Kronecker's limit formula and L-functions of elliptic curves with CM

References:

- Roset's thesis Chapters 2 and 7 ← Main reference!
- Rubin's notes "Elliptic curves with CM and the BSD conjecture".
- Silverman, "Advanced Topics in the Arithmetic of elliptic curves", Chap 2
- Shimura "Intro to Arithmetic Theory of Automorphic Functions", Chap 5.

A quick review of idelic class field theory:

Def^n 1.1: (Artin map) Let K be an imag quad field. and L/K finite abelian extension. Let $\mathfrak{c} \triangleleft \mathcal{O}_K$ ideal divisible by all primes that ramify in L/K, let $I(\mathfrak{c})$ be the frac ideals coprime to $\mathfrak{c}$. The Artin map is

$$(\cdot, L/K) : I(\mathfrak{c}) \longrightarrow \mathrm{Gal}(L/K)$$

$$(\mathfrak{a}, L/K) = \left(\prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}, L/K\right) \longmapsto \prod_{\mathfrak{p}} \sigma_{\mathfrak{p}}^{n_{\mathfrak{p}}}$$

where $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(L/K)$ the Frobenius map, $\sigma_{\mathfrak{p}}(x) \equiv x^q \mod \mathfrak{p}$ $\forall x \in L$, here $q = N_{K/\mathbb{Q}}\mathfrak{p}$.

Def^n/Thm 1.2 (Reciprocity map) Let K imag quad. Then there exists a unique cts homomorphism

$$\mathbb{A}_K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$$

$$s \longmapsto [s, K]$$

such that, for any L/K finite abelian ext^n, and $s \in \mathbb{A}_K^\times$

coprime to primes that ramify in L, we have $[s,K]|_L = ((s), L/K)$ where $(s) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}} s_{\mathfrak{p}}}$.

<u>Properties:</u>
- Reciprocity map is surjective
- $\forall x \in K^*$, $[x, K] = \mathrm{id}$.
- If L/K finite abelian ext^n, $[x, L]|_{K^{ab}} = [N_{L/K}(x), K]$

(see Milne's "Class Field Theory" notes)

<u>Def^n 1.3:</u> Let E be elliptic curve over number field F. Then the L-function of E/F as $L(E/F, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(E/F, N_{\mathfrak{p}}^{-s})^{-1}$ where

$$L_{\mathfrak{p}}(E/F, T) := \begin{cases} 1 - a_{\mathfrak{p}} T + N_{\mathfrak{p}} T^2 & \text{if } \mathfrak{p} \text{ good red}^n \\ 1 - T & \text{if split mult red}^n \\ 1 + T & \text{if non-split mult red}^n \\ 1 & \text{if additive red}^n. \end{cases}$$

where $\#(E(\mathbb{F}_{\mathfrak{p}})) = N_{\mathfrak{p}} + 1 - a_{\mathfrak{p}}$. By Hasse bound, $L(E/F, s)$ converges for $Re(s) > \frac{3}{2}$.

<u>Conjecture:</u> $L(E/F, s)$ has analytic continuation to all of $\mathbb{C}$.

Known for $E/\mathbb{Q}$ (Wiles, et al.), known E/K, K real quad (Freitas - Hung - Siksek) and for <u>E with CM</u> (Deuring & Hecke).

Defⁿ 1.4: Let $\chi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ be a Hecke character. Define the L-series:

$$L(\chi, s) := \prod_{\text{prime } \mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} = \sum_{\mathfrak{b} \triangleleft \mathcal{O}_K} \frac{\chi(\mathfrak{b})}{N\mathfrak{b}^s}$$

Here, $\chi(\mathfrak{p}) = \chi((1, \dots, 1, \overset{\swarrow \mathfrak{p}^{\text{th}} \text{ component}}{\pi}, 1, 1 \dots))$, where $\mathfrak{p} = (\pi)$ if $\mathfrak{p}$ unramified, otherwise $\chi(\mathfrak{p}) = 0$ if $\mathfrak{p}$ ramified.

Defⁿ 1.5: The conductor of χ is the largest ideal $\mathfrak{s}$ of $\mathcal{O}_K$ s.t $\chi(x) = 1$ for all finite ideles $x = (x_\mathfrak{p}) \in \mathbb{A}_K^\times$ s.t $x_\mathfrak{p} \in \mathcal{O}_{K,\mathfrak{p}}^\times$ $\forall \mathfrak{p}$ and $x_\mathfrak{p} \in 1 + \mathfrak{s}\mathcal{O}_{K,\mathfrak{p}}$ $\forall \mathfrak{p} | \mathfrak{s}$.

Defⁿ 1.6 Let $\mathfrak{m} \triangleleft \mathcal{O}_K$ be ideal s.t $\mathfrak{s} | \mathfrak{m}$. Then

$$L_\mathfrak{m}(\chi, s) := \sum_{\substack{\mathfrak{b} \triangleleft \mathcal{O}_K \\ (\mathfrak{b}, \mathfrak{m}) = 1}} \frac{\chi(\mathfrak{b})}{(N\mathfrak{b})^s}$$

More generally, for $\mathfrak{c} \triangleleft \mathcal{O}_K$ coprime to $\mathfrak{m}$, define

$$L_\mathfrak{m}(\chi, s, \mathfrak{c}) := \sum_{\substack{\mathfrak{b} \triangleleft \mathcal{O}_K \\ (\mathfrak{b}, \mathfrak{m}) = 1 \\ (\mathfrak{b}, K(\mathfrak{m})/K) = (\mathfrak{c}, K(\mathfrak{m})/K)}} \frac{\chi(\mathfrak{b})}{N\mathfrak{b}^s}$$

where $K(\mathfrak{m})$ is ray class field mod $\mathfrak{m}$.

Thm 1.7 Let $\chi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ Hecke character.

Then $L(\chi, s)$ has analytic continuation to $\mathbb{C}$

Proof: First proven by Hecke. Reformulated by Tate, proven using Fourier analysis on $\mathbb{A}_K$.

Thm 1.8 (Main Theorem of CM)

Let K be imag quad field, and $E/\mathbb{C}$ elliptic curve with CM by $\mathcal{O}_K$. Let $\underline{a}$ frac ideal s.t.

$f : \mathbb{C}/\underline{a} \xrightarrow{\sim} E(\mathbb{C})$ is isom over $\mathbb{C}$. ⊛ Note that E_{tors} corresponds to $K/\underline{a} \subset \mathbb{C}/\underline{a}$!

Fix $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and idele $x \in \mathbb{A}_K^\times$ s.t. $[x, K] = \sigma|_{K^{ab}}$

Then, there exists a unique isomorphism

$$f' : \mathbb{C}/x^{-1}\underline{a} \xrightarrow{\sim} E^\sigma(\mathbb{C})$$

s.t diagram commutes

$$\begin{tikzcd}
K/\underline{a} \arrow[r, "f"] \arrow[d, "x^{-1}"] & E(\mathbb{C}) \arrow[d, "\sigma"] \\
K/x^{-1}\underline{a} \arrow[r, "f'"] & E^\sigma(\mathbb{C})
\end{tikzcd}$$

⊛ See end of notes for how this is defined!

Proof: See Silverman Thm 8.2, or Shimura Thm 5.4

Def / Thm 1.9: Let E/F elliptic curve with CM by $\mathcal{O}_K$

Then there exists a unique morphism $\underline{\alpha_{E/F} : \mathbb{A}_F^\times \longrightarrow K^\times}$ s.t $\alpha_{E/F}(x)$ generates $(N_{F/K}(x))$ and for any $\underline{a}$ ideal and isom $f : \mathbb{C}/\underline{a} \xrightarrow{\sim} E(\mathbb{C})$, the diagram commutes:

$$\begin{array}{ccc} K/\underline{a} & \xrightarrow{f} & E(F^{ab}) \\ \downarrow{\scriptstyle \alpha_{E/F}(x)(Nx)^{-1}} & & \downarrow{\scriptstyle [x,F]} \\ K/\underline{a} & \xrightarrow{f} & E(F^{ab}) \end{array}$$

Proof: Extend $[x, F]$ to $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$ and let $s = Nm_{F/K}(x)$. Pick $\underline{a}$ s.t $f : \mathbb{C}/\underline{a} \simeq E(\mathbb{C})$

Apply Main Theorem of CM to f, σ, s.

After some technical observations, result follows $\square$.

Def^n/Thm 1.10 (Hecke character associated to E)

Let E/F be an elliptic curve with CM by $\mathcal{O}_K$.

Define the $\psi_{E,F} : \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ given by

$$x \longmapsto \alpha_{E/F}(x)\, N_{F/K}(x^{-1})_\infty$$

where $N_{F/K}(x^{-1})_\infty \in \mathbb{C}^\times$ is the unique infinite component of $N_{F/K}(x^{-1}) \in \mathbb{A}_K^\times$.

Proof: Must check: • χ morphism ✓

• $\chi(F^\times) = 1$.

• χ cts

(See Silverman, Thm 9.2)

Prop 1.11: Let E/F be an elliptic curve with CM by $\mathcal{O}_K$. Let $\mathfrak{p} \lhd \mathcal{O}_F$ of good reduction. Let

$$\underbrace{\begin{aligned} \widetilde{[\chi(\mathfrak{p})]} : \tilde{E} &\longrightarrow \tilde{E} \\ \tilde{P} &\longmapsto \widetilde{[\chi(\mathfrak{p})]P} \end{aligned}}$$

be the reduction of $[\chi(\mathfrak{p})] : E \to E$. Then this coincides with the q-Frobenius endomorphism

$$\varphi_q : \tilde{E} \longrightarrow \tilde{E} \qquad q = N\mathfrak{p}.$$
$$(x, y) \longmapsto (x^q, y^q)$$

Thm 1.12 (Deuring) Let E/K be an elliptic curve with CM by $\mathcal{O}_K$. Then $L(E/K, s) = L(\chi_{E/K}, s) L(\overline{\chi}_{E/K}, s)$

Proof: Let $\mathfrak{p}$ be prime of K. We check equality of local Euler factors

i.e. $L_{\mathfrak{p}}(E/K, T) = (1 - \chi_{E/K}(\mathfrak{p})T)(1 - \overline{\chi}_{E/K}(\mathfrak{p})T)$

Note: E/K has potential good reduction everywhere. Thus either have additive red^n or good red^n at $\mathfrak{p}$.

Case: Additive reduction: $L_p(E/K, T) = 1$ ✓
and $\chi(p) = 0$.

Case: Good reduction: We have to show:

$$1 - a_p T + T^2 N(p) = 1 - (\chi(p) + \bar{\chi}(p))T + \chi(p)\bar{\chi}(p)T^2$$

$\Longleftrightarrow$ $N(p) = N(\chi(p))$ ✓ and $\#E(\mathbb{F}_p) = Np + 1 - (\chi(p) + \bar{\chi}(p))$

For second equality, we have:

from 1.11.

$$\#E(\mathbb{F}_q) = \#\ker(1 - \varphi_q) = \deg(1 - \varphi_q)$$
$$= \deg(1 - \widetilde{[\chi(p)]})$$
$$= \deg(1 - [\chi(p)])$$

Finally, we have:

$$\deg(1 - [\chi(p)]) = \#\ker(1 - [\chi(p)]) = N(1 - \chi(p))$$

as $E[a]$ is free $\mathcal{O}_K/a$-module of rank 1
($\#E[a] = Na$). This proves the result. □

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§2. Eisenstein series and Kronecker limit formula.

Setup: Let E/K be elliptic curve with CM by $\mathcal{O}_K$.
Fix a Weierstrass model for E and a lattice L
given by $\{\int_\gamma \frac{dx}{y} : \gamma \in H_1(E(\mathbb{C}), \mathbb{Z})\}$.

$\xi : \mathbb{C}/L \longrightarrow E(\mathbb{C})$ given by $z \longmapsto (\wp(z), \wp'(z)/2)$

where $\wp$ is Weierstrass $\wp$-function.

Fix ideal $\mathfrak{a} \triangleleft \mathcal{O}_K$ coprime to $6\mathfrak{f}$, where $\mathfrak{f} = \mathrm{cond}(E)$

Choose f, γ generators for $\mathfrak{f}, \mathfrak{a}$ respectively.

Let $S \in E$ be an $\mathcal{O}_K$-generator for $E[\mathfrak{f}]$.

Recall:

$$\Theta_{E,\mathfrak{a}}(Q) := \gamma^{-12} \Delta(E)^{N\mathfrak{a}-1} \prod_{P \in E[\mathfrak{a}] - O} (x(Q) - x(P))^{-6}$$

and $$\Lambda_{E,\mathfrak{a}} := \prod_{\sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)} \Theta_{E,\mathfrak{a}} \circ \tau_{S^\sigma} \quad \text{where } \tau_{S^\sigma}(P) := P + S^\sigma.$$

Def^n 2.1 Define

$$\Theta_{L,\mathfrak{a}}(z) := \Theta_{E,\mathfrak{a}} \circ \xi(z) = \gamma^{-12} \Delta(E)^{N(\mathfrak{a})-1} \prod_{w \in \mathfrak{a}^{-1}L/L} (\wp(z,L) - \wp(w,L))^{-6}.$$

Def^n 2.2 Define the k-Eisenstein series as

$$E_k(z, L) := \lim_{s \to k} \sum_{w \in L} \frac{(\overline{z} + \overline{w})^k}{|z+w|^{2s}} \qquad \left(= \sum_{w \in L} \frac{1}{(z+w)^k} \text{ if } k \geq 3 \right)$$

Def^n 2.3 The Weierstrass σ-function is

$$\sigma(z,L) := z \prod_{\substack{w \in L \\ w \neq 0}} \left(1 - \frac{z}{w}\right) e^{\left(\frac{z}{w}\right) + \frac{1}{2}\left(\frac{z}{w}\right)^2}$$

Defn 2.4: For the lattice $L \subset \mathbb{C}$, define:

$$A(L) := \frac{\operatorname{Area}(\mathbb{C}/L)}{\pi}, \qquad S_2(L) := \lim_{s \to 0^+} \sum_{w \in L} \frac{1}{w^2 |w|^{2s}}$$

$$\eta(z,L) := A(L)^{-1} \bar{z} + S_2(L) z, \qquad \theta(z,L) := \Delta(L) e^{-6\eta(z,L)z} \sigma(z,L)^{12}.$$

Lemma 2.5 We have: $\Theta_{L,\underline{a}}(z) = \dfrac{\theta(z,L)^{N\underline{a}}}{\theta(z, \underline{a}^{-1}L)}$

Proof: Note: $\operatorname{div}(x - x(P)) = (P) + (-P) - 2(0)$, thus

$$\operatorname{div}(\Theta_{E,\underline{a}}) = 12 N\underline{a}\,(0) - 12 \sum_{P \in E[\underline{a}]} (P)$$

$$\Rightarrow \quad \operatorname{div}(\Theta_{L,\underline{a}}) = 12 N\underline{a}\,(0) - 12 \sum_{P \in \underline{a}^{-1}L/L} (P)$$

Also: $$RHS = \frac{\Delta(L)}{\Delta(\underline{a}^{-1}L)} \frac{\sigma(z,L)^{12N\underline{a}}}{\sigma(z,\underline{a}^{-1}L)^{12}} e^{-6z^2\left(N\underline{a}\, S_2(L) - S_2(\underline{a}^{-1}L)\right)}$$

Therefore:

$$\operatorname{div}(RHS) = 12 N\underline{a} \operatorname{div}(\sigma(z,L)) - 12 \operatorname{div}(\sigma(z,\underline{a}^{-1}L))$$

$$= 12 N\underline{a}\,(0) - 12 \sum_{P \in \underline{a}^{-1}L/L} (P)$$

$\therefore$ $\operatorname{div}(\Theta_{L,\underline{a}}) = \operatorname{div}(RHS)$

Thus $\Theta_{L,\underline{a}} = \lambda\,(RHS)$ for some $\lambda \in \mathbb{C}$.

Compare Taylor expansions of both sides at $z=0$ to get $p^{-12}\Delta(E)^{N\underline{a}-1} z^{12(N\underline{a}-1)}(1 + O(z))$. $\square$

<u>Lemma</u> 2.6:

- $E_1(z,L) = \zeta(z,L) - \eta(z,L)$
- $E_2(z,L) = \wp(z,L) + s_2(L)$

For $k \geq 3$,

- $E_k(z,L) = \frac{(-1)^k}{(k-1)!}\left(\frac{d}{dz}\right)^{k-2} \wp(z,L)$.

<u>Proof</u>: For $k \geq 3$, use $E_k(z,L) = \sum_{w \in L}(z+w)^{-k}$

<u>Corollary</u>: $E_{k-1}(z,L)' = -(k-1)E_k(z,L) \quad \forall k \geq 2$.

<u>Prop 2.7</u> If $k \geq 1$, then:

$$\left(\frac{d}{dz}\right)^k \log \Theta_{L,\underline{a}}(z) = 12(-1)^{k-1}(k-1)!\left(N\underline{a}\, E_k(z,L) - E_k(z,\underline{a}^{-1}L)\right)$$

Take logs of lemma 2.5 to get

$$\log \Theta_{L,\underline{a}}(z) = N\underline{a} \log\Theta(z,L) - \log\Theta(z,\underline{a}^{-1}L)$$

and by defn of Θ we get:

$$\log\Theta(z,L) = \log(\Delta(L)) - 6\eta(z,L)z + 12\log\sigma(z,L)$$

$$\Rightarrow \quad \log \Theta(z,L)' = -6 s_2(L) z - 6\eta(z,L) + 12 \zeta(z,L)$$
$$= 12 E_1(z,L) + 6 A(L)^{-1} \bar{z}$$

$$\therefore \frac{d}{dz} \log \Theta_{L,\mathfrak{a}}(z) = N\mathfrak{a} \left(12 E_1(z,L) + 6A(L)^{-1}\bar{z}\right) - \left(12E_1(z, \mathfrak{a}^{-1}L) + 6A(\mathfrak{a}^{-1}L)^{-1}\bar{z}\right)$$
$$= 12\left(N\mathfrak{a}\, E_1(z,L) - E_1(z, \mathfrak{a}^{-1}L)\right)$$

since $A(L) = N\mathfrak{a}\, A(\mathfrak{a}^{-1}L)$.

By repeatedly differentiating, we get the result! $\square$

<u>Lemma 2.8</u> If $\beta \in \mathcal{O}_K$ coprime to $\mathfrak{f}$, then $\frac{\chi((\beta))}{\beta} \in \mathcal{O}_K^\times$. Moreover, if $\alpha \in \mathcal{O}_K$ s.t $\alpha \equiv \beta \mod \mathfrak{f}$

Then $$\frac{\chi((\alpha))}{\alpha} = \frac{\chi((\beta))}{\beta}.$$

<u>Proof:</u> By def^n of χ, we have $(\chi((\beta))) = (\beta)$ which proves $\frac{\chi((\beta))}{\beta} \in \mathcal{O}_K^\times$. Now let $x = \frac{\alpha}{\beta}$.

- Let $x := (x, x, \dots) \in \mathbb{A}_K^\times$. As $\chi(K^\times) = 1$, clearly $\chi(x) = 1$.

- Let $a := (x, 1, 1, \dots)$, where x is in ∞-component. Then, by def^n of χ, we have $\chi(a) = x^{-1}$.

- Let $b := (b_p) \in \mathbb{A}_K^\times$ s.t $b_p = x$ if $\mathrm{ord}_p(x) \neq 0$

and $b_p = 1$ otherwise. By def^n of χ on ideals
$\chi(b) = \chi((x))$.

- Let $c := xb^{-1}a^{-1} = (c_p)$. By def^n $c_p = 1$ if $\mathrm{ord}_p(x) \neq 0$ and $c_p = x$ otherwise. As $x \equiv 1 \bmod \mathfrak{f}$ this $\chi(c) = 1$ (by def^n of $\mathfrak{f}$)

$$\therefore \quad \frac{\chi((\alpha))}{\chi((\beta))} = \chi\big((x)\big) = \chi(b) = \frac{\chi(x)}{\chi(a)\chi(c)} = x = \frac{\alpha}{\beta}.$$

$\square$

Prop 2.9: Let $\mathfrak{m} \triangleleft \mathcal{O}_K$ divisible by $\mathfrak{f}$ and let $v \in KL/L$ of order exactly $\mathfrak{m}$. Then, for $k \geq 1$,

$$E_k(v, L) = v^{-k}\chi(\mathfrak{c})^k L_{\mathfrak{m}}(\bar{\chi}^k, k, \mathfrak{c})$$

where $\mathfrak{c} = \Omega^{-1} v \mathfrak{m}$, where $L = \Omega\mathcal{O}_K$.

Proof: As $[\mathfrak{m}]v \in L$, there exists $\alpha \in \mathcal{O}_K$ coprime to $\mathfrak{m}$ s.t. $v = \alpha\Omega/\mu$ for μ a generator of $\mathfrak{m}$.
Thus, can write $E_k(v, L)$ as

$$E_k(v, L) := \lim_{s \to k} \sum_{w \in L} \frac{(\bar{v} + \bar{w})^k}{|v+w|^{2s}} = \lim_{s \to k} \frac{N\mu^s}{\bar{\mu}^k} \frac{\bar{\Omega}^k}{|\Omega|^{2s}} \sum \frac{(\bar{\alpha} + \overline{w\mu}/\bar{\Omega})^k}{(\alpha + w\mu/\Omega)^{2s}}$$

$$= \lim_{s \to k} \frac{N\mu^s}{\bar{\mu}^k} \frac{\bar{\Omega}^k}{|\Omega|^{2s}} \sum \frac{(\bar{\alpha} + \overline{\mu x})^k}{(\alpha + \mu x)^{2s}}$$

as $L = \Omega\mathcal{O}_K$. By letting $\beta := \alpha + \mu x$, we can rewrite the last sum as

$$\sum_{x \in \mathcal{O}_K} \frac{(\overline{\alpha} + \overline{\mu x})^k}{|\alpha + \mu x|^{2s}} = \sum_{\substack{\beta \in \mathcal{O}_K \\ \beta \equiv \alpha \bmod \mathfrak{m}}} \frac{\overline{\beta}^k}{N\beta^s}$$

Using Lemma 2.8, we have $\frac{\psi((\alpha))}{\alpha} = \frac{\psi((\beta))}{\beta}$, and thus $\overline{\beta} = \frac{\psi((\alpha))}{\alpha} \overline{\psi}((\beta))$ and so

$$\sum_{\substack{\beta \in \mathcal{O}_K \\ \beta \equiv \alpha \bmod \mathfrak{m}}} \frac{\overline{\beta}^k}{N\beta^s} = \frac{\psi^k((\alpha))}{\alpha^k} \sum_{\substack{\beta \in \mathcal{O}_K \\ \beta \equiv \alpha \bmod \mathfrak{m}}} \frac{\overline{\psi}((\beta))^k}{N\beta^s}$$

Want to write sum over ideals $\mathfrak{b} \triangleleft \mathcal{O}_K$. Can check the following map is a bijection:

$$\{\beta \in \mathcal{O}_K \mid \beta \equiv \alpha \bmod \mathfrak{m}\} \longrightarrow \{\mathfrak{b} \subset \mathcal{O}_K \mid (\mathfrak{b}, K(\mathfrak{m})/K) = ((\alpha), K(\mathfrak{m})/K)\}.$$
$$\beta \longmapsto (\beta)$$

Therefore:
$$\sum_{\substack{\beta \in \mathcal{O}_K \\ \beta \equiv a \bmod \mathfrak{m}}} \frac{\overline{\psi}((\beta))^k}{N\beta^s} = \sum_{\substack{\mathfrak{b} \triangleleft \mathcal{O}_K \\ (\mathfrak{b}, K(\mathfrak{m})/K) = ((\alpha), K(\mathfrak{m})/K)}} \frac{\overline{\psi}^k(\mathfrak{b})}{N\mathfrak{b}^s} = L_{\mathfrak{m}}(\overline{\psi}^k, s, (\alpha))$$

Putting everything together and evaluating at $s=k$ yields
$$E_k(v, L) = v^{-k} \psi(\underline{c})^k L_{\mathfrak{m}}(\overline{\psi}^k, k, \underline{c}) \qquad \square.$$

<u>Def^n/Thm 2.10</u> Define $\Lambda_{L,\underline{a}} := \Lambda_{E,\underline{a}} \circ \xi$.

Let $B \subset I_K(\mathfrak{f})$ s.t. $\underline{b} \longmapsto (\underline{b}, K(\mathfrak{f})/K)$ is a bijection $B \longrightarrow \mathrm{Gal}(K(\mathfrak{f})/K)$. We have:

$$\Lambda_{L,\underline{a}}(z) = \prod_{\underline{b} \in B} \Theta_{L,\underline{a}}(\psi(\underline{b})u + z)$$

where $u = \Omega / \mathfrak{f}$ is a $\mathcal{O}_K$-generator for $\mathfrak{f}$-torsion on $\mathbb{C}/L$.

Thm 2.11: "Kronecker limit formula"

For all $k \geqslant 1$

$$\frac{d^k}{dz^k} \log \Lambda_{L,\underline{a}}(z) \Big|_{z=0} = 12(-1)^k (k-1)!\, \mathfrak{f}^k (N\underline{a} - \psi(\underline{a})^k) \cdot \Omega^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k)$$

elliptic units

L-function

Proof: We apply prev thms:

$$\frac{d^k}{dz^k} \log \Lambda_{L,a}(z) \Big|_{z=0} = \sum_{b \in B} \frac{d^k}{dz^k} \log \Theta_{L,a}(z) \Big|_{z=\psi(\underline{b})u}$$

$$= 12(-1)^k (k-1)! \sum_{\underline{b} \in B} \left(N\underline{a}\, E_k(\psi(\underline{b})u, L) - E_k(\psi(\underline{b})u, \underline{a}^{-1}L) \right)$$

Applying Prop 2.9 with $\psi(\underline{b})u$ and $\mathfrak{f}$

$$E_k(\psi(\underline{b})u; L) = u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k, \underline{b})$$

and summing over all $\underline{b}$ yields:

$$\sum_{b \in B} E_k(\psi(\underline{b}u; L) = u^{-k} \sum_{b \in B} L_{\mathfrak{f}}(\psi^k, k, \underline{b})$$

$$= u^{-k} L_{\mathfrak{f}}(\bar{\psi}^k, k)$$

$\square$

Corollary 2.12 (Damerell) $\dfrac{L(\bar{\chi}^k, k)}{\Omega^k} \in K \quad \forall k \geq 1$.

Proof: Note: $\wp$ satisfies $\wp'(z,L)^2 = 4\wp(z,L)^3 + 4A\wp(z,L) + 4B$

This implies: $\left(\frac{d}{dz}\right)^k \wp(z,L) \in K(\wp(z,L), \wp(z,L)')$

and so $\Delta^{(k)}_{L,\mathfrak{a}}(z) \in K(\wp(z,L), \wp(z,L)')$. Result follows from Thm 2.11 and evaluating at $z=0$. $\square$.

Thm 2.13: Let $\mathfrak{p}$ be prime not dividing $6\mathfrak{a}\mathfrak{f}$.

Choose minimal model for E with coords x, y.

Consider $\hat{E}$, formal group of E over $\mathcal{O}_{K,\mathfrak{p}}$

with $x(z) \in z^{-2}\mathcal{O}_{K,\mathfrak{p}}[[z]]$, $y(z) \in z^{-3}\mathcal{O}_{K,\mathfrak{p}}[[z]]$

Let $\Lambda_{\mathfrak{p},\mathfrak{a}}(z) := \Lambda_{E,\mathfrak{a}}(x(z), y(z)) \in K_{\mathfrak{p}}((z))$.

Define $D := \dfrac{1}{\log'_{\hat{E}}(z)} \cdot \dfrac{d}{dz}$ (where $\log_{\hat{E}}(z)$ is log of $\hat{E}$)

Given $\Lambda_{E,\mathfrak{a}}(z)$, the power series $\Lambda_{\mathfrak{p},\mathfrak{a}}(z)$ is a <u>unit</u> in $\mathcal{O}_K[[z]]$ and satisfies:

$$D^k \log \Lambda_{\mathfrak{p},\mathfrak{a}}(z)\Big|_{z=0} = 12(-1)^{k-1}(k-1)!\, f^k (N\mathfrak{a} - \chi(\mathfrak{a})^k) \cdot \Omega^{-k} L_{\mathfrak{f}}(\bar{\chi}^k, k)$$

⊛ Def^n of the map $K/\underline{a} \xrightarrow{\cdot x} K/(x)\underline{a}$ in Thm 1.8

Let $x = (x_p) \in \mathbb{A}_K^\times$, and let $(x) = \left(\prod_p p^{\mathrm{ord}_p x_p}\right)$ be the corresponding ideal of x. For each prime p, define the map $\cdot x_p$ as:

$$\begin{aligned} \cdot x_p : K_p/\underline{a}_p &\longrightarrow K_p/x_p \underline{a}_p \\ y &\longmapsto x_p y \end{aligned}$$

(i.e. just used multp by x_p)

Now note the decomposition $K/\underline{a} \cong \bigoplus_{\text{prime } p} K_p/\underline{a}_p$

We define the map $K/\underline{a} \xrightarrow{\cdot x} K/(x)\underline{a}$ by the following commutative diagram:

$$\begin{tikzcd} K/\underline{a} \arrow[r, "\cdot x"] \arrow[d, "\wr"] & K/(x)\underline{a} \arrow[d, "\wr"] \\ \bigoplus_p K_p/\underline{a}_p \arrow[r, "(\cdot x_p)"] & \bigoplus_p K_p/x_p\underline{a}_p \end{tikzcd}$$