry over to the torus.

$$\|Lw\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} + c_1\delta^{\frac{1-2b}{2}}\|(V * |w_\delta|^2)w_\delta\|_{X^{s,b-1}} \tag{4.129}$$

Similarly, we obtain:

$$\|\mathcal{D}Lw\|_{X^{0,b}} \leq c\delta^{\frac{1-2b}{2}}\|\mathcal{D}\Phi\|_{L^2} + c_1\delta^{\frac{1-2b}{2}}\|\mathcal{D}((V * |w_\delta|^2)w_\delta)\|_{X^{0,b-1}} \tag{4.130}$$

We now estimate $\|(V*|w_\delta|^2)w_\delta\|_{X^{s,b-1}}$ by duality. Namely, suppose that we are given $c = c(n,\tau)$ such that:

$$\sum_n \int d\tau |c(n,\tau)|^2 = 1.$$

We want to estimate:

$$I := \sum_{n_1-n_2+n_3-n_4=0} \int_{\tau_1-\tau_2+\tau_3-\tau_4=0} \frac{|c(n_4,\tau_4)|}{(1+|\tau_4+|n_4|^2|)^{1-b}}(1+|n_4|)^s|\widetilde{w_\delta}(n_1,\tau_1)|$$

$$|\widetilde{w_\delta}(n_2,\tau_2)||\widetilde{w_\delta}(n_3,\tau_3)||\widehat{V}(n_1+n_2)|d\tau_j$$

Since $\widehat{V} \in L^\infty(\mathbb{Z}^2)$, this expression is:

$$\lesssim \sum_{n_1-n_2+n_3-n_4=0} \int_{\tau_1-\tau_2+\tau_3-\tau_4=0} \frac{|c(n_4,\tau_4)|}{(1+|\tau_4+|n_4|^2|)^{1-b}}(1+|n_4|)^s|\widetilde{w_\delta}(n_1,\tau_1)|$$

$$|\widetilde{w_\delta}(n_2,\tau_2)||\widetilde{w_\delta}(n_3,\tau_3)|d\tau_j$$

Let us write:

$$\mathbb{Z}^2 = \bigcup_{k=0}^{\infty} D_k;\ D_k = \{n \in \mathbb{Z}^2; |n| \sim 2^k\}$$

Let I_{k_1,k_2,k_3} denote the contribution to I with $n_j \in D_{k_j}$, for $j = 1, 2, 3$. Let us consider WLOG the case when:

$$k_1 \geq k_2 \geq k_3. \tag{4.131}$$

The contributions from other cases are bounded analogously.

Following [15], we write:

$$D_{k_1} \subseteq \bigcup_{\alpha} Q_\alpha$$

Here, Q_α are balls of radius 2^{k_2}. We can choose this cover so that each element of D_{k_1} lies in a fixed finite number of Q_α. This number is independent of k_1 and k_2.

If $n_1 \in Q_\alpha$, then since $n_4 = n_1 - n_2 + n_3$, $|n_2|, |n_3| \lesssim 2^{k_2}$, it follows that n_4 lies in $\widetilde{Q}_\alpha$, a dilate of Q_α. Thus, the term that we want to estimate is:

$$J_{k_1,k_2,k_3} := 2^{k_1 s} \sum_{\alpha} \sum_{n_1 \in Q_\alpha, n_2 \in D_{k_2}, n_3 \in D_{k_3}, n_4 \in \widetilde{Q}_\alpha, n_1 - n_2 + n_3 - n_4 = 0} \int_{\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0}$$

$$|\widetilde{w_\delta}(n_1, \tau_1)||\widetilde{w_\delta}(n_2, \tau_2)||\widetilde{w_\delta}(n_3, \tau_3)| \frac{|c(n_4, \tau_4)|}{(1 + |\tau_4 + |n_4|^2|)^{1-b}} d\tau_j$$

We now define:

$$F_\alpha(x,t) := \sum_{n \in \widetilde{Q}_\alpha} \int d\tau \frac{|c(n,\tau)|}{(1 + |\tau + |n|^2|)^{1-b}} e^{i(\langle n,x \rangle + \tau t)} \tag{4.132}$$

$$G_\alpha(x,t) := \sum_{n \in Q_\alpha} \int d\tau |\widetilde{w_\delta}(n,\tau)| e^{i(\langle n,x \rangle + \tau t)} \tag{4.133}$$

$$H_j(x,t) := \sum_{n \in D_{k_j}} \int d\tau |\widetilde{w_\delta}(n,\tau)| e^{i(\langle n,x \rangle + \tau t)} \tag{4.134}$$

By Parseval's identity and Hölder's inequality, we deduce:

$$J_{k_1,k_2,k_3} \lesssim 2^{k_1 s} \sum_{\alpha} \int_{\mathbb{R}} \int_{\mathbb{T}^2} \overline{F_\alpha} G_\alpha \overline{H_2} H_3 dx dt$$

$$\leq 2^{k_1 s} \sum_{\alpha} \|F_\alpha\|_{L^4_{t,x}} \|G_\alpha\|_{L^4_{t,x}} \|H_2\|_{L^4_{t,x}} \|H_3\|_{L^4_{t,x}}.$$

Now, from Lemma 4.2.3, with s_1, b_1 as in the assumptions of Lemma 4.2.3, we have:

$$\|H_2\|_{L^4_{t,x}} \lesssim 2^{k_2 s_1} \Big(\sum_{n \in D_{k_2}} d\tau (1 + |\tau + |n|^2|)^{2b_1} |\widetilde{w_\delta}(n,\tau)|^2 \Big)^{\frac{1}{2}}$$

$$\lesssim 2^{k_2 s_1} \|(w_\delta)_{2^{k_2}}\|_{X^{0,b_1}}$$

Here $(w_\delta)_M$ is defined by: $((w_\delta)_M)\widehat{\ } = \widehat{w_\delta} \chi_{D_M}$, and we note that localization in t and in n commute. This is a slight abuse of notation, but we interpret w_δ as a localization in time if $\delta > 0$ is small, and we interpret w_N as a localization in frequency if N is a dyadic integer.

By interpolation, it follows that:

$$\|(w_\delta)_{2^{k_2}}\|_{X^{0,b_1}} \lesssim \|(w_\delta)_{2^{k_2}}\|^{\theta}_{X^{0,0}} \|(w_\delta)_{2^{k_2}}\|^{1-\theta}_{X^{0,b}}$$

Here:

$$\theta := 1 - \frac{b_1}{b} \tag{4.135}$$

By construction of ψ_δ, we obtain:

$$\|(w_\delta)_{2^{k_2}}\|_{X^{0,0}} = \|(w_\delta)_{2^{k_2}}\|_{L^2_{t,x}} = \|(w_\delta)_{2^{k_2}} \psi_\delta\|_{L^2_{t,x}}$$

We now use Hölder's inequality and (4.15) to see that this expression is:

$$\lesssim \|(w_\delta)_{2^{k_2}}\|_{L^4_t L^2_x} \|\psi_\delta\|_{L^4_t} \lesssim \delta^{\frac{1}{4}} \|(w_\delta)_{2^{k_2}}\|_{X^{0,\frac{1}{4}+}} \leq \delta^{\frac{1}{4}} \|(w_\delta)_{2^{k_2}}\|_{X^{0,b}}.$$

Consequently:

$$\|H_2\|_{L^4_{t,x}} \lesssim 2^{k_2 s_1} \delta^{\frac{\theta}{4}} \|(w_\delta)_{2^{k_2}}\|_{X^{0,b}}$$

$$\lesssim 2^{k_2 s_1} \delta^{\frac{\theta}{4}+\frac{1-2b}{2}} \|w_{2^{k_2}}\|_{X^{0,b}} \tag{4.136}$$

In the last inequality, we used appropriate time-localization in $X^{0,b}$.

Analogously:

$$\|H_3\|_{L^4_{t,x}} \lesssim 2^{k_3 s_1} \delta^{\frac{\theta}{4}+\frac{1-2b}{2}} \|w_{2^{k_3}}\|_{X^{0,b}} \tag{4.137}$$

Given an index α, we define $(w_\delta)_\alpha$, and w_α to be the restriction to $n \in Q_\alpha$ of w_δ and w respectively. We note that this is a different localization than the ones we used before. Since each Q_α has radius 2^{k_2}, Lemma 4.2.3 implies that:

$$\|G_\alpha\|_{L^4_{t,x}} \lesssim 2^{k_2 s_1} \Big(\sum_{n \in Q_\alpha} d\tau (1 + |\tau + |n|^2|)^{2b_1} |\widetilde{w_\delta}(n,\tau)|^2 \Big)^{\frac{1}{2}}$$

$$\lesssim 2^{k_2 s_1} \|(w_\delta)_\alpha\|_{X^{0,b_1}}$$

Arguing as in (4.136),(4.137), we obtain:

$$\|G_\alpha\|_{L^4_{t,x}} \lesssim 2^{k_2 s_1} \delta^{\frac{\theta}{4}+\frac{1-2b}{2}} \|w_\alpha\|_{X^{0,b}} \tag{4.138}$$

Furthermore, each Q_α is of radius $\sim 2^{k_2}$. Let c_α be the restriction of c to $n \in \widetilde{Q}_\alpha$. Let us also choose b_1 such that:

$$b_1 \leq 1 - b. \tag{4.139}$$

From Lemma 4.2.3, and the previous definitions, we obtain:

$$\|F_\alpha\|_{L^4_{t,x}} \lesssim 2^{k_2 s_1} \|F_\alpha\|_{X^{0,b_1}} \leq 2^{k_2 s_1} \|F_\alpha\|_{X^{0,1-b}}$$

$$\lesssim 2^{k_2 s_1} \|c_\alpha\|_{L^2_{\tau,n}}. \tag{4.140}$$

From (4.136), (4.137), (4.138), (4.140), it follows that:

$$J_{k_1,k_2,k_3} \lesssim \sum_\alpha \delta^{\frac{3\theta}{4}+\frac{3(1-2b)}{2}} 2^{k_1 s} 8^{k_2 s_1} 2^{k_3 s_1} \|w_{2^{k_2}}\|_{X^{0,b}} \|w_{2^{k_3}}\|_{X^{0,b}} \|w_\alpha\|_{X^{0,b}} \|c_\alpha\|_{L^2_{\tau,n}}$$

We apply the Cauchy-Schwarz inequality in α to deduce that the previous expression is [4]:

$$\lesssim \delta^{\frac{3\theta}{4}+\frac{3(1-2b)}{2}} 2^{k_1 s} 8^{k_2 s_1} 2^{k_3 s_1} \|w_{2^{k_1}}\|_{X^{0,b}} \|w_{2^{k_2}}\|_{X^{0,b}} \|w_{2^{k_3}}\|_{X^{0,b}} \|c_{2^{k_1}}\|_{L^2_{\tau,n}}$$

We write $8^{k_2 s_1} = (8^{k_2 s_1})^{0-}(8^{k_2 s_1})^{1+}, 2^{k_3 s_1} = (2^{k_3 s_1})^{0-}(2^{k_3 s_1})^{1+}$, and we sum a geometric series in k_2, k_3 to deduce that:

$$\sum_{k_j \text{ satisfying } (4.131)} J_{k_1,k_2,k_3} \lesssim$$

$$\lesssim \sum_{k_1} \delta^{\frac{3\theta}{4}+\frac{3(1-2b)}{2}} \|w_{2^{k_1}}\|_{X^{s,b}} \|c_{2^{k_1}}\|_{L^2_{\tau,n}} \|w\|_{X^{3s_1+,b}} \|w\|_{X^{s_1+,b}}$$

Using the Cauchy-Schwarz inequality in k_1, this expression is:

$$\lesssim \delta^{\frac{3\theta}{4}+\frac{3(1-2b)}{2}} \|w\|_{X^{s,b}} \|c\|_{L^2_{\tau,n}} \|w\|_{X^{3s_1+,b}} \|w\|_{X^{s_1+,b}}$$

[4] Strictly speaking, we are making the annulus $|n| \sim 2^{k_1}$ a little bit larger, but we write the localization in the same way as before.

$$\lesssim \delta^{\frac{3\theta}{4}+\frac{3(1-2b)}{2}} \|w\|_{X^{s,b}} \|w\|^2_{X^{3s_1+,b}} \tag{4.141}$$

Let us take $s_1 = \frac{1}{3}-$. Then, the assumptions of Lemma 4.2.3 will be satisfied if we take $b_1 = \frac{1-(\frac{1}{3}-)}{2}+ = \frac{1}{3}+$. Since $b = \frac{1}{2}+$, (4.139) is then satisfied. By our construction in (4.135), one has: $\theta = 1 - \frac{\frac{1}{3}+}{\frac{1}{2}+} > \frac{1}{4}$. Hence, $\rho_0 := \frac{3\theta}{4} + 3(1-2b) > 0$.

Thus, by (4.129), and by definition of B it follows that for $w \in B$:

$$\|Lw\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} + c_2 \delta^{\frac{3\theta}{4}+2(1-2b)} \|w\|_{X^{s,b}} \|w\|^2_{X^{1,b}}$$

$$\leq c\delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} + c_3 \delta^{\frac{1-2b}{2}} \|\Phi\|_{H^s} \delta^{\frac{3\theta}{4}+3(1-2b)} \|\Phi\|^2_{H^1}$$

Similarly, for $v, w \in B$, one has:

$$\|Lv - Lw\|_{X^{1,b}} \leq c_1 \delta^{\frac{3\theta}{4}+2(1-2b)} (\|v\|_{X^{1,b}} + \|w\|_{X^{1,b}})^2 \|v - w\|_{X^{1,b}}$$

$$\leq c_2 \delta^{\frac{3\theta}{4}+3(1-2b)} \|\Phi\|^2_{H^1} \|v - w\|_{X^{1,b}}$$

We now argue as in Chapter 2 to obtain a fixed point $v \in B$. We then take $\mathcal{D}$'s of both sides and use (4.130). Now, we have to estimate:

$$\|\mathcal{D}((V * |v_\delta|^2) v_\delta)\|_{X^{0,b-1}}.$$

Arguing as before, it follows that this expression is:

$$\lesssim \delta^{\rho_0} \|\mathcal{D}v\|_{X^{0,b}} \|v\|^2_{X^{1,b}}$$

Namely, in the analogue of J_{k_1,k_2,k_3}, we can replace the $2^{k_1 s}$ by $\theta_{2^{k_1}}$, which is equal to $\frac{2^{k_1 s}}{N^s}$ if $2^{k_1} \geq N$, and 1 otherwise. One then argues as in Chapter 2, and (4.40), (4.41) immediately follow.

We now check uniqueness, i.e. (4.39). Namely, we suppose that:

$$\begin{cases} iu_t + \Delta u = (V * |u|^2)u, x \in \mathbb{T}^2, t \in \mathbb{R} \\ iv_t + \Delta v = (V * |v|^2)v, x \in \mathbb{T}^2, t \in \mathbb{R} \\ u|_{t=0} = v|_{t=0} \in H^s(\mathbb{T}^2), s > 1. \end{cases} \tag{4.142}$$

We are assuming that u is a well-posed solution to (4.1) on $\mathbb{T}^2$, and hence $\|u(t)\|_{H^s}$ satisfies exponential bounds, as was noted in the Introduction. Furthermore, since $v \in X^{s,\frac{1}{2}+}$, by Sobolev embedding in time, it follows that $v \in L_t^\infty H_x^s$. Consequently, there exist $A, B > 0$ such that, for all $t \in \mathbb{R}$, one has:

$$\|u(t)\|_{H^s}, \|v(t)\|_{H^s} \leq Ae^{B|t|} \tag{4.143}$$

We observe:

$$u(t) - v(t) = -i \int_0^t S(t-t')((V * |u|^2)u - (V * |v|^2)v)(t')dt'$$

We take L^2 norms in x and use Minkowski's inequality to deduce:

$$\|u(t) - v(t)\|_{L_x^2} \leq \int_0^t \|(V * |u|^2)u - (V * |v|^2)v\|_{L_x^2} dt' \tag{4.144}$$

In order to bound the integral, we need the two following bounds, which follow from Hölder's inequality, Young's inequality, and Sobolev embedding [5].

$$\|(V * (u_1 u_2))u_3\|_{L_x^2} \leq \|V * (u_1 u_2)\|_{L_x^\infty} \|u_3\|_{L_x^2} \leq \|V\|_{L_x^1} \|u_1\|_{L_x^\infty} \|u_2\|_{L_x^\infty} \|u_3\|_{L_x^2}$$

$$\leq \|u_1\|_{H_x^s} \|u_2\|_{H_x^s} \|u_3\|_{L_x^2} \tag{4.145}$$

Also:

[5] Note that we are considering $s > 1$.

$$\|(V*(u_1u_2))u_3\|_{L^2_x} \leq \|V*(u_1u_2)\|_{L^2_x}\|u_3\|_{L^\infty_x} \leq \|V\|_{L^1_x}\|u_1u_2\|_{L^2_x}\|u_3\|_{L^\infty_x}$$

$$\leq \|V\|_{L^1_x}\|u_1\|_{L^2_x}\|u_2\|_{L^\infty_x}\|u_3\|_{L^\infty_x} \leq \|u_1\|_{L^2_x}\|u_2\|_{H^s_x}\|u_3\|_{H^s_x} \tag{4.146}$$

Substituting (4.145) and (4.146) into (4.144), and using (4.143, it follows that:

$$\|u(t)-v(t)\|_{L^2_x} \lesssim \int_0^t (\|u\|_{H^s}+\|v\|_{H^s})^2\|u-v\|_{L^2_x}dt' \lesssim \int_0^t e^{2\beta t'}\|u-v\|_{L^2_x}dt'$$

By Gronwall's inequality, it follows that on $[0,t]$, one has $\|u-v\|_{L^2_x}=0$, hence $u=v$. The same argument works for negative times. (4.39) now follows.

Arguing as in Chapter 2, we note that all the implied constants depend on s, energy, and mass, and that they are continuous in energy and mass.

This proves Proposition 4.3.1. $\square$

Chapter 5

Bounds on $\mathbb{R}^3$; the Gross-Pitaevskii Equation for dipolar quantum gases

5.1 Introduction

In this chapter, we shall consider the initial value problem of the *Gross-Pitaevskii equation for dipolar quantum gases*:

$$\begin{cases} iu_t + \Delta u = \mu_1 |u|^2 u + \mu_2 (K * |u|^2) u,\ x \in \mathbb{R}^3,\ t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}^3) \end{cases} \tag{5.1}$$

Here, we take: $K := \frac{1-3\cos^2\phi}{|x|^3}$, where $\phi = \phi(x)$ is the angle between $x \in \mathbb{R}^3$ and the fixed dipole axis $\vec{n} = (0,0,1)$. One can check that then:

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5} \tag{5.2}$$

We are assuming that:

$$\mu_1 \geq \frac{4\pi}{3}\mu_2 \geq 0 \tag{5.3}$$

This corresponds to the *stable regime* for (5.1). This condition is discussed in detail in [1, 26]. Furthermore, we are assuming that $s > \frac{3}{2}$ is a real number.

The Cauchy Problem (5.1) was used in [115, 116] to model the time evolution of a dipolar quantum gas (with appropriate scaling constants). A rigorous mathematical treatment regarding global well-posedness was given in [26]. This line of study was continued in [1] in which the authors prove the existence of solitons in certain unstable regimes.

The equation has the following conserved quantities:

$$M(u) := \int |u|^2 dx \,(\text{Mass})$$

and

$$E(u) := \frac{1}{2} \int |\nabla u|^2 dx + \frac{1}{4}\mu_1 \int |u|^4 dx + \frac{1}{4}\mu_2 \int (K * |u|^2)|u|^2 dx \,(\text{Energy})$$

A key feature of the convolution potential K is the fact that we can compute $\widehat{K}$ explicitly and we find that this is a bounded function. More precisely, in [26], it is shown that:

$$\widehat{K}(\xi) = \frac{4\pi}{3}(3\cos^2\phi - 1) \tag{5.4}$$

Here, $\phi = \phi(\xi)$ is the angle between $\xi \in \mathbb{R}^3$ and the dipole axis $\vec{n} = (0, 0, 1)$. In particular, we obtain:

$$\widehat{K} \in L^\infty(\mathbb{R}^3). \tag{5.5}$$

and

$$\widehat{K}(\xi) \geq -\frac{4\pi}{3}. \tag{5.6}$$

As in [26], we let $\rho := |u|^2$, and we observe that then:

$$\|\nabla u\|_{L^2}^2 = 2E - \frac{1}{2}\mu_1 \|u\|_{L^4}^4 - \frac{1}{2}\mu_2 \int (K * |u|^2)|u|^2 dx = 2E - \frac{1}{16\pi^3} \int \big(\mu_1 + \mu_2 \widehat{K}(\xi)\big)|\widehat{\rho}(\xi)|^2 d\xi$$

$$\leq 2E - \frac{1}{16\pi^3} \int \left(\mu_1 - \frac{4\pi}{3}\mu_2\right) |\widehat{\rho}(\xi)|^2 d\xi \leq 2E \tag{5.7}$$

Here, we used 5.3 and 5.6). It is proved in [26] that (5.1) has a global solution u, with $\|u(t)\|_{H^1} \leq C(\Phi)$.

5.1.1 Statement of the main result

Let u denote the solution to (5.1). As in previous chapters, it makes sense to study the growth of $\|u(t)\|_{H^s}$. The result that we prove is:

Theorem 5.1.1. *Let u and s be as above. There exists $C = C(\Phi) > 0$ such that, for all $t \in \mathbb{R}$:*

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{s+} \|\Phi\|_{H^s} \tag{5.8}$$

Remark 5.1.2. *We note that the growth of high Sobolev norms of solutions to* (5.1) *has not been studied so far. The scattering results for the defocusing cubic NLS on $\mathbb{R}^3$ proved in [35] rely on the existence of a Morawetz action functional. It is not immediately clear how one could modify this construction to the equation* (5.1)*.*

Remark 5.1.3. *From the proof, we note that the same bounds hold for solutions of the defocusing cubic NLS on $\mathbb{R}^3$:*

$$\begin{cases} iu_t + \Delta u = |u|^2 u,\, x \in \mathbb{R}^3,\, t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}^3) \end{cases} \tag{5.9}$$

However, from the scattering result in [35], one can obtain uniform bounds on $\|u(t)\|_{H^s}$.

5.1.2 Main ideas of the proof

The main idea of the proof of Theorem 5.1.1 is to modify the methods in the previous chapters to three dimensions. We again use the *upside-down I-method*. Namely, in (5.33), we construct an operator $\mathcal{D}$ such that $\|\mathcal{D}u(t)\|_{L^2_x}$ is equivalent to $\|u(t)\|_{H^s_x}$. We then show that the quantity $\|\mathcal{D}u(t)\|^2_{L^2_x}$ is *slowly varying*. The main reason why we can use this approach is the fact that K is even, and $\widehat{K} \in L^\infty(\mathbb{R}^3)$. Let us remark that before, we assumed that the convolution potential V was in L^1. This was just needed in order to ensure that $\widehat{V} \in L^\infty$.

5.2 Facts from harmonic analysis

A key tool will be the following variant of *Strichartz Estimates*, [69, 101, 106]:

Proposition 5.2.1. *Suppose that $2 \leq q, r \leq \infty$, and $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$. Then, one has:*

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.10}$$

In particular, we can take $q = r = \frac{10}{3}$ and deduce that:

$$\|u\|_{L_{t,x}^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.11}$$

From Sobolev Embedding, we know:

$$\|u\|_{L_{t,x}^{\infty}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{3}{2}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.12}$$

We note that, if $k = \frac{1}{3}$, then:

$$\frac{k}{\frac{10}{3}} + \frac{1-k}{\infty} = \frac{1}{10}.$$

Hence, we can interpolate and deduce that:

$$\|u\|_{L_{t,x}^{10}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{1+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.13}$$

and

$$\|u\|_{L_{t,x}^{10-}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{1,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.14}$$

Similarly, interpolation between (5.11) and (5.12) allows us to deduce that:

$$\|u\|_{L^{\frac{10}{3}+}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.15}$$

Furthermore, we can interpolate between (5.11) and the fact that $\|u\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \sim \|u\|_{X^{0,0}(\mathbb{R}\times\mathbb{R}^3)}$ to deduce that:

$$\|u\|_{L^{\frac{10}{3}-}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{0,\frac{1}{2}-}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.16}$$

We can also interpolate between the following consequence of Sobolev embedding in time (i.e. Proposition 5.2.1 with $q=\infty, r=2$):

$$\|u\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^3)}$$

and the fact that $\|u\|L^2_{t,x}(\mathbb{R}\times\mathbb{R}^3) \sim \|u\|_{X^{0,0}(\mathbb{R}\times\mathbb{R}^3)}$ to deduce that:

$$\|u\|_{L^4_t L^2_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{0,\frac{1}{4}+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.17}$$

The following estimate will be useful:

Lemma 5.2.2. *Let $c<d$ be real numbers, and let us denote by $\chi=\chi(t)=\chi_{[c,d]}(t)$. One then has, for all $s\in\mathbb{R}$, and for all $b<\frac{1}{2}$:*

$$\|\chi u\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{X^{s,b+}(\mathbb{R}\times\mathbb{R}^3)} \tag{5.18}$$

The proof of Lemma 5.2.2 is the same as the proof of Lemma 2.2.1 in Chapter 2 (see also [24, 36]). From the proof, we note that the implied constant is independent of c and d. We omit the details.

5.2.1 An improved Strichartz estimate

We recall the following result, which was first proved by Bourgain in [14]:

Proposition 5.2.3. *(Improved Strichartz Estimate) Suppose that N_1, N_2 are dyadic integers such that $N_1 \gg N_2$, and suppose that $u, v \in X^{0,\frac{1}{2}+}(\mathbb{R}^2\times\mathbb{R})$ satisfy, for all t: $supp\,\widehat{u}(t) \subseteq \{|\xi|\sim N_1\}$, and $supp\,\widehat{v}(t) \subseteq \{|\xi|\sim N_2\}$. Then, one has:*

$$\|uv\|_{L^2_{t,x}} \lesssim \frac{N_2}{N_1^{\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \lesssim \frac{1}{N_1^{\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}} \tag{5.19}$$

An alternative proof (in the 1D case) is given in [31].

We want to obtain a similar estimate in $L^{2+}_t L^2_x$. Let us observe that: $\frac{1}{4} = \frac{1}{6} \cdot 0 + \frac{5}{6} \cdot \frac{3}{10}$ and $\frac{1}{6} \cdot \frac{3}{2} + \frac{5}{6} \cdot 0 = \frac{1}{4}$.

We now interpolate between (5.11), (5.12), and the estimate: $\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}$ to deduce that:

$$\|u\|_{L^{4+}_t L^4_x} \lesssim \|u\|_{X^{\frac{1}{4}+,\frac{1}{2}+}} \tag{5.20}$$

Proposition 5.2.4. *Suppose that u, v are as in the assumption of 5.2.3. One then has:*

$$\|uv\|_{L^{2+}_t L^2_x} \lesssim \frac{N_2}{N_1^{\frac{1}{2}-}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \lesssim \frac{1}{N_1^{\frac{1}{2}-}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{1,\frac{1}{2}+}} \tag{5.21}$$

Proof. We use Hölder's Inequality and (5.20) to deduce that:

$$\begin{aligned} \|uv\|_{L^{2++}_t L^2_x} &\leq \|u\|_{L^{4+}_t L^4_x} \|v\|_{L^{4+}_t L^4_x} \lesssim \|u\|_{X^{\frac{1}{4}+,\frac{1}{2}+}} \|v\|_{X^{\frac{1}{4}+,\frac{1}{2}+}} \\ &\lesssim N_1^{\frac{1}{4}+} N_2 \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \end{aligned} \tag{5.22}$$

The Proposition now follows by interpolating between (5.22) and (5.19). □

Finally, we are interested in a version of the improved Strichartz Estimate with a rough cut-off in time. As before, given $\phi \in C^\infty_0(\mathbb{R})$, such that: $0 \leq \phi \leq 1$, $\int_{\mathbb{R}} \phi(t)\, dt = 1$, and $\lambda > 0$, we recall that the *rescaling* ϕ_λ of ϕ is defined by:

$$\phi_\lambda(t) := \frac{1}{\lambda}\, \phi(\frac{t}{\lambda}).$$

Having defined the rescaling, we write, for the scale $N > 1$:

$$\chi(t) = a(t) + b(t), \text{ for } a := \chi * \phi_{N^{-1}}. \tag{5.23}$$

We recall Lemma 8.2. of [31], the authors note the following estimate, which holds in all space dimensions:

$$\|a(t)f\|_{X^{0,\frac{1}{2}+}} \lesssim N^{0+}\|f\|_{X^{0,\frac{1}{2}+}}. \tag{5.24}$$

(The implied constant here is independent of N.)

On the other hand, for any $M \in (1,+\infty)$, one obtains:

$$\|b\|_{L^M_t} \leq C(M,\Phi).$$

If one defines,

$$b_1(t) := \int_{\mathbb{R}} |\hat{b}(\tau)| e^{it\tau} d\tau. \tag{5.25}$$

then, one also has:

$$\|b_1\|_{L^M_t} \leq C(M,\Phi). \tag{5.26}$$

Hence, we can prove:

Proposition 5.2.5. *(Improved Strichartz Estimate with rough cut-off in time) Let $u,v \in X^{0,\frac{1}{2}+}(\mathbb{R}^3 \times \mathbb{R})$ satisfy the assumptions of Proposition 5.2.3. Suppose that $N_1 \gtrsim N$. Let u_1, v_1 be given by:*

$$\widetilde{u_1} := |(\chi u)\widetilde{\ }|, \widetilde{v_1} := |\widetilde{v}\,|.$$

Then one has:

$$\|u_1 v_1\|_{L^2_{t,x}} \lesssim \frac{N_2}{N_1^{\frac{1}{2}-}} \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \tag{5.27}$$

The same bound holds if

$$\widetilde{u_1} := |\widetilde{u}\,|, \widetilde{v_1} := |(\chi v)\widetilde{\ }|.$$

Proof. Let's consider the case when $\widetilde{u_1} = |(\chi u)\widetilde{\ }|, \widetilde{v_1} = |\widetilde{v}\,|$. With notation as earlier, let F_1, F_2 be given by:

$$\widetilde{F_1} := |(au)\widetilde{\ }|, \widetilde{F_2} := |(bu)\widetilde{\ }|.$$

Then, by the triangle inequality, one has:

$$\widetilde{u_1} \leq \widetilde{F_1} + \widetilde{F_2}.$$

Since $\widetilde{u_1}, \widetilde{v_1} \geq 0$, Plancherel's Theorem and duality imply that:

$$\|u_1 v_1\|_{L^2_{t,x}} \sim \sup_{\|c\|_{L^2_{\tau,\xi}}=1} \int_{\tau_1+\tau_2+\tau_3=0} \int_{\xi_1+\xi_2+\xi_3=0} \widetilde{u_1}(\xi_1,\tau_1)\widetilde{v_1}(\xi_2,\tau_2)|c(\xi_3,\tau_3)| d\xi_j d\tau_j$$

$$\leq \sup_{\|c\|_{L^2_{\tau,\xi}}=1} \int_{\tau_1+\tau_2+\tau_3=0} \int_{\xi_1+\xi_2+\xi_3=0} \widetilde{F_1}(\xi_1,\tau_1)\widetilde{v_1}(\xi_2,\tau_2)|c(\xi_3,\tau_3)| d\xi_j d\tau_j +$$

$$\sup_{\|c\|_{L^2_{\tau,\xi}}=1} \int_{\tau_1+\tau_2+\tau_3=0} \int_{\xi_1+\xi_2+\xi_3=0} \widetilde{F_2}(\xi_1,\tau_1)\widetilde{v_1}(\xi_2,\tau_2)|c(\xi_3,\tau_3)| d\xi_j d\tau_j$$

Since $\widetilde{F_1}, \widetilde{F_2}, \widetilde{v_1} \geq 0$, it follows that the latter expression is $\sim \|F_1 v_1\|_{L^2_{t,x}} + \|F_2 v_1\|_{L^2_{t,x}}$. Hence, it follows that:

$$\|u_1 v_1\|_{L^2_{t,x}} \lesssim \|F_1 v_1\|_{L^2_{t,x}} + \|F_2 v_1\|_{L^2_{t,x}}$$

By Proposition 5.2.3, by the frequency assumptions on F_1 and v_1, and by the fact that taking absolute values in the spacetime Fourier transform doesn't change the $X^{s,b}$ norms, we know that:

$$\|F_1 v_1\|_{L^2_{t,x}} \lesssim \frac{N_2}{N_1^{\frac{1}{2}}} \|au\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}}$$

We now use (5.24) to deduce that this expression is:

$$\lesssim \frac{N_2}{N_1^{\frac{1}{2}}} (N^{0+} \|u\|_{X^{0,\frac{1}{2}+}}) \|v\|_{X^{0,\frac{1}{2}+}}$$

Since $N_1 \gtrsim N$, this expression is:

$$\lesssim \frac{N_2}{N_1^{\frac{1}{2}-}}\|u\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}} \tag{5.28}$$

On the other hand, let us consider $c \in L^2_{\tau,\xi}$. With notation as before, one has:

$$|\int_{\tau_1+\tau_2=0}\int_{\xi_1+\xi_2=0}(F_2v_1)\tilde{}(\xi_1,\tau_1)c(\xi_2,\tau_2)d\xi_j d\tau_j|$$

$$= |\int_{\tau_1+\tau_2+\tau_3=0}\int_{\xi_1+\xi_2+\xi_3=0}|(bu)\tilde{}(\xi_1,\tau_1)|\widetilde{v_1}(\xi_2,\tau_2)c(\xi_3,\tau_3)d\xi_j d\tau_j|$$

$$\leq \int_{\tau_0+\tau_1+\tau_2+\tau_3=0}\int_{\xi_1+\xi_2+\xi_3=0}|\widehat{b}(\tau_0)||\widetilde{u}(\xi_1,\tau_1)||\widetilde{v_1}(\xi_2,\tau_2)||c(\xi_3,\tau_3)|d\xi_j d\tau_j := I$$

We then define the functions $G_j, j = 1,\ldots,3$ by:

$$\widetilde{G_1} := |\widetilde{u}|, \widetilde{G_2} := |\widetilde{v_1}|, \widetilde{G_3} := |c|$$

Recalling (5.25), and using Parseval's identity, it follows that:

$$I \lesssim \int_{\mathbb{R}\times\mathbb{R}^3} b_1(t)G_1(x,t)G_2(x,t)G_3(x,t)dxdt$$

We choose $M \in (1,\infty)$, and $2+$ such that: $\frac{1}{M}+\frac{1}{2+}=1$. By an $L^M_t, L^{2+}_t L^2_x, L^2_{t,x}$ Hölder inequality, we deduce that:

$$I \lesssim \|b_1\|_{L^M_t}\|G_1G_2\|_{L^{2+}_t L^2_x}\|G_4\|_{L^2_{t,x}}$$

We use (5.26), Proposition 5.2.4, and Plancherel's theorem to deduce that:

$$I \lesssim \frac{N_2}{N_1^{\frac{1}{2}-}}\|u\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}}\|c\|_{L^2_{\tau,\xi}}.$$

By duality and by Plancherel's theorem, it follows that:

$$\|F_2v_1\|_{L^2_{t,x}} \lesssim \frac{N_2}{N_1^{\frac{1}{2}-}}\|u\|_{X^{0,\frac{1}{2}+}}\|v\|_{X^{0,\frac{1}{2}+}} \tag{5.29}$$

The case when $\widetilde{u_1} := |\widetilde{u}\,|$, $\widetilde{v_1} := |(\chi v)\widetilde{\ }|$ is treated analogously. The Proposition now follows from (5.28) and (5.29).

□

5.2.2 A frequency localized Strichartz estimate

We will need to use the following Strichartz estimate, which assumes that the function which we are estimating satisfies appropriate localization in frequency. A similar result was proved in two dimensions in [15], and we had to use its modification in Chapter 4:

Lemma 5.2.6. *Suppose that Q is a ball in $\mathbb{R}^3$ of radius N, and center ξ_0. Suppose that u satisfies $supp\,\widehat{u} \subseteq Q$. Then, one has:*

$$\|u\|_{L^{\frac{10}{3}}_{t,x}} \lesssim N^{\frac{3}{5}} \|u\|_{X^{0,\frac{1}{5}+}} \tag{5.30}$$

Proof. Suppose that u is as in the assumption of the Lemma. Suppose that $b' > \frac{1}{4}$. Then, one has, by the Hausdorff-Young Inequality in space and time:

$$\|u\|_{L^{\frac{10}{3}}_{t,x}} \leq$$

$$\Big(\int_Q \int_{\mathbb{R}} |\widetilde{u}(\xi,\tau)|^{\frac{10}{7}} d\tau d\xi\Big)^{\frac{7}{10}} =$$

$$\Big(\int_Q \int_{\mathbb{R}} (1+|\tau+|\xi|^2)^{\frac{10b'}{7}} |\widetilde{u}(\xi,\tau)|^{\frac{10}{7}} (1+|\tau+|\xi|^2)^{-\frac{10b'}{7}} d\tau d\xi\Big)^{\frac{7}{10}}$$

We use an $L^{\frac{7}{5}}$, $L^{\frac{7}{2}}$ Hölder inequality in τ to deduce that this is:

$$\leq \Big(\int_Q \Big(\int_{\mathbb{R}} (1+|\tau+|\xi|^2)^{2b'} |\widetilde{u}(\xi,\tau)|^2 d\tau\Big)^{\frac{5}{7}} \Big(\int_{\mathbb{R}} (1+|\tau+|\xi|^2)^{-5b'} d\tau\Big)^{\frac{2}{7}} d\xi\Big)^{\frac{7}{10}}$$

Since $b' > \frac{1}{5}$, this expression is:

$$\lesssim \Big(\int_Q \Big(\int_{\mathbb{R}} (1+|\tau+|\xi|^2)^{2b'} |\widetilde{u}(\xi,\tau)|^2 d\tau\Big)^{\frac{5}{7}} d\xi\Big)^{\frac{7}{10}}$$

$$= \Big(\int_Q \big((\int_{\mathbb{R}} (1+|\tau+|\xi|^2)^{2b'} |\widetilde{u}(\xi,\tau)|^2 d\tau)^{\frac{1}{2}} \big)^{\frac{10}{7}} d\xi \Big)^{\frac{7}{10}}$$

By an L^2, L^5 Hölder inequality in ξ, this expression is:

$$\leq \big(\int_Q \int (1+|\tau+|\xi|^2)^{2b'} |\widetilde{u}(\xi,\tau)|^2 d\tau d\xi \big)^{\frac{1}{2}} \big(\int_Q 1 d\xi \big)^{\frac{1}{5}}$$

$$\lesssim N^{\frac{3}{5}} \|u\|_{X^{0,b'}}$$

since we are working in three dimensions.

□

We can now interpolate between the bounds (5.11) and (5.30) to deduce:

Proposition 5.2.7. *Suppose that u is as in the assumption of Lemma 5.2.6. Suppose, furthermore, that* $b_1, s_1 \in \mathbb{R}$ *satisfy:* $\frac{1}{5} < b_1 < \frac{1}{2}+$, *and* $s_1 > 1 - 2b_1$. *Then, one has:*

$$\|u\|_{L^{\frac{10}{3}}_{t,x}} \lesssim N^{s_1} \|u\|_{X^{0,b_1}} \tag{5.31}$$

5.3 Proof of the Main Result

5.3.1 Definition of the $\mathcal{D}$ operator

We start by defining an appropriate multiplier:

Suppose $N > 1$ is given. Let $\theta : \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by:

$$\theta(\xi) := \begin{cases} \big(\frac{|\xi|}{N}\big)^s, \text{if } |\xi| \geq N \\ 1, \text{ if } |\xi| \leq N \end{cases} \tag{5.32}$$

Then, if $f : \mathbb{R}^3 \rightarrow \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(\xi) := \theta(\xi)\hat{f}(\xi). \tag{5.33}$$

We observe that:

$$\|\mathcal{D}f\|_{L^2_x} \lesssim_s \|f\|_{H^s_x} \lesssim_s N^s \|\mathcal{D}f\|_{L^2_x}. \tag{5.34}$$

Our goal is to then estimate $\|\mathcal{D}u(t)\|_{L^2_x}$, from which we can estimate $\|u(t)\|_{H^s_x}$ by (5.34). In order to do this, we first need to have good local-in-time bounds.

5.3.2 Local-in-time bounds

Let u denote the global solution to (5.1). One then has:

Proposition 5.3.1. *(Local-in-time bounds) There exist $\delta = \delta(s, E(\Phi), M(\Phi)), C = C(s, E(\Phi), M(\Phi)) > 0$, which are continuous in energy and mass, such that for all $t_0 \in \mathbb{R}$, there exists a globally defined function $v : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ such that:*

$$v|_{[t_0, t_0+\delta]} = u|_{[t_0, t_0+\delta]}. \tag{5.35}$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \tag{5.36}$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(\Phi), M(\Phi)) \|\mathcal{D}u(t_0)\|_{L^2}. \tag{5.37}$$

We prove Proposition 5.3.1 in the Appendix of this chapter.

Remark 5.3.2. *We note that mass and energy are continuous on $H_x^{\frac{3}{2}+}$. To note that energy is continuous on $H_x^{\frac{3}{2}+}$, we use the fact that:*

$$\|(K * (w_1 w_2)) w_3 w_4\|_{L^1_x} \leq \|(K * (w_1 w_2))\|_{L^2_x} \|w_3 w_4\|_{L^2_x} \lesssim \|w_1 w_2\|_{L^2_x} \|w_3 w_4\|_{L^2_x}$$

$$\leq \|w_1\|_{L^2_x} \|w_2\|_{L^\infty_x} \|w_3\|_{L^2_x} \|w_4\|_{L^\infty_x} \lesssim \|w_1\|_{H_x^{\frac{3}{2}+}} \|w_2\|_{H_x^{\frac{3}{2}+}} \|w_3\|_{H_x^{\frac{3}{2}+}} \|w_4\|_{H_x^{\frac{3}{2}+}}$$

Hence, we can use smooth functions as initial data (and hence work with smooth solutions), and use density to deduce the general solution as in the previous chapters.

5.3.3 Estimate on the growth of $\|\mathcal{D}u(t)\|_{L_x^2}^2$

We use the equation to deduce that:

$$\frac{d}{dt}\|\mathcal{D}u(t)\|_{L_x^2}^2 \sim \int_{\xi_1+\xi_2=0} (\mathcal{D}u_t)\widehat{\ }(\xi_1)(\mathcal{D}\overline{u})\widehat{\ }(\xi_2)d\xi_j + \int_{\xi_1+\xi_2=0} (\mathcal{D}u)\widehat{\ }(\xi_1)(\mathcal{D}\overline{u}_t)\widehat{\ }(\xi_2)d\xi_j$$

$$=: (I) + (II)$$

We observe that then:

$$(I) = \int_{\xi_1+\xi_2=0} (\mathcal{D}i\Delta u - i\mathcal{D}(|u|^2u) - i\mathcal{D}((K * |u|^2)u))\widehat{\ }(\xi_1)(\mathcal{D}\overline{u})\widehat{\ }(\xi_2)d\xi_j$$

$$= i\int_{\xi_1+\xi_2=0} (\theta(\xi_1))^2(-\xi_1^2)\widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)d\xi_j - i\int_{\xi_1+\xi_2=0} (|u|^2u)\widehat{\ }(\xi_1)(\theta(\xi_2))^2\widehat{\overline{u}}(\xi_2)d\xi_j$$

$$-i\int_{\xi_1+\xi_2=0} ((K * |u|^2)u)\widehat{\ }(\xi_1)(\theta(\xi_2))^2\widehat{\overline{u}}(\xi_2)d\xi_j$$

$$= i\int_{\xi_1+\xi_2=0} (\theta(\xi_1))^2(-\xi_1^2)\widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)d\xi_j - i\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)(\theta(\xi_4))^2\widehat{\overline{u}}(\xi_4)d\xi_j$$

$$-i\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \widehat{K}(\xi_1+\xi_2)\widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)(\theta(\xi_4))^2\widehat{\overline{u}}(\xi_4)d\xi_j$$

$$= i\int_{\xi_1+\xi_2=0} (\theta(\xi_1))^2(-\xi_1^2)\widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)d\xi_j - \frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \big((\theta(\xi_2))^2+(\theta(\xi_4))^2\big)\widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\overline{u}}(\xi_4)d\xi_j$$

$$-\frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \widehat{K}(\xi_1+\xi_2)(\theta(\xi_4))^2\widehat{u}(\xi_1)\widehat{\overline{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\overline{u}}(\xi_4)d\xi_j$$

$$-\frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\widehat{K}(\xi_3+\xi_4)(\theta(\xi_2))^2\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j$$

$$=i\int_{\xi_1+\xi_2=0}(\theta(\xi_1))^2(-\xi_1^2)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)d\xi_j-\frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\big((\theta(\xi_2))^2+(\theta(\xi_4))^2\big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j$$

$$-\frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\widehat{K}(\xi_3+\xi_4)\big((\theta(\xi_2))^2+(\theta(\xi_4))^2\big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j \qquad (5.38)$$

Here, we used the appropriate symmetrization $\xi_2 \leftrightarrow \xi_4$ and the fact that K is even, and hence $\widehat{K}$ is also even, so we have: $\widehat{K}(\xi_1+\xi_2)=\widehat{K}(\xi_3+\xi_4)$.

Analogously:

$$(II)=i\int_{\xi_1+\xi_2=0}(\theta(\xi_2))^2\xi_2^2\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)d\xi_j+\frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\big((\theta(\xi_1))^2+(\theta(\xi_3))^2\big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j$$

$$+\frac{i}{2}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\widehat{K}(\xi_3+\xi_4)\big((\theta(\xi_1))^2+(\theta(\xi_3))^2\big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j \qquad (5.39)$$

From (5.38), (5.39), we deduce that, for some fixed $c\in\mathbb{R}$, one has:

$$\frac{d}{dt}\|\mathcal{D}u(t)\|_{L_x^2}^2=ci\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\big((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2\big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j$$

$$+ci\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\widehat{K}(\xi_3+\xi_4)\big((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2\big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j$$

$$=ci\lambda_4((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2;u)$$

$$+ci\lambda_4(((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2)\widehat{K}(\xi_3+\xi_4);u)$$

$$=\lambda_4(M_4;u) \qquad (5.40)$$

Here, $M_4 : \Gamma_4 \to \mathbb{C}$ is given by:

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) := ic\big((\theta(\xi_1))^2 - (\theta(\xi_2))^2 + (\theta(\xi_3))^2 - (\theta(\xi_4))^2\big)\widehat{K}(\xi_3 + \xi_4) \quad (5.41)$$

Let $\delta > 0$ be as in Proposition 5.3.1. Let $t_0 \in \mathbb{R}$ be given. The quantity we want to estimate is:

$$\|\mathcal{D}u(t_0 + \delta)\|_{L^2_x}^2 - \|\mathcal{D}u(t_0)\|_{L^2_x}^2 \quad (5.42)$$

The bound that we prove is:

Lemma 5.3.3. *The following bound holds for all $t_0 \in \mathbb{R}$:*

$$\big|\|\mathcal{D}u(t_0 + \delta)\|_{L^2_x}^2 - \|\mathcal{D}u(t_0)\|_{L^2_x}^2\big| \lesssim \frac{1}{N^{1-}}\|\mathcal{D}u(t_0)\|_{L^2_x}^2$$

The implied constant is independent of t_0.

As in previous chapters, we see that Lemma 5.3.3 implies Theorem 5.1.1.

We now prove Lemma 5.3.3

Proof. Let us consider WLOG the case when $t_0 = 0$. The proof in the general case is the same. Let v be the function obtained by Proposition 5.3.1 when we let $t_0 = 0$. By (5.40), we then have to estimate:

$$\int_0^\delta \lambda_4(M_4; u(t))dt = \int_0^\delta \lambda_4(M_4; v(t))dt$$

We now use a dyadic localization. We suppose that $|\xi_j| \sim N_j$, where N_j are dyadic integers. Let us WLOG suppose that $N_1 \geq N_2 \geq N_3 \geq N_4$. The other cases are bounded in an analogous way. By construction, we know that $M_4 = 0$ unless one has:

$$N_1 \sim N_2; N_1 \gtrsim N \quad (5.43)$$

We henceforth consider only such cases.

Furthermore, by construction of θ, and by (5.5), we note that [1]:

$$M_4 = O(\theta(N_1)\theta(N_2)) \tag{5.44}$$

Let $\chi := \chi_{[0,\delta]}(t)$. By the triangle inequality and by (5.44), we have to estimate the following quantity:

$$I_{N_1,N_2,N_3,N_4} := \tag{5.45}$$

$$\int_{\tau_1+\tau_2+\tau_3+\tau_4=0}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\theta(N_1)\theta(N_2)|\widetilde{(\chi v)}_{N_1}(\xi_1,\tau_1)||\widetilde{\overline{v}}_{N_2}(\xi_2,\tau_2)||\widetilde{v}_{N_3}(\xi_3,\tau_3)||\widetilde{\overline{v}}_{N_4}(\xi_4,\tau_4)|d\xi_j d\tau_j$$

We consider two cases:

Case 1: $N_3 \sim N_1$

Let us define the functions $F_j, j = 1,\ldots,4$ by:

$$\widetilde{F_1} := |\widetilde{(\chi v)}_{N_1}|, \widetilde{F_2} := |\widetilde{\overline{v}}_{N_2}|, \widetilde{F_3} := |\widetilde{v}_{N_3}|, \widetilde{F_4} := |\widetilde{\overline{v}}_{N_4}|$$

By Parseval's Identity, one then has:

$$I_{N_1,N_2,N_3,N_4} \sim \int_{\mathbb{R}}\int_{\mathbb{R}^3}\theta(N_1)\theta(N_2)F_1\overline{F_2}F_3\overline{F_4}dxdt$$

We can now use an $L^{\frac{10}{3}-}_{t,x}, L^{\frac{10}{3}+}_{t,x}, L^{\frac{10}{3}}_{t,x}, L^{10-}_{t,x}$ Hölder Inequality to deduce that:

$$|I_{N_1,N_2,N_3,N_4}| \leq \theta(N_1)\theta(N_2)\|F_1\|_{L^{\frac{10}{3}-}_{t,x}}\|F_2\|_{L^{\frac{10}{3}+}_{t,x}}\|F_3\|_{L^{\frac{10}{3}}_{t,x}}\|F_4\|_{L^{10-}_{t,x}}$$

which by (5.16), (5.15), (5.11), and (5.14) is:

$$\lesssim \theta(N_1)\theta(N_2)\|F_1\|_{X^{0,\frac{1}{2}-}}\|F_2\|_{X^{0+,\frac{1}{2}+}}\|F_3\|_{X^{0,\frac{1}{2}+}}\|F_4\|_{X^{1,\frac{1}{2}+}}$$

Since taking absolute values in the spacetime Fourier Transform doesn't change the

[1] Here, we are slightly abusing notation by writing $\theta(N_j)$ instead of $\theta(N_j,0)$. We recall that θ is a radial function, so this doesn't matter.

$X^{s,b}$ norm, it follows that the preceding quantity equals:

$$\lesssim \theta(N_1)\theta(N_2)\|(\chi v)_{N_1}\|_{X^{0,\frac{1}{2}-}}\|v_{N_2}\|_{X^{0+,\frac{1}{2}+}}\|v_{N_3}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{1,\frac{1}{2}+}}$$

$$\sim \|(\chi\mathcal{D}v)_{N_1}\|_{X^{0,\frac{1}{2}-}}\big(N_2^{0+}\|\mathcal{D}v_{N_2}\|_{X^{0,\frac{1}{2}+}}\big)\big(\frac{1}{N_3}\|v_{N_3}\|_{X^{1,\frac{1}{2}+}}\big)\|v_{N_4}\|_{X^{1,\frac{1}{2}+}}$$

which by using Proposition 5.2.2, localization in frequency, (5.45), and the assumption of Case 1 is:

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}}$$

By using Proposition 5.3.1, and the uniform bounds on $\|u(t)\|_{H^1}$, this expression is:

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}\Phi\|^2_{L^2} \tag{5.46}$$

Case 2: $N_1 \gg N_3$

In this case, we let:

$$\widetilde{G_1} := |(\chi v)_{N_1}\widetilde{\ }|, \widetilde{G_2} := |\widetilde{v}_{N_2}|, \widetilde{G_3} := |\widetilde{v}_{N_3}|, \widetilde{G_4} := |\widetilde{v}_{N_4}|$$

By Parseval's Identity, it follows that:

$$I_{N_1,N_2,N_3,N_4} \sim \int_{\mathbb{R}}\int_{\mathbb{R}^3}\theta(N_1)\theta(N_2)G_1\overline{G_2}G_3\overline{G_4}dxdt$$

We use the Cauchy-Schwarz Inequality to deduce that the previous expression is:

$$\leq \theta(N_1)\theta(N_2)\|G_1G_3\|_{L^2_{t,x}}\|G_2G_4\|_{L^2_{t,x}}$$

By the frequency assumptions, we know that:

$$N_1 \gg N_3, N_2 \gg N_4, N_1 \sim N_2 \gtrsim N.$$

Hence, we can use Proposition 5.2.5 and Proposition 5.2.3 to deduce that the above expression is:

$$\lesssim \theta(N_1)\theta(N_2)\big(\frac{N_3}{N_1^{\frac{1}{2}-}}\|G_1\|_{X^{0,\frac{1}{2}+}}\|G_3\|_{X^{0,\frac{1}{2}+}}\big)\big(\frac{N_4}{N_2^{\frac{1}{2}-}}\|G_2\|_{X^{0,\frac{1}{2}+}}\|G_4\|_{X^{0,\frac{1}{2}+}}\big)$$

which by frequency localization is:

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}v_{N_1}\|_{X^{0,\frac{1}{2}+}}\|v_{N_2}\|_{X^{1,\frac{1}{2}+}}\|\mathcal{D}v_{N_3}\|_{X^{0,\frac{1}{2}+}}\|v_{N_4}\|_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}v\|^2_{X^{0,\frac{1}{2}+}}\|v\|^2_{X^{1,\frac{1}{2}+}}$$

By Proposition 5.3.1, this expression is:

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}\Phi\|^2_{L^2} \tag{5.47}$$

We combine (5.46) and (5.47), and we sum in the N_j keeping in mind the condition (5.43) to obtain the Lemma.

□

5.4 Appendix: Proof of Proposition 5.3.1

Proof. The proof is based on a fixed-point argument, and is a modification of the proof of the similar result in two dimensions which is given in Chapter 4. The latter proof is a slight modification of a proof from [15]. Let us WLOG look at $t_0 = 0$. The general proof is analogous. As before, we take: $\chi, \phi, \psi \in C_0^\infty(\mathbb{R})$, with $0 \leq \chi, \phi, \psi \leq 1$, such that:

$$\chi = 1 \text{ on } [-1,1]\ , \chi = 0 \text{ outside } [-2,2]. \tag{5.48}$$

$$\phi = 1 \text{ on } [-2,2]\ , \phi = 0 \text{ on } [-4,4]. \tag{5.49}$$

$$\psi = 1 \text{ on } [-4,4]\ , \psi = 0 \text{ on } [-8,8]. \tag{5.50}$$

We let:

$$\chi_\delta := \chi(\frac{\cdot}{\delta}), \phi_\delta := \phi(\frac{\cdot}{\delta}), \psi_\delta := \psi(\frac{\cdot}{\delta}). \tag{5.51}$$

Then:

$$\chi_\delta = 1 \text{ on } [-\delta, \delta]\ , \chi_\delta = 0 \text{ outside } [-2\delta, 2\delta]. \tag{5.52}$$

$$\phi_\delta = 1 \text{ on } [-2\delta, 2\delta]\ , \phi_\delta = 0 \text{ outside } [-4\delta, 4\delta]. \tag{5.53}$$

$$\psi_\delta = 1 \text{ on } [-4\delta, 4\delta]\ , \psi_\delta = 0 \text{ outside } [-8\delta, 8\delta]. \tag{5.54}$$

Similarly as in Chapter 2, we denote by w_δ the function $\phi_\delta w$, and we consider the operator L defined by:

$$Lw := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t)\int_0^t S(t-t')\big(\mu_1|w_\delta|^2 w_\delta(t') + \mu_2(K * |w_\delta|^2)w_\delta(t')\big)dt' \tag{5.55}$$

Let $c > 0$ be the constant[2] such that $\|\chi_\delta S(t)\Phi\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s}$. Such a constant exists by using arguments from [71]. Let us take $b := \frac{1}{2}+$. We then define:

$$B := \{w; \|w\|_{X^{1,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^1}, \|w\|_{X^{s,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s}\}$$

Arguing as in Chapter 2, B is complete w.r.t $\|\cdot\|_{X^{1,b}}$. For $w \in B$, we obtain:

$$\|Lw\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} + c_1\delta^{\frac{1-2b}{2}}\||w_\delta|^2 w_\delta\|_{X^{s,b-1}} + c_1\delta^{\frac{1-2b}{2}}\|(K * |w_\delta|^2)w_\delta\|_{X^{s,b-1}} \tag{5.56}$$

Similarly, we obtain:

$$\|\mathcal{D}Lw\|_{X^{0,b}} \leq c\delta^{\frac{1-2b}{2}}\|\mathcal{D}\Phi\|_{L^2} + c_1\delta^{\frac{1-2b}{2}}\|\mathcal{D}(|w_\delta|^2 w_\delta)\|_{X^{0,b-1}} + c_1\delta^{\frac{1-2b}{2}}\|\mathcal{D}((K*|w_\delta|^2)w_\delta)\|_{X^{0,b-1}} \tag{5.57}$$

[2] All previous localization estimates in time carry over to $\mathbb{R}^3$.

We now estimate $\|(K * |w_\delta|^2)w_\delta\|_{X^{s,b-1}}$ by duality. The term $\||w_\delta|^2 w_\delta\|_{X^{s,b-1}}$ is estimated in the same way. We suppose that we are given $c = c(\xi, \tau)$ such that:

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}} |c(\xi,\tau)|^2 d\tau d\xi = 1.$$

We want to estimate:

$$I := \int_{\xi_1-\xi_2+\xi_3-\xi_4=0} \int_{\tau_1-\tau_2+\tau_3-\tau_4=0} \frac{|c(\xi_4,\tau_4)|}{(1+|\tau_4+|\xi_4|^2|)^{1-b}} (1+|\xi_4|)^s |\widetilde{w_\delta}(\xi_1,\tau_1)|$$

$$|\widetilde{w_\delta}(\xi_2,\tau_2)||\widetilde{w_\delta}(\xi_3,\tau_3)||\widehat{K}(\xi_1+\xi_2)|d\tau_j d\xi_j$$

Since $\widehat{K} \in L^\infty(\mathbb{R}^3)$, by (5.5), this expression is:

$$\lesssim \sum_{\xi_1-\xi_2+\xi_3-\xi_4=0} \int_{\tau_1-\tau_2+\tau_3-\tau_4=0} \frac{|c(\xi_4,\tau_4)|}{(1+|\tau_4+|\xi_4|^2|)^{1-b}} (1+|\xi_4|)^s |\widetilde{w_\delta}(\xi_1,\tau_1)|$$

$$|\widetilde{w_\delta}(\xi_2,\tau_2)||\widetilde{w_\delta}(\xi_3,\tau_3)|d\tau_j d\xi_j$$

Let us write:

$$\mathbb{R}^3 = \bigcup_{k=0}^{\infty} D_k;\ D_k = \{\xi \in \mathbb{R}^3; |\xi| \sim 2^k\}$$

Let I_{k_1,k_2,k_3} denote the contribution to I with $\xi_j \in D_{k_j}$, for $j = 1, 2, 3$. Let us consider WLOG the case when:

$$k_1 \geq k_2 \geq k_3. \tag{5.58}$$

The contributions from other cases are bounded analogously.

Following [15], we write:

$$D_{k_1} \subseteq \bigcup_{\alpha} Q_\alpha$$

Here, Q_α are balls of radius 2^{k_2}. We can choose this cover so that each element of D_{k_1} lies in a fixed finite number of Q_α. This number is independent of k_1 and k_2.

If $\xi_1 \in Q_\alpha$, then since $\xi_4 = \xi_1 - \xi_2 + \xi_3$, $|\xi_2|, |\xi_3| \lesssim 2^{k_2}$, it follows that ξ_4 lies in $\widetilde{Q}_\alpha$, a dilate of Q_α. Thus, the term that we want to estimate is:

$$J_{k_1,k_2,k_3} := 2^{k_1 s} \sum_{\alpha} \int_{\xi_1 \in Q_\alpha, \xi_2 \in D_{k_2}, \xi_3 \in D_{k_3}, \xi_4 \in \widetilde{Q}_\alpha, \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0} \int_{\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0}$$

$$|\widetilde{w_\delta}(\xi_1,\tau_1)||\widetilde{w_\delta}(\xi_2,\tau_2)||\widetilde{w_\delta}(\xi_3,\tau_3)| \frac{|c(\xi_4,\tau_4)|}{(1+|\tau_4 + |\xi_4|^2|)^{1-b}} d\tau_j d\xi_j$$

We now define:

$$F_\alpha(x,t) := \int_{\xi \in \widetilde{Q}_\alpha} \int_{\mathbb{R}} \frac{|c(\xi,\tau)|}{(1+|\tau+|\xi|^2|)^{1-b}} e^{i(\langle x,\xi\rangle + t\tau)} d\tau d\xi \tag{5.59}$$

$$G_\alpha(x,t) := \int_{\xi \in Q_\alpha} \int_{\mathbb{R}} |\widetilde{w_\delta}(\xi,\tau)| e^{i(\langle x,\xi\rangle + t\tau)} d\tau d\xi \tag{5.60}$$

$$H_j(x,t) := \sum_{\xi \in D_{k_j}} |\widetilde{w_\delta}(\xi,\tau)| e^{i(\langle x,\xi\rangle + t\tau)} d\tau d\xi \tag{5.61}$$

for $j = 2, 3$.

By Parseval's identity and Hölder's inequality, we deduce:

$$J_{k_1,k_2,k_3} \lesssim 2^{k_1 s} \sum_{\alpha} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \overline{F_\alpha} G_\alpha \overline{H_2} H_3 dx dt$$

$$\leq 2^{k_1 s} \sum_{\alpha} \|F_\alpha\|_{L^{\frac{10}{3}}_{t,x}} \|G_\alpha\|_{L^{\frac{10}{3}}_{t,x}} \|H_2\|_{L^{\frac{10}{3}}_{t,x}} \|H_3\|_{L^{10}_{t,x}}. \tag{5.62}$$

Given a dyadic integer M, let us define $(w_\delta)_M$ by: $((w_\delta)_M)\widehat{\ } = \widehat{w_\delta}\chi_{D_M}$, and we note that localization in t and in ξ commute. This is a slight abuse of notation, but we interpret w_δ as a localization in time if $\delta > 0$ is small, and we interpret w_N as a localization in frequency if N is a dyadic integer. We similarly abuse notation by writing $(w_\delta)_\alpha$ for the inverse Fourier transform of $\widehat{w_\delta}\chi_{Q_\alpha}$.

We first note by Proposition 5.2.7, with s_1, b_1 as in the assumptions of the Proposition:

$$\|G_\alpha\|_{L^{\frac{10}{3}}_{t,x}} \lesssim 2^{k_2 s_1} \|(w_\delta)_\alpha\|_{X^{0,b_1}}$$

By interpolation, it follows that:

$$\|(w_\delta)_\alpha\|_{X^{0,b_1}} \lesssim \|(w_\delta)_\alpha\|^{\theta}_{X^{0,0}} \|(w_\delta)_\alpha\|^{1-\theta_0}_{X^{0,b}}$$

Here:

$$\theta_0 := 1 - \frac{b_1}{b} \tag{5.63}$$

By construction of ψ_δ, we obtain:

$$\|(w_\delta)_\alpha\|_{X^{0,0}} \sim \|(w_\delta)_\alpha\|_{L^2_{t,x}} = \|(w_\delta)_\alpha \psi_\delta\|_{L^2_{t,x}}$$

We now use Hölder's inequality and (5.17) to see that this expression is:

$$\lesssim \|(w_\delta)_\alpha\|_{L^4_t L^2_x} \|\psi_\delta\|_{L^4_t} \lesssim \delta^{\frac{1}{4}} \|(w_\delta)_\alpha\|_{X^{0,\frac{1}{4}+}} \leq \delta^{\frac{1}{4}} \|(w_\delta)_\alpha\|_{X^{0,b}}.$$

Consequently:

$$\|G_\alpha\|_{L^{\frac{10}{3}}_{t,x}} \lesssim 2^{k_2 s_1} \delta^{\frac{\theta}{4}} \|(w_\delta)_\alpha\|_{X^{0,b}}$$

$$\lesssim 2^{k_2 s_1} \delta^{\frac{\theta_0}{4} + \frac{1-2b}{2}} \|w_\alpha\|_{X^{0,b}} \tag{5.64}$$

In the last inequality, we used appropriate time-localization in $X^{0,b}$.

Analogously:

$$\|H_2\|_{L^{\frac{10}{3}}_{t,x}} \lesssim 2^{k_2 s_1} \delta^{\frac{\theta_0}{4}+\frac{1-2b}{2}} \|w_{2^{k_2}}\|_{X^{0,b}} \tag{5.65}$$

In order to estimate $\|H_3\|_{L^{10}_{t,x}}$, we recall by (5.13) that:

$$\|H_3\|_{L^{10}_{t,x}} \lesssim \|H_3\|_{X^{1+,\frac{1}{2}+}} \sim (2^{k_3})^{1+}\|H_3\|_{X^{0,b}}$$

$$\sim (2^{k_3})^{1+}\|(w_\delta)_{2^{k_3}}\|_{X^{0,b}}$$

$$\lesssim (2^{k_3})^{1+}\delta^{\frac{1-2b}{2}}\|w_{2^{k_3}}\|_{X^{0,b}} \tag{5.66}$$

Finally, we want to estimate $\|F_\alpha\|_{L^{\frac{10}{3}}_{t,x}}$. In order to do this, let c_α denote the localization of $c = c(\xi,\tau)$ to $\widetilde{Q}_\alpha$. We use Proposition 5.2.7 with s_1, b_1 as in the assumption of this Proposition to deduce that:

$$\|F_\alpha\|_{L^{\frac{10}{3}}_{t,x}} \lesssim 2^{k_2 s_1}\|\mathcal{F}^{-1}\Big(\frac{c_\alpha(\xi,\tau)}{(1+|\tau+|\xi|^2)^{1-b}}\Big)\|_{X^{0,b_1}}$$

Here, $\mathcal{F}^{-1}$ denotes the inverse Spacetime Fourier transform. Consequently, if the condition:

$$b_1 \leq 1-b \tag{5.67}$$

one has:

$$\|F_\alpha\|_{L^{\frac{10}{3}}_{t,x}} \lesssim 2^{k_2 s_1}\|c_\alpha\|_{L^2_{\tau,\xi}} \tag{5.68}$$

We now combine (5.62),(5.64),(5.65),(5.66), and (5.68) to deduce that, assuming the condition (5.67), one has:

$$|J_{k_1,k_2,k_3}| \lesssim \sum_\alpha \delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}} 2^{k_1 s} 8^{k_2 s_1} (2^{k_3})^{1+} \|w_\alpha\|_{X^{0,b}} \|w_{2^{k_2}}\|_{X^{0,b}} \|w_{2^{k_3}}\|_{X^{0,b}} \|c_\alpha\|_{L^2_{\tau,\xi}}$$

$$\lesssim \sum_{\alpha} \delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}} (2^{-k_2})^{0+}(2^{-k_3})^{0+}\|w_\alpha\|_{X^{s,b}}\|w\|_{X^{3s_1+,b}}\|w\|_{X^{1,b}}\|c_\alpha\|_{L^2_{\tau,\xi}}$$

which by using the Cauchy-Schwarz inequality in α is:

$$\lesssim \delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}} (2^{-k_2})^{0+}(2^{-k_3})^{0+}\|w_{2^{k_1}}\|_{X^{s,b}}\|w\|_{X^{3s_1+,b}}\|w\|_{X^{1,b}}\|c_{2^{k_1}}\|_{L^2_{\tau,\xi}}$$

Here $c_{2^{k_1}}$ denotes the localization to the region obtained by the union of the $\widetilde{Q}_\alpha$. The notation is justified by the fact that on this region, one has $|\xi| \sim 2^{k_1}$.

We now sum a geometric series in k_2 and k_3, and we use the Cauchy-Schwarz inequality in k_1 to deduce that:

$$\sum_{k_1,k_2,k_3} |J_{k_1,k_2,k_3}| \lesssim \delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}} \|w\|_{X^{s,b}}\|w\|_{X^{3s_1+,b}}\|w\|_{X^{1,b}} \tag{5.69}$$

We now choose $s_1 := \frac{1}{3}-$ such that:

$$3s_1+ = 1 \tag{5.70}$$

By Proposition 5.2.7, we can take $b_1 := \frac{1-\frac{1}{3}-}{2}+ = \frac{1}{3}+ \in (\frac{1}{5}, \frac{1}{2}+)$. We must check now that θ_0 as defined in (5.63) belongs to $(0,1)$, and that the condition (5.67) holds. We note indeed:

$$\theta_0 = 1 - \frac{b_1}{b} = 1 - \frac{\frac{1}{3}+}{\frac{1}{2}+} > \frac{1}{4}$$

hence $\theta_0 \in (0,1)$ and:

$$1 - b = \frac{1}{2}- \geq \frac{1}{3}+ = b_1$$

By the preceding arguments, it follows that:

$$\|(K * |w_\delta|^2)w_\delta\|_{X^{s,b-1}} \lesssim \delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}} \|w\|_{X^{s,b}}\|w\|^2_{X^{1,b}}$$

An analogous bound holds for $\||w_\delta|^2 w_\delta\|_{X^{s,b-1}}$. Consequently, by (5.56), it follows that for $w \in B$, one has:

$$\|Lw\|_{X^{s,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} + c_2\delta^{\frac{\theta_0}{2}+3(1-2b)}\|\Phi\|^2_{H^1}\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^s} \tag{5.71}$$

We note that $\frac{\theta_0}{2} + 3(1-2b) > 0$, if $b = \frac{1}{2}+$ is sufficiently close to $\frac{1}{2}$.

The argument works for $s = 1$, and we obtain:

$$\|Lw\|_{X^{1,b}} \leq c\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^1} + c_2\delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}}\|\Phi\|_{H^1}^2\delta^{\frac{1-2b}{2}}\|\Phi\|_{H^1} \tag{5.72}$$

Furthermore, the same arguments that we used to obtain (5.71) and (5.72) imply that, for $v, w \in B$, one has:

$$\begin{aligned}\|Lv - Lw\|_{X^{1,b}} &\leq c_3\delta^{\frac{\theta_0}{2}+2(1-2b)}(\|v\|_{X^{1,b}} + \|w\|_{X^{1,b}})^2\|v-w\|_{X^{1,b}} \\ &\leq c_4\delta^{\frac{\theta_0}{2}+3(1-2b)}\|\Phi\|_{H^1}^2\|v-w\|_{X^{1,b}} \end{aligned} \tag{5.73}$$

We now choose $\delta > 0$ sufficiently small so that: $c_2\delta^{\frac{\theta_0}{2}+\frac{3(1-2b)}{2}}\|\Phi\|_{H^1}^2 \leq c$, and $c_4\delta^{\frac{\theta_0}{2}+3(1-2b)}\|\Phi\|_{H^1}^2 \leq \frac{1}{2}$. (5.71), (5.72), and (5.73) will then imply that L is a contraction on (B, d), where $d(v, w) := \|v - w\|_{X^{1,b}}$. We recall that in Chapter 2, we proved that (B, d) is a Banach space in the 1D periodic setting (cf. Proposition 2.3.2 in Chapter 2). The proof was based on the use of Theorem 1.2.5. from [28]. The same proof works in the 3D non-periodic setting. Hence, we can apply the Banach fixed point Theorem to deduce that there exists a fixed point $v \in B$ of L.

For this fixed point $v \in B$, we have:

$$v := \chi_\delta(t)S(t)\Phi - i\chi_\delta(t)\int_0^t S(t-t')\big(|v_\delta|^2 v_\delta(t') + (K * |v_\delta|^2)v_\delta(t')\big)dt'$$

We take $\mathcal{D}$'s of both sides, and argue as before to deduce that:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq c\|\mathcal{D}\Phi\|_{L^2} + c_5\delta^{\frac{\theta_0}{2}+3(1-2b)}\|\Phi\|_{H^1}^2\|\mathcal{D}v\|_{X^{0,b}}$$

We choose $\delta > 0$ possibly even smaller than the one we found earlier such that:

$$c_5\delta^{\frac{\theta_0}{2}+3(1-2b)}\|\Phi\|_{H^1}^2 \leq \frac{1}{2}$$

Let us note that this doesn't affect any of the previous estimates. Since $\|\mathcal{D}v\|_{X^{0,b}} \leq$

$\|v\|_{X^{s,b}} < \infty$, it follows that:

$$\|\mathcal{D}v\|_{X^{0,b}} \leq 2c\delta^{\frac{1-2b}{2}}\|\mathcal{D}\Phi\|_{L^2}$$

We now have to check uniqueness. Let us note that we are considering the solution u of (5.1) which satisfies exponential in time bounds, i.e., for $s \geq 1$, there exists $C = C(s), K = K(s) > 0$ s.t. for all $t \in \mathbb{R}$, one has:

$$\|u(t)\|_{H^s} \leq Ce^{K|t|}$$

. Let us now suppose that:

$$\begin{cases} iu_t + \Delta u = \mu_1|u|^2u + \mu_2(K * |u|^2)u, \; x \in \mathbb{R}^3, \; t \in [0,\delta] \\ iv_t + \Delta v = \mu_1|v|^2v + \mu_2(K * |v|^2)v, \; x \in \mathbb{R}^3, \; t \in [0,\delta] \\ u|_{t=0} = v|_{t=0} = \Phi \in H^s(\mathbb{R}^3) \end{cases} \tag{5.74}$$

We want to argue that $u = v$ on $[0,\delta]$. To do this, we note that, for all $t \in [0,\delta]$, one has:

$$u(t) - v(t) = \int_0^t S(t-t')\big(\mu_1(|u|^2u - |v|^2v)(t') + \mu_2((K * |u|^2)u - (K * |v|^2)v(t'))\big)dt'$$

and hence, by Minkowski's inequality and unitarity:

$$\|u(t) - v(t)\|_{L^2_x} \lesssim \int_0^t \big(\|(|u|^2u - |v|^2v)(t')\|_{L^2_x} + \|(K * |u|^2)u - (K * |v|^2)v(t')\|_{L^2_x}\big)dt' \tag{5.75}$$

We note that:

$$\begin{aligned} \|w_1w_2w_3\|_{L^2_x} &\leq \|w_1\|_{L^\infty_x}\|w_2\|_{L^\infty_x}\|w_3\|_{L^2_x} \\ &\lesssim \|w_1\|_{H_x^{\frac{3}{2}+}}\|w_2\|_{H_x^{\frac{3}{2}+}}\|w_3\|_{L^2_x} \end{aligned} \tag{5.76}$$

$$\|(K*(w_1w_2))w_3\|_{L^2_x} \leq \|K*(w_1w_2)\|_{L^\infty_x}\|w_3\|_{L^2_x}$$

$$\lesssim \|K*(w_1w_2)\|_{H_x^{\frac{3}{2}+}}\|w_3\|_{L^2_x} \lesssim \|w_1w_2\|_{H_x^{\frac{3}{2}+}}$$

$$\lesssim \|w_1\|_{H_x^{\frac{3}{2}+}}\|w_2\|_{H_x^{\frac{3}{2}+}}\|w_3\|_{L^2_x} \tag{5.77}$$

Here, we used the boundedness of $\widehat{K}$ and the fact that $H^{\frac{3}{2}+}(\mathbb{R}^3)$ is an algebra.

$$\|(K*(w_1w_2)w_3\|_{L^2_x} \leq \|K*(w_1w_2)\|_{L^2_x}\|w_3\|_{L^\infty_x}$$

$$\lesssim \|w_1w_2\|_{L^2_x}\|w_3\|_{L^\infty_x} \leq \|w_1\|_{L^2_x}\|w_2\|_{L^\infty_x}\|w_3\|_{L^\infty_x}$$

$$\lesssim \|w_1\|_{L^2_x}\|w_2\|_{H_x^{\frac{3}{2}+}}\|w_3\|_{H_x^{\frac{3}{2}+}} \tag{5.78}$$

We use (5.76), (5.77), and (5.78) and recall (5.75) to deduce that, for all $t \in [0,\delta]$, one has:

$$\|u(t)-v(t)\|_{L^2_x} \lesssim \int_0^t \|u(t')-v(t')\|_{L^2_x}(\|u(t')\|_{H_x^{\frac{3}{2}+}} + \|v(t')\|_{H_x^{\frac{3}{2}+}})^2 dt'$$

Hence, from previous arguments, there exists a non-negative continuous function $f:[0,\delta]\to[0,+\infty)$ such that for all $t\in[0,\delta]$, one has:

$$\|u(t)-v(t)\|_{L^2_x} \leq \int_0^t \|u(t')-v(t')\|_{L^2_x} f(t')dt'$$

By Gronwall's inequality, it follows that on $[0,\delta]$, one has $\|u(t)-v(t)\|_{L^2_x}=0$. Hence, $u=v$ on $[0,\delta]$. Uniqueness on $[0,\delta]$ now follows. Uniqueness for $[t_0,t_0+\delta]$, give $t_0\in\mathbb{R}$ is proved analogously.

We note that all the implied constants that we obtained depend continuously on $\|u(t)\|$, and hence depend continuously on energy and mass.

The Proposition now follows.

□

5.5 Comments and further results

5.5.1 The unstable regime

It makes sense to consider (5.1) in the unstable regime. We consider the case when:

$$\mu_1 < \frac{4\pi}{3}\mu_2, \text{ and } \mu_2 \geq 0.$$

It is shown in [26] that in this regime, there exist global solutions in $H^1(\mathbb{R}^3)$ if one assumes that $\|\nabla\Phi\|_{L^2}$ is sufficiently small, depending on $\|\Phi\|_{L^2}, \mu_1, \mu_2$. The key is to observe that the energy and mass again control $\|u(t)\|_{H^1}$. The bounds as in Theorem 5.1.1 then also hold with the same proof.

5.5.2 Adding a potential

One can also add a potential real-valued potential $V = V(x)$ to consider:

$$\begin{cases} iu_t + \Delta u = Vu + \mu_1|u|^2u + \mu_2(K * |u|^2)u, \; x \in \mathbb{R}^3, \; t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}^3) \end{cases} \tag{5.79}$$

The energy then becomes:

$$E(u) := \frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{2}\int V(x)|u(x)|^2 dx + \frac{1}{4}\mu_1 \int |u|^4 dx + \frac{1}{4}\mu_2 \int (K * |u|^2)|u|^2 dx$$

A formal modification of the arguments Chapter 2 can give us that the result of Theorem 5.1.1 still holds if we take $V \in \mathcal{S}(\mathbb{R}^3)$ and $V \geq 0$. This sort of potential is not of the same sort as the one used in [26]. Namely, the potential used in [26] is assumed to be quadratic:

$$V = \omega_1 x_1^2 + \omega_2 x_2^2 + \omega_3 x_3^2$$

for some $\omega_1, \omega_2, \omega_3 \geq 0$, which are not all equal to zero. Physically, the term obtained by adding the V corresponds to adding a trapping potential. The presence of the potential V forces us to work in weighted Sobolev spaces, i.e. we have to assume that $x\Phi \in L^2(\mathbb{R}^3)$ in order to be able to define the energy and to obtain a global solution. Our methods don't seem to apply in this setting, and a different approach would be needed here.

5.5.3 Higher modified energies

With notation as in Section 2, let us suppose that M_4 is a function on Γ_4. We want to compute $\frac{d}{dt}\lambda_4(M_4; u)$. One can compute that:

$$\frac{d}{dt}\lambda_4(M_4; u) = -i\lambda_4\big(M_4(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2); u\big) - i\lambda_6\big(M_6; u\big)$$

$$-i\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} M_4(\xi_1, \xi_2, \xi_3, \xi_4)[(Vu)\widehat{\ }(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4) \tag{5.80}$$

where:

$$M_6(\xi_1, \ldots, \xi_6) := M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)(1+\widehat{K}(\xi_1+\xi_2)) - M_4(\xi_1, \xi_{234}, \xi_5, \xi_6)(1+\widehat{K}(\xi_2+\xi_3))$$

$$+M_4(\xi_1, \xi_2, \xi_{345}, \xi_6)(1 + \widehat{K}(\xi_3 + \xi_4))M_4(\xi_1, \xi_2, \xi_3, \xi_{456})(1 + \widehat{K}(\xi_4 + \xi_5))$$

As in the two-dimensional setting, we can compute that for $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4$, one has:

$$|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 = 2\xi_{12} \cdot \xi_{14} \tag{5.81}$$

As before, we decompose:

$$\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.$$

Here, the set Ω_{nr} of *non-resonant* frequencies is defined by:

$$\Omega_{nr} := \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4; \xi_{12}, \xi_{14} \neq 0, |cos\angle(\xi_{12}, \xi_{14})| > \beta_0\} \tag{5.82}$$

and the set Ω_r of *resonant* frequencies Ω_r is defined to be its complement in Γ_4.

We now define the multiplier M_4 by:

$$M_4(\xi_1,\xi_2,\xi_3,\xi_4) := \begin{cases} c\frac{((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2)}{|\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2}\widehat{V}(\xi_3+\xi_4)\,, \text{if } (\xi_1,\xi_2,\xi_3,\xi_4)\in\Omega_{nr} \\ 0, \text{ if } (\xi_1,\xi_2,\xi_3,\xi_4)\in\Omega_r \end{cases} \tag{5.83}$$

Let us now define the multiplier M_6 on Γ_6 by:

$$M_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) := \tag{5.84}$$

$$M_4(\xi_{123},\xi_4,\xi_5,\xi_6)+M_4(\xi_1,\xi_{234},\xi_5,\xi_6)+M_4(\xi_1,\xi_2,\xi_{345},\xi_6)+M_4(\xi_1,\xi_2,\xi_3,\xi_{456})$$

Furthermore, let us define:

$$E^2(u) := \|\mathcal{D}u\|_{L^2}^2 + \lambda_4(M_4;u).$$

We now use (5.40), (5.80), and the construction of M_4 to deduce that [3]:

$$\frac{d}{dt}E^2(u) =$$

$$\int_{\xi_1+\xi_2+\xi_3+\xi_4=0,|cos\angle(\xi_{12},\xi_{14})|\leq\beta_0} \big((\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2\big)\widehat{V}(\xi_3+\xi_4)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)d\xi_j+$$

$$+\int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} M_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4)\widehat{u}(\xi_5)\widehat{\bar{u}}(\xi_6)$$

$$=: I + II \tag{5.85}$$

If we now argue as in the two dimensional setting, we see that, in order to bound I, we have to essentially bound [4]:

$$I_{N_1,N_2,N_3,N_4} :=$$

[3] We recall that $(\theta(\xi_1))^2-(\theta(\xi_2))^2+(\theta(\xi_3))^2-(\theta(\xi_4))^2=0$, whenever $\xi_{12}=0$ or $\xi_{14}=0$, hence the corresponding terms again don't contribute to the quadrilinear term. Therefore, we don't have to worry about defining the quantity $\cos(0,\cdot)$.

[4] Ignoring the fact that the integral is over a finite time interval and the fact that all of the estimates in Proposition 5.3.1 are local in time.

$$\int_{\tau_1+\tau_2+\tau_3+\tau_4=0}\int_{\xi_1+\xi_2+\xi_3+\xi_4=0}\frac{N_3}{N_1}\theta(N_1)\theta(N_2)\widetilde{u}_{N_1}(\xi_1,\tau_1)\widetilde{\overline{u}}_{N_2}(\xi_2,\tau_2)\widetilde{u}_{N_3}(\xi_3,\tau_3)\widetilde{\overline{u}}_{N_4}(\xi_4,\tau_4)d\xi_j$$

where $N_1 \geq N_2 \geq N_3 \geq N_4$. One would want to use an $L^{\frac{10}{3}}_{t,x}, L^{\frac{10}{3}}_{t,x}, L^{\frac{10}{3}}_{t,x}, L^{10}_{t,x}$ Hölder inequality to deduce that:

$$|I_{N_1,N_2,N_3,N_4}| \lesssim \frac{1}{N_1}\|\mathcal{D}u_{N_1}\|_{L^2_{t,x}}\|\mathcal{D}u_{N_2}\|_{L^2_{t,x}}\|\nabla u_{N_3}\|_{L^2_{t,x}}\|u_{N_4}\|_{L^{10}_{t,x}}$$

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}u\|^2_{X^{0,\frac{1}{2}+}}\|u\|^2_{X^{1,\frac{1}{2}+}}$$

$$\lesssim \frac{1}{N_1^{1-}}\|\mathcal{D}u\|^2_{X^{0,\frac{1}{2}+}}$$

Hence, in this way, we don't seem to be getting a better decay factor than $\frac{1}{N^{1-}}$. Similarly, if we group other Strichartz norms, we can't get a better decay factor.

5.5.4 Lower dimensional results

Two dimensional results

Let $s > 1$ be a real number, and let us consider:

$$\begin{cases} iu_t + \Delta u = |u|^2 u + (K_2 * |u|^2)u,\ x \in \mathbb{R}^2,\ t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}^2) \end{cases} \tag{5.86}$$

Here, we are assuming that $K_2 = K_2(x_1, x_2)$ is real-valued, and $\widehat{K_2} \in L^\infty(\mathbb{R}^2)$. The conserved mass and energy for (5.86) are then the same as before. Let us note that in 2D, one has:

$$\|(K_2 * |u|^2)|u|^2\|_{L^1_x} \leq \|K_2 * |u|^2\|_{L^2_x}\||u|^2\|_{L^2_x}$$

$$\lesssim \||u|^2\|_{L^2_x}\||u|^2\|_{L^2_x} \lesssim \|u\|^4_{L^4_x} \lesssim \|u\|^2_{L^2_x}\|\nabla u\|^2_{L^2_x}$$

Hence, if $\|\Phi\|_{L^2_x}$ is sufficiently small, it follows that conservation of mass and energy gives us uniform bounds on $\|u(t)\|_{H^1_x}$. By using the same construction as in the previous subsection and the arguments as in Chapter 4, we obtain:

Proposition 5.5.1. *Suppose that* $\|\Phi\|_{L^2_x}$ *is sufficiently small that* (5.86) *has a global solution. Let* u *denote the global solution of* (5.86)*. Then, there exists* $C = C(s, \Phi) > 0$ *such that for all* $t \in \mathbb{R}$*, one has:*

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{\frac{4s}{7}+}\|\Phi\|_{H^s} \tag{5.87}$$

The equation (5.86) occurs in [26] when one wants to find specific solutions to (5.1) by dimension reduction. The specific convolution potential K_2 is given by: $K_2(x_1, x_2) := \int_{\mathbb{R}} K(x_1, x_2, x_3)\phi(x_3)dx_3$, where K is the convolution potential used in (5.1), and where $\phi \in \mathcal{S}(\mathbb{R})$ is an appropriate real-valued Schwartz function. One can then check that K_2 satisfies the wanted conditions. Strictly speaking, from the solution of (5.86), we can construct a function which is close in L^2_x on a finite time interval to an exact solution to (5.1) with the same initial data (for details, see Section 6 in [26]). Hence, by this method, we can't deduce that a nontrivial solution of (5.1) satisfies (5.87).

One dimensional results

The method of higher modified energies works for:

$$\begin{cases} iu_t + \Delta u = |u|^2 u + (K_1 * |u|^2)u, \, x \in \mathbb{R}, \, t \in \mathbb{R} \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}) \end{cases} \tag{5.88}$$

Here, we are assuming that $s \geq 1$ is real, and that $K_1 = K_1(x_1)$ is real, and $\widehat{K_1} \in L^\infty(\mathbb{R})$. One can obtain global existence for sufficiently small initial data in H^1. This sort of model also arises in a particular dimension reduction in [26] by taking: $K_1(x_1) := \int_{\mathbb{R}^2} K(x_1, x_2, x_3)\psi(x_2, x_3)dx_2dx_3$, where $\psi \in \mathcal{S}(\mathbb{R}^2)$ is a real-valued Schwartz function. We can again use solutions to (5.88) to obtain approximate solutions of (5.1), which are only close to the exact solution in the L^2 sense.

By a modification of the arguments of previous chapters, we use the higher modified energies to deduce:

Proposition 5.5.2. *Suppose that* $\|\Phi\|_{H^1}$ *is sufficiently small that* (5.88) *has a global solution. Let* u *be the global solution to* (5.88). *There exists* $C = C(s,\Phi) > 0$ *such that, for all* $t \in \mathbb{R}$, *one has:*

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{\frac{s}{3}+}\|\Phi\|_{H^s}. \tag{5.89}$$

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