Cluster pictures for hyperelliptic curves

Warwick Postgraduate Seminar

Robin Visser
Mathematics Institute
University of Warwick

4 May 2022

Hyperelliptic curves

Hyperelliptic curves

Weighted projective space

Given some field K and $g \geq 2$, we define **weighted projective space** $\mathbb{P}^2_g$ as the ambient space whose points over K are equivalence classes of $(x, y, z) \in K^3 \backslash \{(0, 0, 0)\}$ where $(x, y, z) \sim (x', y', z')$ iff there exists $\lambda \in K^\times$ such that $(x', y', z') = (\lambda x, \lambda^{g+1} y, \lambda z)$.

Hyperelliptic curves

Weighted projective space

Given some field K and $g \geq 2$, we define **weighted projective space** $\mathbb{P}^2_g$ as the ambient space whose points over K are equivalence classes of $(x, y, z) \in K^3 \setminus \{(0, 0, 0)\}$ where $(x, y, z) \sim (x', y', z')$ iff there exists $\lambda \in K^\times$ such that $(x', y', z') = (\lambda x, \lambda^{g+1} y, \lambda z)$.

Hyperelliptic curve

Let K be a field with $\mathrm{char}(K) \neq 2$. A **hyperelliptic curve** of genus g is a subvariety of $\mathbb{P}^2_g$ defined by

$$Y^2 = F(X, Z)$$

where $F \in K[X, Z]$ is squarefree and homogeneous of degree $2g + 2$.

Hyperelliptic curves

Weighted projective space

Given some field K and $g \geq 2$, we define **weighted projective space** $\mathbb{P}_g^2$ as the ambient space whose points over K are equivalence classes of $(x, y, z) \in K^3 \backslash \{(0, 0, 0)\}$ where $(x, y, z) \sim (x', y', z')$ iff there exists $\lambda \in K^\times$ such that $(x', y', z') = (\lambda x, \lambda^{g+1} y, \lambda z)$.

Hyperelliptic curve

Let K be a field with $\mathrm{char}(K) \neq 2$. A **hyperelliptic curve** of genus g is a subvariety of $\mathbb{P}_g^2$ defined by

$$Y^2 = F(X, Z)$$

where $F \in K[X, Z]$ is squarefree and homogeneous of degree $2g + 2$.

- For the most part, we can simply think of hyperelliptic curves as $y^2 = f(x)$.

Cluster pictures

Cluster pictures

Definition

Let C/K be a hyperelliptic curve of genus g given by

$$y^2 = f(x) = c(x - a_1)(x - a_2)\cdots(x - a_n)$$

Let $\mathcal{R} = \{a_1, \ldots, a_n\}$ denote the set of roots of f. Let $\mathfrak{p}$ be an odd prime in K. We define the **cluster picture** $\Sigma_\mathfrak{p}$ associated to C with respect to $\mathfrak{p}$ as

$$\Sigma_\mathfrak{p} := \left\{\mathfrak{s} \in \mathcal{P}(\mathcal{R}) \mid \mathfrak{s} = D_{z,d} \cap \mathcal{R} \text{ for some } z \in \overline{K}, d \in \mathbb{Q}\right\}$$

where $D_{z,d} := \{x \in \overline{K} \mid v_\mathfrak{p}(x - z) \geq d\}$.

Cluster pictures

Definition

Let C/K be a hyperelliptic curve of genus g given by

$$y^2 = f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n)$$

Let $\mathcal{R} = \{a_1, \ldots, a_n\}$ denote the set of roots of f. Let $\mathfrak{p}$ be an odd prime in K. We define the **cluster picture** $\Sigma_{\mathfrak{p}}$ associated to C with respect to $\mathfrak{p}$ as

$$\Sigma_{\mathfrak{p}} := \left\{\mathfrak{s} \in \mathcal{P}(\mathcal{R}) \mid \mathfrak{s} = D_{z,d} \cap \mathcal{R} \text{ for some } z \in \overline{K}, d \in \mathbb{Q}\right\}$$

where $D_{z,d} := \{x \in \overline{K} \mid v_{\mathfrak{p}}(x - z) \geq d\}$.

- In short, $\Sigma_{\mathfrak{p}}$ are the subsets of $\mathcal{R}$ which are cut out by bounded p-adic discs in K.

Cluster pictures

- We call the elements $\mathfrak{s}$ of Σ_p **clusters**.

Cluster pictures

- We call the elements $\mathfrak{s}$ of Σ_p **clusters**.

Depth

The **depth** of a cluster $\mathfrak{s}$ is

$$d_{\mathfrak{s}} := \min_{r,r' \in \mathfrak{s}} v_{\mathfrak{p}}(r - r')$$

i.e. the maximal valuation which cuts out $\mathfrak{s}$.

Cluster pictures

- We call the elements $\mathfrak{s}$ of Σ_p **clusters**.

Depth

The **depth** of a cluster $\mathfrak{s}$ is

$$d_{\mathfrak{s}} := \min_{r,r' \in \mathfrak{s}} v_{\mathfrak{p}}(r - r')$$

i.e. the maximal valuation which cuts out $\mathfrak{s}$.

Leading Depth

The **leading depth** of a cluster $\mathfrak{s}$ is

$$\nu_{\mathfrak{s}} := v_{\mathfrak{p}}(c) + \sum_{r \in \mathcal{R}} d_{r \wedge \mathfrak{s}}$$

where $r \wedge \mathfrak{s}$ denotes the smallest cluster in $\Sigma_{\mathfrak{p}}$ containing r and $\mathfrak{s}$.

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$$

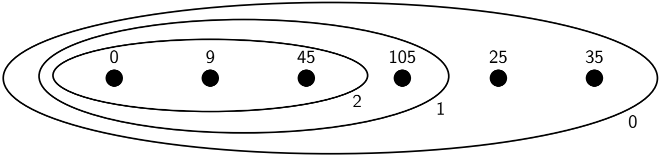
Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$$

- For $p = 3$, cluster picture Σ_3 is



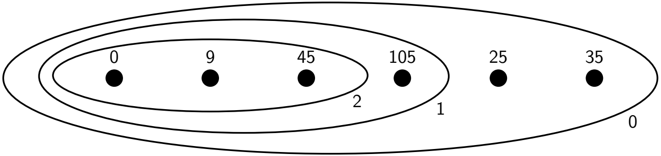
Example 1

Example

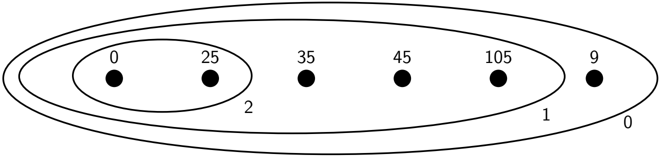
Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$$

- For $p = 3$, cluster picture Σ_3 is

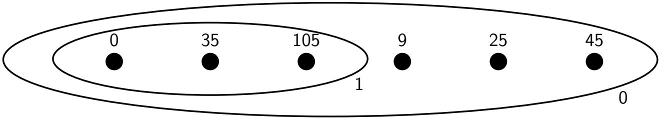


- For $p = 5$, the cluster picture Σ_5 is



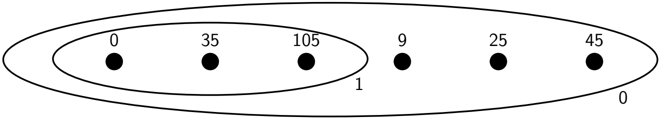
Example 1

- For $p = 7$, the cluster picture Σ_7 is

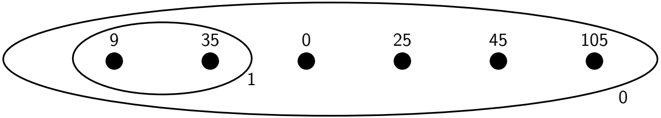


Example 1

- For $p = 7$, the cluster picture Σ_7 is

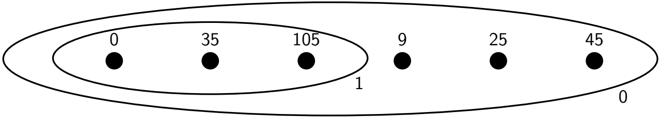


- For $p = 13$, the cluster picture Σ_{13} is

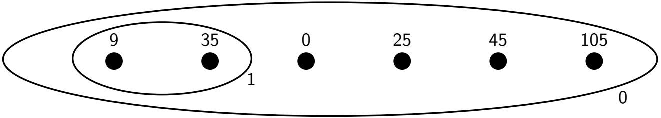


Example 1

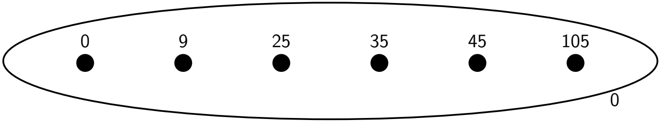
- For $p = 7$, the cluster picture Σ_7 is



- For $p = 13$, the cluster picture Σ_{13} is



- For every other odd prime p, the cluster picture Σ_p is trivial:



Cluster pictures

Observations:

Cluster pictures

Observations:

- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contain the singleton elements $\{r_i\}$.

Cluster pictures

Observations:

- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contain the singleton elements $\{r_i\}$.
- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contains $\mathcal{R}$.

Cluster pictures

Observations:

- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contain the singleton elements $\{r_i\}$.
- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contains $\mathcal{R}$.

Definition

If $\Sigma_{\mathfrak{p}}$ consists only of the singletons and $\mathcal{R}$, then we say that $\Sigma_{\mathfrak{p}}$ is **trivial**.

Cluster pictures

Observations:

- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contain the singleton elements $\{r_i\}$.
- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contains $\mathcal{R}$.

Definition

If $\Sigma_{\mathfrak{p}}$ consists only of the singletons and $\mathcal{R}$, then we say that $\Sigma_{\mathfrak{p}}$ is **trivial**.

- $\Sigma_{\mathfrak{p}}$ is trivial for all but finitely many $\mathfrak{p}$.

Cluster pictures

Observations:

- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contain the singleton elements $\{r_i\}$.
- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contains $\mathcal{R}$.

Definition

If $\Sigma_{\mathfrak{p}}$ consists only of the singletons and $\mathcal{R}$, then we say that $\Sigma_{\mathfrak{p}}$ is **trivial**.

- $\Sigma_{\mathfrak{p}}$ is trivial for all but finitely many $\mathfrak{p}$.
- Two clusters in $\Sigma_{\mathfrak{p}}$ are either disjoint or contained in one another.

Cluster pictures

Observations:

- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contain the singleton elements $\{r_i\}$.
- Every cluster picture $\Sigma_{\mathfrak{p}}$ always contains $\mathcal{R}$.

Definition

If $\Sigma_{\mathfrak{p}}$ consists only of the singletons and $\mathcal{R}$, then we say that $\Sigma_{\mathfrak{p}}$ is **trivial**.

- $\Sigma_{\mathfrak{p}}$ is trivial for all but finitely many $\mathfrak{p}$.
- Two clusters in $\Sigma_{\mathfrak{p}}$ are either disjoint or contained in one another.
- If $\mathfrak{s}_1 \subsetneq \mathfrak{s}_2$, then $d_{\mathfrak{s}_1} > d_{\mathfrak{s}_2}$ and $\nu_{\mathfrak{s}_1} > \nu_{\mathfrak{s}_2}$.

Example 2

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = f(x) = (x^2 - x + 1)(2x^2 - x + 2)(3x^2 - 2x + 3)$$

Example 2

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = f(x) = (x^2 - x + 1)(2x^2 - x + 2)(3x^2 - 2x + 3)$$

- We first factorise $f(x)$ over $\mathbb{Q}(\mathcal{R}) = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}, \sqrt{5})$ to get $C : y^2 = 6(x - a_1) \cdots (x - a_6)$ where

$$a_1 = \tfrac{1+i\sqrt{3}}{2},\ a_2 = \tfrac{1-i\sqrt{3}}{2},\ a_3 = \tfrac{1+i\sqrt{15}}{4},\ a_4 = \tfrac{1-i\sqrt{15}}{4},\ a_5 = \tfrac{1+2i\sqrt{2}}{3},\ a_6 = \tfrac{1-2i\sqrt{2}}{3}.$$

Example 2

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = f(x) = (x^2 - x + 1)(2x^2 - x + 2)(3x^2 - 2x + 3)$$

- We first factorise $f(x)$ over $\mathbb{Q}(\mathcal{R}) = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}, \sqrt{5})$ to get $C : y^2 = 6(x - a_1) \cdots (x - a_6)$ where

$$a_1 = \tfrac{1+i\sqrt{3}}{2},\ a_2 = \tfrac{1-i\sqrt{3}}{2},\ a_3 = \tfrac{1+i\sqrt{15}}{4},\ a_4 = \tfrac{1-i\sqrt{15}}{4},\ a_5 = \tfrac{1+2i\sqrt{2}}{3},\ a_6 = \tfrac{1-2i\sqrt{2}}{3}.$$

- Let's compute Σ_3. We check:

$$v_3(a_1 - a_2) = v_3(i\sqrt{3}) = \tfrac{1}{2},\ \ v_3(a_3 - a_4) = v_3(\tfrac{i\sqrt{15}}{2}) = \tfrac{1}{2},\ \ v_3(a_5 - a_6) = v_3(\tfrac{4i\sqrt{2}}{3}) = -1.$$

Example 2

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by

$$C : y^2 = f(x) = (x^2 - x + 1)(2x^2 - x + 2)(3x^2 - 2x + 3)$$

- We first factorise $f(x)$ over $\mathbb{Q}(\mathcal{R}) = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}, \sqrt{5})$ to get $C : y^2 = 6(x - a_1) \cdots (x - a_6)$ where

$$a_1 = \tfrac{1+i\sqrt{3}}{2},\ a_2 = \tfrac{1-i\sqrt{3}}{2},\ a_3 = \tfrac{1+i\sqrt{15}}{4},\ a_4 = \tfrac{1-i\sqrt{15}}{4},\ a_5 = \tfrac{1+2i\sqrt{2}}{3},\ a_6 = \tfrac{1-2i\sqrt{2}}{3}.$$

- Let's compute Σ_3. We check:

$$v_3(a_1 - a_2) = v_3(i\sqrt{3}) = \tfrac{1}{2},\ \ v_3(a_3 - a_4) = v_3(\tfrac{i\sqrt{15}}{2}) = \tfrac{1}{2},\ \ v_3(a_5 - a_6) = v_3(\tfrac{4i\sqrt{2}}{3}) = -1.$$

- To compute $v_3(a_1 - a_5)$, we can choose to extend v_3 to $\mathbb{Q}(\mathcal{R})$ by letting $v_3(1 - i\sqrt{2}) = 1$ (note $3 = (1 - i\sqrt{2})(1 + i\sqrt{2})$). This yields $v_3(a_1 - a_5) = 0$.

Example 2

Table: $v_3(a_i - a_j)$

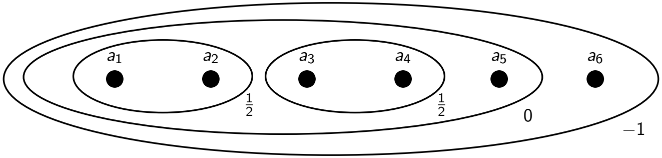
	a_1	a_2	a_3	a_4	a_5	a_6
a_1	$\cdot$	$\frac{1}{2}$	0	0	0	-1
a_2		$\cdot$	0	0	0	-1
a_3			$\cdot$	$\frac{1}{2}$	0	-1
a_4				$\cdot$	0	-1
a_5					$\cdot$	-1
a_6						$\cdot$

Example 2

Table: $v_3(a_i - a_j)$

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	$\cdot$	$\frac{1}{2}$	0	0	0	-1
a_2		$\cdot$	0	0	0	-1
a_3			$\cdot$	$\frac{1}{2}$	0	-1
a_4				$\cdot$	0	-1
a_5					$\cdot$	-1
a_6						$\cdot$

The cluster picture Σ_3 is:



Reduction

Good/Bad reduction

A hyperelliptic curve $C/K : y^2 = f(x)$ of genus g has **good reduction** at a prime $\mathfrak{p}$ if it has an integral model whose reduction mod $\mathfrak{p}$ defines a smooth curve of genus g (i.e. $\tilde{F}(X, Z)$ is squarefree). Otherwise we say C/K has **bad reduction** at $\mathfrak{p}$.

Reduction

Good/Bad reduction

A hyperelliptic curve $C/K : y^2 = f(x)$ of genus g has **good reduction** at a prime $\mathfrak{p}$ if it has an integral model whose reduction mod $\mathfrak{p}$ defines a smooth curve of genus g (i.e. $\tilde{F}(X, Z)$ is squarefree). Otherwise we say C/K has **bad reduction** at $\mathfrak{p}$.

Potential good reduction

A hyperelliptic curve C/K has **potential good reduction** at a prime $\mathfrak{p}$ if there exists a finite extension $L \supset K$ such that the base change curve C_L/L has good reduction at all primes of L above $\mathfrak{p}$.

Reduction

Good/Bad reduction

A hyperelliptic curve $C/K : y^2 = f(x)$ of genus g has **good reduction** at a prime $\mathfrak{p}$ if it has an integral model whose reduction mod $\mathfrak{p}$ defines a smooth curve of genus g (i.e. $\tilde{F}(X, Z)$ is squarefree). Otherwise we say C/K has **bad reduction** at $\mathfrak{p}$.

Potential good reduction

A hyperelliptic curve C/K has **potential good reduction** at a prime $\mathfrak{p}$ if there exists a finite extension $L \supset K$ such that the base change curve C_L/L has good reduction at all primes of L above $\mathfrak{p}$.

Example

Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$. C has bad reduction at 2, 3, 5, and 7, but has potential good reduction at 7.

Semistable Reduction

We can further refine types of bad reduction!

Semistable Reduction

We can further refine types of bad reduction!

Example

Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$.

Semistable Reduction

We can further refine types of bad reduction!

Example

Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$.

The reduction $\widetilde{C}$ mod 3 is
$\tilde{C} : y^2 = x^3(x-1)(x+1)$.

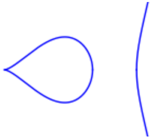
Semistable Reduction

We can further refine types of bad reduction!

Example

Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$.

The reduction $\widetilde{C}$ mod 3 is
$\tilde{C} : y^2 = x^3(x-1)(x+1)$.



This yields a **cusp** (triple root).

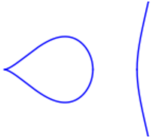
Semistable Reduction

We can further refine types of bad reduction!

Example

Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$.

The reduction $\widetilde{C}$ mod 3 is
$\tilde{C} : y^2 = x^3(x-1)(x+1)$.



This yields a **cusp** (triple root).

The reduction $\widetilde{C}$ mod 5 is
$\tilde{C} : y^2 = 2x^2(x-1)^2(x+2)$.

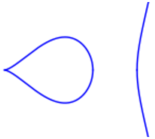
Semistable Reduction

We can further refine types of bad reduction!

Example

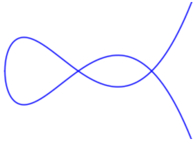
Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$.

The reduction $\widetilde{C}$ mod 3 is
$\tilde{C} : y^2 = x^3(x-1)(x+1)$.



This yields a **cusp** (triple root).

The reduction $\widetilde{C}$ mod 5 is
$\tilde{C} : y^2 = 2x^2(x-1)^2(x+2)$.



This yields two **nodes** (double root).

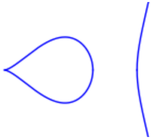
Semistable Reduction

We can further refine types of bad reduction!

Example

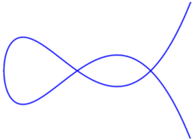
Let $C/\mathbb{Q}$ be defined by $C : y^2 = 7x(x-1)(x-3)(x-5)(x-6)$.

The reduction $\widetilde{C}$ mod 3 is
$\tilde{C} : y^2 = x^3(x-1)(x+1)$.



This yields a **cusp** (triple root).

The reduction $\widetilde{C}$ mod 5 is
$\tilde{C} : y^2 = 2x^2(x-1)^2(x+2)$.



This yields two **nodes** (double root).

Definition: Semistable reduction

A hyperelliptic curve $C/K : y^2 = f(x)$ has **semistable reduction** at $\mathfrak{p}$ if its reduction $\tilde{C}$ mod $\mathfrak{p}$ has (at worst) nodes as singularities.

Cluster pictures

More definitions!

Cluster pictures

More definitions!

Parent/Child

If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster, we say that $\mathfrak{s}'$ is a **child** of $\mathfrak{s}$ and that $\mathfrak{s}$ is the **parent** of $\mathfrak{s}'$

Cluster pictures

More definitions!

Parent/Child

If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster, we say that $\mathfrak{s}'$ is a **child** of $\mathfrak{s}$ and that $\mathfrak{s}$ is the **parent** of $\mathfrak{s}'$

Principal clusters

A cluster $\mathfrak{s} \in \Sigma_p$ is **principal** if $|s| \geq 3$, except if either $\mathfrak{s} = \mathcal{R}$ is even and has exactly two children, or if $\mathfrak{s}$ has a child of size $2g$.

Cluster pictures

More definitions!

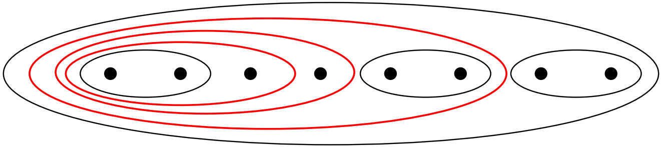
Parent/Child

If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster, we say that $\mathfrak{s}'$ is a **child** of $\mathfrak{s}$ and that $\mathfrak{s}$ is the **parent** of $\mathfrak{s}'$

Principal clusters

A cluster $\mathfrak{s} \in \Sigma_p$ is **principal** if $|s| \geq 3$, except if either $\mathfrak{s} = \mathcal{R}$ is even and has exactly two children, or if $\mathfrak{s}$ has a child of size $2g$.

Genus 3 example:



Cluster pictures

Theorem (Dokchitser–Dokchitser–Maistret–Morgan, 2017)

Let C/K be a hyperelliptic curve of genus g with Weierstrass points in K. Let $\mathfrak{p}$ be an odd prime of K. Then

1. *C is semistable at $\mathfrak{p}$ (equivalently $Jac(C)$)*
   $\iff$ *every principal cluster $\mathfrak{s}$ of $\Sigma_{\mathfrak{p}}$ has $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.*
2. *C has potential good reduction at $\mathfrak{p}$*
   $\iff$ *$\Sigma_{\mathfrak{p}}$ has no proper clusters of size $< 2g+1$.*
3. *C has good reduction at $\mathfrak{p}$*
   $\iff$ *$\Sigma_{\mathfrak{p}}$ has no proper clusters of size $< 2g+1$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$ for the unique principal cluster.*
4. *$Jac(C)$ has potential good reduction at $\mathfrak{p}$*
   $\iff$ *all clusters $\mathfrak{s} \neq \mathcal{R}$ are odd.*
5. *$Jac(C)$ has good reduction at $\mathfrak{p}$*
   $\iff$ *all clusters $\mathfrak{s} \neq \mathcal{R}$ are odd, and principal clusters have $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.*

Example 1

Example

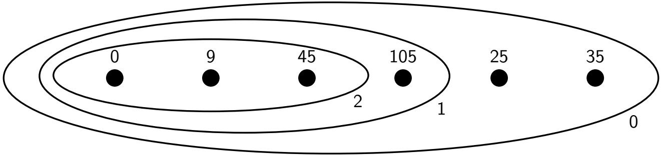
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 3$, cluster picture Σ_3 is

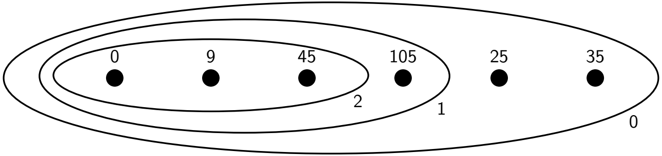


Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 3$, cluster picture Σ_3 is



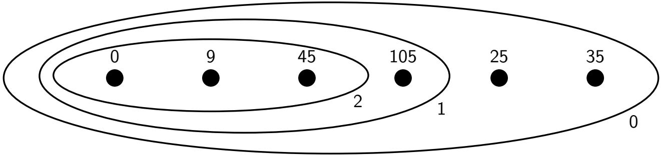
There are two principal clusters: $\mathfrak{s}_1 = \{0, 9, 45\}$, $d_{\mathfrak{s}_1} = 2$, $\nu_{\mathfrak{s}_1} = 7$ and $\mathfrak{s}_2 = \{0, 9, 45, 105\}$ with $d_{\mathfrak{s}_2} = 1$, $\nu_{\mathfrak{s}_2} = 4$. So at $p = 3$:

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 3$, cluster picture Σ_3 is



There are two principal clusters: $\mathfrak{s}_1 = \{0, 9, 45\}$, $d_{\mathfrak{s}_1} = 2$, $\nu_{\mathfrak{s}_1} = 7$ and $\mathfrak{s}_2 = \{0, 9, 45, 105\}$ with $d_{\mathfrak{s}_2} = 1$, $\nu_{\mathfrak{s}_2} = 4$. So at $p = 3$:

- C does not have potential good reduction.
- $\mathrm{Jac}(C)$ does not have potential good reduction.
- C (and $\mathrm{Jac}(C)$) is not semistable.

Example 1

Example

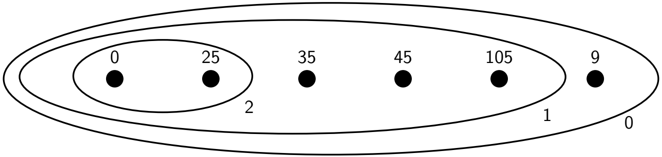
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 5$, cluster picture Σ_5 is

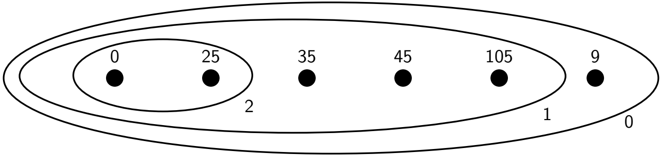


Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 5$, cluster picture Σ_5 is



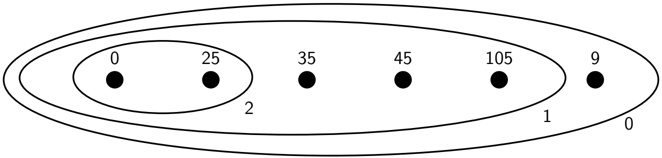
There is one principal cluster: $\mathfrak{s} = \{0, 25, 35, 45, 105\}$, with $d_{\mathfrak{s}} = 1$, $\nu_{\mathfrak{s}} = 5$. So at $p = 5$:

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 5$, cluster picture Σ_5 is



There is one principal cluster: $\mathfrak{s} = \{0, 25, 35, 45, 105\}$, with $d_{\mathfrak{s}} = 1$, $\nu_{\mathfrak{s}} = 5$. So at $p = 5$:

- C does not have potential good reduction.
- $\mathrm{Jac}(C)$ does not have potential good reduction.
- C (and $\mathrm{Jac}(C)$) is not semistable.

Example 1

Example

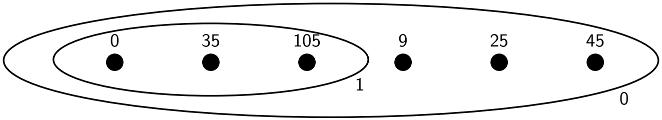
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 7$, cluster picture Σ_7 is

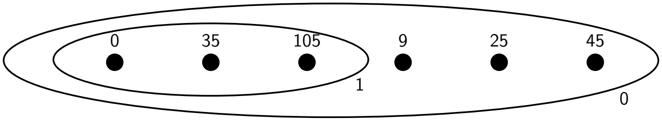


Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 7$, cluster picture Σ_7 is



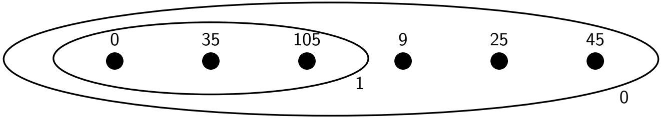
There are two principal clusters: $\mathfrak{s}_1 = \{0, 35, 105\}$, with $d_{\mathfrak{s}_1} = 1$, $\nu_{\mathfrak{s}_1} = 3$, and $\mathfrak{s}_2 = \mathcal{R}$, with $d_{\mathfrak{s}_2} = 0$, $\nu_{\mathfrak{s}_2} = 0$. So at $p = 7$:

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 7$, cluster picture Σ_7 is



There are two principal clusters: $\mathfrak{s}_1 = \{0, 35, 105\}$, with $d_{\mathfrak{s}_1} = 1$, $\nu_{\mathfrak{s}_1} = 3$, and $\mathfrak{s}_2 = \mathcal{R}$, with $d_{\mathfrak{s}_2} = 0$, $\nu_{\mathfrak{s}_2} = 0$. So at $p = 7$:

- C does not have potential good reduction.
- $\mathrm{Jac}(C)$ does have potential good reduction, but not good reduction.
- C (and $\mathrm{Jac}(C)$) is not semistable.

Example 1

Example

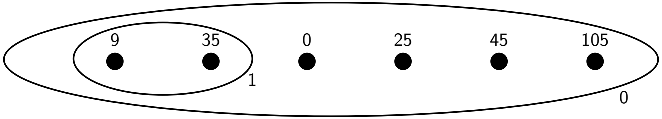
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 13$, cluster picture Σ_{13} is

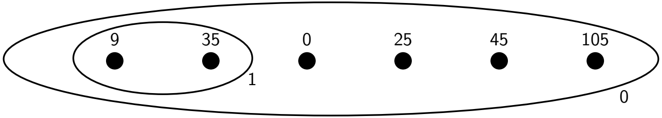


Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 13$, cluster picture Σ_{13} is



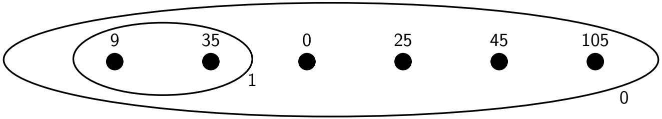
There is one principal clusters: $\mathfrak{s} = \mathcal{R}$, with $d_{\mathfrak{s}} = 0$, $\nu_{\mathfrak{s}} = 0$. So at $p = 13$:

Example 1

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-9)(x-25)(x-35)(x-45)(x-105)$.

For $p = 13$, cluster picture Σ_{13} is



There is one principal clusters: $\mathfrak{s} = \mathcal{R}$, with $d_{\mathfrak{s}} = 0$, $\nu_{\mathfrak{s}} = 0$. So at $p = 13$:

- C does not have potential good reduction.
- $\mathrm{Jac}(C)$ does not have potential good reduction.
- C (and $\mathrm{Jac}(C)$) is semistable.

Some theorems

Lemma

Let $C/\mathbb{Q}$ be a hyperelliptic curve with rational Weierstrass points. Then C cannot have potential good reduction at any odd prime $p < 2g$.

Some theorems

Lemma

Let $C/\mathbb{Q}$ be a hyperelliptic curve with rational Weierstrass points. Then C cannot have potential good reduction at any odd prime $p < 2g$.

Proof:

- Let $C : y^2 = cx(x - a_1)\cdots(x - a_{2g})$ be an integral model such that $\gcd(a_1, \ldots, a_{2g}) = 1$, and let $p < 2g$ be an odd prime.

Some theorems

Lemma

Let $C/\mathbb{Q}$ be a hyperelliptic curve with rational Weierstrass points. Then C cannot have potential good reduction at any odd prime $p < 2g$.

Proof:

- Let $C : y^2 = cx(x - a_1) \cdots (x - a_{2g})$ be an integral model such that $\gcd(a_1, \ldots, a_{2g}) = 1$, and let $p < 2g$ be an odd prime.
- Note that Σ_p has $d_{\mathcal{R}} = 0$.

Some theorems

Lemma

Let $C/\mathbb{Q}$ be a hyperelliptic curve with rational Weierstrass points. Then C cannot have potential good reduction at any odd prime $p < 2g$.

Proof:

- Let $C : y^2 = cx(x - a_1)\cdots(x - a_{2g})$ be an integral model such that $\gcd(a_1, \ldots, a_{2g}) = 1$, and let $p < 2g$ be an odd prime.
- Note that Σ_p has $d_{\mathcal{R}} = 0$.
- At least two of the roots $a_1, \ldots, a_{2g+1}$ must have the same image under the reduction $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Some theorems

Lemma

Let $C/\mathbb{Q}$ be a hyperelliptic curve with rational Weierstrass points. Then C cannot have potential good reduction at any odd prime $p < 2g$.

Proof:

- Let $C : y^2 = cx(x - a_1) \cdots (x - a_{2g})$ be an integral model such that $\gcd(a_1, \ldots, a_{2g}) = 1$, and let $p < 2g$ be an odd prime.
- Note that Σ_p has $d_{\mathcal{R}} = 0$.
- At least two of the roots $a_1, \ldots, a_{2g+1}$ must have the same image under the reduction $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.
- This forms a cluster $\mathfrak{s} \neq \mathcal{R}$ with $d_{\mathfrak{s}} \geq 1$, thus Σ_p is non-trivial. □

Some theorems

Lemma

Let $C/\mathbb{Q}$ be a hyperelliptic curve with rational Weierstrass points. Then C cannot have potential good reduction at any odd prime $p < 2g$.

Proof:

- Let $C : y^2 = cx(x - a_1)\cdots(x - a_{2g})$ be an integral model such that $\gcd(a_1, \ldots, a_{2g}) = 1$, and let $p < 2g$ be an odd prime.
- Note that Σ_p has $d_{\mathcal{R}} = 0$.
- At least two of the roots $a_1, \ldots, a_{2g+1}$ must have the same image under the reduction $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.
- This forms a cluster $\mathfrak{s} \neq \mathcal{R}$ with $d_{\mathfrak{s}} \geq 1$, thus Σ_p is non-trivial. $\square$

The same argument proves that a hyperelliptic curve C/K with Weierstrass points in K cannot have potential good reduction at any odd prime $\mathfrak{p}$ in K satisfying $N_{K/\mathbb{Q}}(\mathfrak{p}) < 2g$.

Some theorems

Theorem

There are infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ with rational Weierstrass points such that $Jac(C)$ has potential good reduction outside $\{2\}$.

Some theorems

Theorem

There are infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ with rational Weierstrass points such that $Jac(C)$ has potential good reduction outside $\{2\}$.

Proof: We can prove this by direct construction! For any positive integer r, let $C_r/\mathbb{Q}$ be the genus 2 curve

$$C_r : y^2 = x(x-1)(x-2^r-1)(x-\tfrac{2^r+1}{2})(x-2^r)$$

Some theorems

Theorem

There are infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ with rational Weierstrass points such that $Jac(C)$ has potential good reduction outside $\{2\}$.

Proof: We can prove this by direct construction! For any positive integer r, let $C_r/\mathbb{Q}$ be the genus 2 curve

$$C_r : y^2 = x(x-1)(x-2^r-1)(x-\tfrac{2^r+1}{2})(x-2^r)$$

Let $\lambda_1 = 2^r + 1, \lambda_2 = \frac{2^r+1}{2}$, and $\lambda_3 = 2^r$. We consider three cases:

Some theorems

Theorem

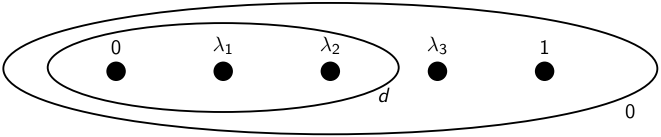
There are infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ with rational Weierstrass points such that $Jac(C)$ has potential good reduction outside $\{2\}$.

Proof: We can prove this by direct construction! For any positive integer r, let $C_r/\mathbb{Q}$ be the genus 2 curve

$$C_r : y^2 = x(x-1)(x-2^r-1)(x-\tfrac{2^r+1}{2})(x-2^r)$$

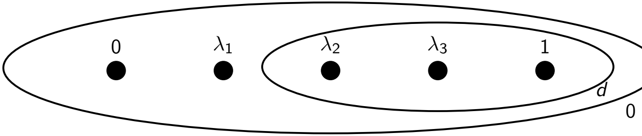
Let $\lambda_1 = 2^r+1, \lambda_2 = \frac{2^r+1}{2}$, and $\lambda_3 = 2^r$. We consider three cases:

- **Case 1:** p divides 2^r+1. Let $d := v_p(\lambda_1) \geq 1$. Then the cluster picture Σ_p is



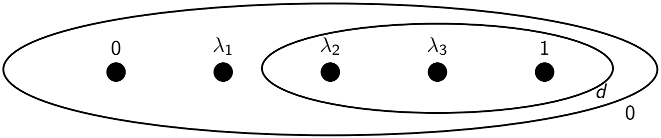
Some theorems

- **Case 2:** p divides $2^r - 1$. Let $d := v_p(2^r - 1) \geq 1$. Then the cluster picture Σ_p is

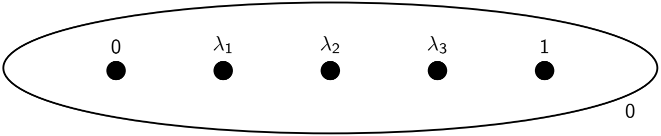


Some theorems

- **Case 2:** p divides $2^r - 1$. Let $d := v_p(2^r - 1) \geq 1$. Then the cluster picture Σ_p is

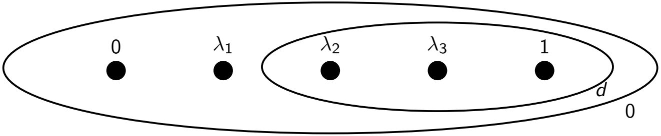


- **Case 3:** All other odd p. The cluster Σ_p is trivial:

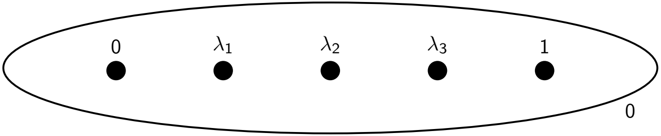


Some theorems

- **Case 2:** p divides $2^r - 1$. Let $d := v_p(2^r - 1) \geq 1$. Then the cluster picture Σ_p is



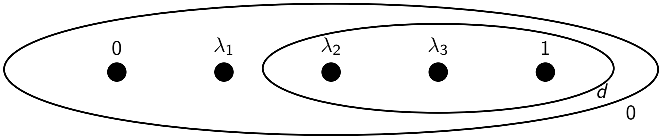
- **Case 3:** All other odd p. The cluster Σ_p is trivial:



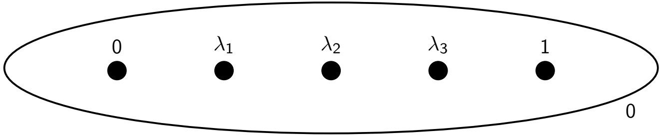
Thus for every odd prime p, $\mathrm{Jac}(C)$ has potential good reduction at p. □

Some theorems

- **Case 2:** p divides $2^r - 1$. Let $d := v_p(2^r - 1) \geq 1$. Then the cluster picture Σ_p is



- **Case 3:** All other odd p. The cluster Σ_p is trivial:



Thus for every odd prime p, $\mathrm{Jac}(C)$ has potential good reduction at p. □

Question: Is this true for genus 3 hyperelliptic curves over $\mathbb{Q}$?

Conductor

Even/Übereven

A cluster $\mathfrak{s} \in \Sigma_p$ is **even** if $|\mathfrak{s}|$ is even. Furthermore, $\mathfrak{s}$ is **übereven** if all children of $\mathfrak{s}$ are even.

Conductor

Even/Übereven

A cluster $\mathfrak{s} \in \Sigma_p$ is **even** if $|\mathfrak{s}|$ is even. Furthermore, $\mathfrak{s}$ is **übereven** if all children of $\mathfrak{s}$ are even.

Theorem (Dokchitser–Dokchitser–Maistret–Morgan, 2017)

Let C/K be a hyperelliptic curve which is semistable at $\mathfrak{p}$. Then the conductor exponent $f_{\mathfrak{p}}$ is

$$f_{\mathfrak{p}} = \begin{cases} \#A - 1 & \text{if } \mathcal{R} \text{ is übereven} \\ \#A & \text{otherwise} \end{cases}$$

where $A = \{\text{even clusters } \mathfrak{s} \neq \mathcal{R} \mid \mathfrak{s} \text{ is not übereven}\}$.

Example 3

Example

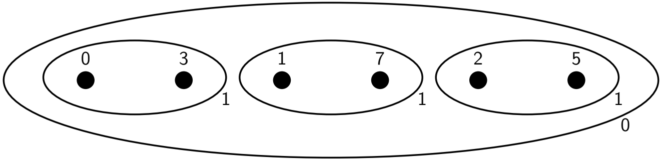
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 3$, cluster picture Σ_3 is

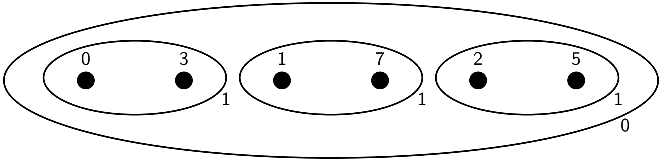


Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 3$, cluster picture Σ_3 is



$C/\mathbb{Q}$ is semistable at 3. We have $\#A = 3$. Here, $\mathcal{R}$ is übereven, so $f_3 = 2$.

Example 3

Example

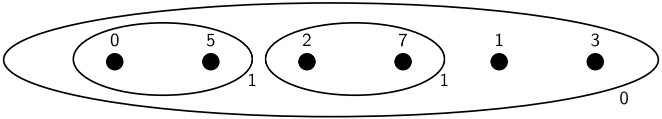
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 5$, cluster picture Σ_5 is

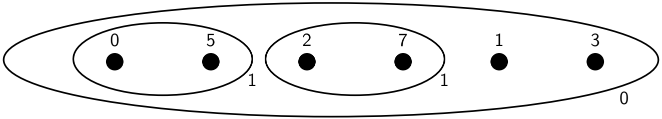


Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 5$, cluster picture Σ_5 is



$C/\mathbb{Q}$ is semistable at 5. We have $\#A = 2$. Here, $\mathcal{R}$ is not übereven, so $f_5 = 2$.

Example 3

Example

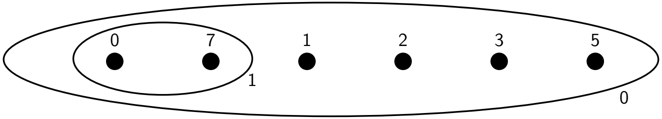
Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 7$, cluster picture Σ_7 is

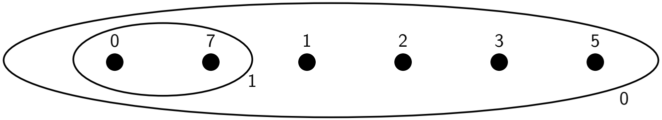


Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 7$, cluster picture Σ_7 is



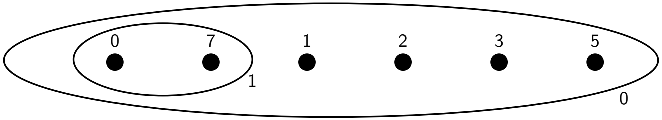
$C/\mathbb{Q}$ is semistable at 7. We have $\#A = 1$. Here, $\mathcal{R}$ is not übereven, so $f_7 = 1$.

Example 3

Example

Let $C/\mathbb{Q}$ be a genus 2 curve defined by $y^2 = x(x-1)(x-2)(x-3)(x-5)(x-7)$.

For $p = 7$, cluster picture Σ_7 is



$C/\mathbb{Q}$ is semistable at 7. We have $\#A = 1$. Here, $\mathcal{R}$ is not übereven, so $f_7 = 1$.

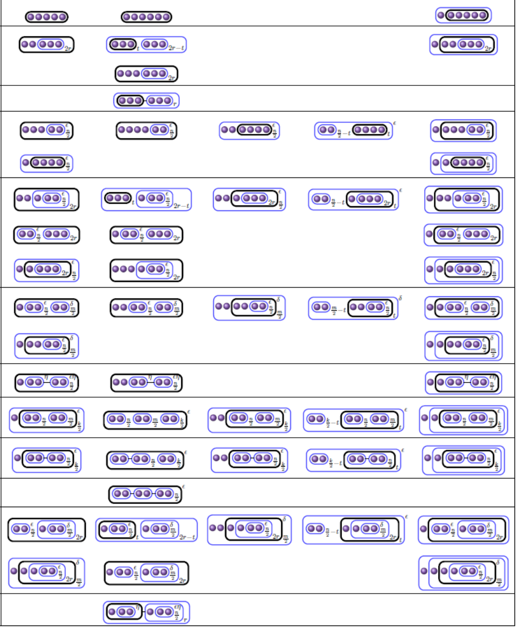
For all other odd primes, Σ_p is trivial and thus $f_p = 0$. Therefore, the odd part of the conductor of $C/\mathbb{Q}$ is $3^2 \cdot 5^2 \cdot 7$.

...and more!

Cluster pictures are useful for many more things! This includes:

- Calculating Tamagawa numbers c_p.
- The tame and wild part of the conductor exponent (for general non-semistable case).
- Root numbers for $\mathrm{Jac}(C)$.
- Discriminant for any given model.
- A basis of integral differentials of C.
- Criteria if a given model is a minimal Weierstrass model.
- Decomposition of the ℓ-adic Galois representation $H^1_{\text{ét}}(C/\overline{K}, \mathbb{Q}_\ell)$.

N-U Type	Type	$\lvert\mathcal{R}\rvert = 5$	$\mathfrak{s} < \mathcal{R}$ sizes ≤ 3	$\mathfrak{s} < \mathcal{R}$ sizes $4, 1, 1$	$\mathfrak{s} < \mathcal{R}$ sizes $4, 2$	$\mathfrak{s} < \mathcal{R}$ sizes $5, 1$
I_{0-0-0}	2					
I_0–I_0–r	$1 \times_r 1$					
	$1 \tilde{\times}_r 1$					
I_{n-0-0}	1_n^{ϵ}					
I_n–I_0–r	$1 \times_r \mathrm{I}_n^{\epsilon}$					
I_{n-m-0}	$\mathrm{I}_{n,m}^{\epsilon,\delta}$					
	$\mathrm{I}_{n\text{-}n}^{\epsilon}$					
I_{n-m-k}	$\mathrm{U}_{n,m,k}^{\epsilon}$					
	$\mathrm{U}_{n\text{-}n,k}^{\epsilon}$					
	$\mathrm{U}_{n\text{-}n\text{-}n}^{\epsilon}$					
I_n–I_m–r	$\mathrm{I}_n^{\epsilon} \times_r \mathrm{I}_m^{\delta}$					
	$\mathrm{I}_n^{\epsilon} \tilde{\times}_r \mathrm{I}_n$					



References

Best, A.J., Betts, A., Bisatt, M., van Bommel, R., Dokchitser, V., Faraggi, O., Kunzweiler, S., Maistret, C., Morgan, A., Muselli S., Nowell, S. (2020)
A user's guide to the local arithmetic of hyperelliptic curves
Preprint, Available at: arXiv:2007.01749.

Dokchitser, T., Dokchitser, V., Maistret, C., Morgan, A. (2018)
Arithmetic of hyperelliptic curves over local fields
Preprint, Available at: arXiv:1808.02936.

Dokchitser, T., Dokchitser, V., Maistret, C., Morgan, A. (2019)
Semistable types of hyperelliptic curves
Algebraic curves and their applications, 73-135, Contemp. Math., 724.

Questions?