L-values of Elliptic Curves twisted by Hecke Grössencharacters

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Abstract

Let E/K be an elliptic curve over an imaginary quadratic field K, and let ψ be a Hecke Grössencharacter of K. We shall study the values of the L-function $L(E/K, \psi, 1)$, and show both conjecturally and numerically that $L(E/K, \psi, 1)$ is an algebraic multiple of $\pi p(\psi)^2$ whenever ψ has non-trivial infinity type, where $p(\psi)$ denotes the period of ψ. We shall also give a brief overview of existing results for base change E/K using the method of Rankin-Selberg convolutions, as well as outline a possible approach to proving analogous results for arbitrary curves E/K.

Introduction

The study of L-series and its corresponding L-function have played a fundamental role in many areas of number theory over the past two centuries. Indeed, given some sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{C}$, we can consider the following series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which thus opens the door to an endless list of questions one can ask, such as whether this has a meromorphic (or analytic) continuation to the complex plane, whether it satisfies a suitable functional equation, what form does its Euler product take, etc.

Euler was one of the first to seriously study these objects. Also using these ideas, Dirichlet [Dir37] first famously used L-functions of Dirichlet characters in 1837 to prove there are infinitely many primes in arithmetic progressions. Later on, Riemann [Rie59] then studied the well-known zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and proved a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

In the early 20th century, Hecke generalised these notions of L-functions to number fields, and proceeded to study Hecke Grössencharacters (i.e. characters of infinite order) [Hec18, Hec20]. To give a complete overview of the history of L-functions is far beyond the scope of this project[1], however this introduction would not be complete without at least mentioning the immense profundity that the Langlands programme has now brought to studying L-function in the last few decades.

For this project, our aim is to study the L-functions of elliptic curves twisted by Hecke Grössencharacters. We first explicitly calculate the values $L(E/K, \psi, 1)$ for various elliptic curves E and Grössencharacters ψ to a high level of precision, as well as prove that our computations are consistent with Deligne's period conjecture.

Whilst a full unconditional proof is not presented within the scope of this project, we do provide an overview of some results which have been proven for elliptic curves E/K which are base change from $\mathbb{Q}$ using the method of Rankin-Selberg convolutions. Finally, we give a brief outline of some possible avenues one could investigate to extend these results for arbitrary curves E/K.

Preliminaries

We begin by stating a few preliminary definitions, first including the L-function of an elliptic curve E/K:

[1] For those interested in further history of L-functions, [BCF16] gives a neat overview of the topic.

Definition 1: [Sil09, p. 449] Let E be an elliptic curve over some number field K. We recall the **L-function** of E/K is given by the Euler product:

$$L(E/K,\, s) := \prod_{\mathfrak{p} \lhd \mathcal{O}_K} L_{\mathfrak{p}}(N(\mathfrak{p})^{-s}) \tag{1}$$

where $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$, and where the product is taken over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ (or equivalently finite places of K).

Each of the local Euler factors $L_{\mathfrak{p}}(T)$ essentially depends on the reduction type of E at $\mathfrak{p}$. For arbitrary curves, the local Euler factor $L_{\mathfrak{p}}(T)$ can be defined generally in terms of the geometric Frobenius in a decomposition group at $\mathfrak{p}$, however in the case of elliptic curves, one can give the following explicit definition:

For a prime $\mathfrak{p}$ of good reduction, let $\#\widetilde{E}_{\mathfrak{p}}$ denote the number of points in the reduction of E modulo $\mathfrak{p}$, and let $a_{\mathfrak{p}} = N(\mathfrak{p}) + 1 - \#\widetilde{E}_{\mathfrak{p}}$. Then we define $L_{\mathfrak{p}}(T)^{-1}$ as:

$$L_{\mathfrak{p}}(T)^{-1} = \begin{cases} 1 - a_{\mathfrak{p}}T + N(\mathfrak{p})T^2 & \text{if } E \text{ has good reduction at } \mathfrak{p} \\ 1 - T & \text{if } E \text{ has split multiplicative reduction at } \mathfrak{p} \\ 1 + T & \text{if } E \text{ has non-split multiplicative reduction at } \mathfrak{p} \\ 1 & \text{if } E \text{ has additive reduction at } \mathfrak{p} \end{cases}$$

To obtain a full set of Fourier coefficients $a_{\mathfrak{n}}$ of E, for all ideals $\mathfrak{n}$ in $\mathcal{O}_K$, we do as follows: We first set $a_1 = 1$, and for all prime ideals $\mathfrak{p}$ we set $a_{\mathfrak{p}}$ to be the linear coefficient of $L_{\mathfrak{p}}(T)$ considered as a power series in T, i.e.

$$a_{\mathfrak{p}} = \begin{cases} N(\mathfrak{p}) + 1 - \#\widetilde{E}_{\mathfrak{p}} & \text{if } E \text{ has good reduction at } \mathfrak{p} \\ 1 & \text{if } E \text{ has split multiplicative reduction at } \mathfrak{p} \\ -1 & \text{if } E \text{ has non-split multiplicative reduction at } \mathfrak{p} \\ 0 & \text{if } E \text{ has additive reduction at } \mathfrak{p} \end{cases}$$

For powers of primes, we can then define $a_{\mathfrak{p}^r}$ recursively by letting, for $r \geq 2$:

$$a_{\mathfrak{p}^r} = \begin{cases} a_{\mathfrak{p}} \cdot a_{\mathfrak{p}^{r-1}} - N(\mathfrak{p}) \cdot a_{\mathfrak{p}^{r-2}} & \text{if } E \text{ has good reduction at } \mathfrak{p} \\ (a_{\mathfrak{p}})^r & \text{if } E \text{ has bad reduction at } \mathfrak{p} \end{cases}$$

and then finally define $a_{\mathfrak{n}}$ for all ideals $\mathfrak{n}$ by multiplicativity (i.e. if $(\mathfrak{m}, \mathfrak{n}) = 1$, then $a_{\mathfrak{mn}} = a_{\mathfrak{m}} \cdot a_{\mathfrak{n}}$).

This therefore yields the same L-function given by taking the sum

$$L(E/K, s) = \sum_{\mathfrak{n} \lhd \mathcal{O}_K} \frac{a_{\mathfrak{n}}}{N(\mathfrak{n})^s}$$

where the equivalence with the definition given in (1) can be shown by an easy argument using geometric series.

By Hasse's theorem, we note that we have the bound $|a_{\mathfrak{p}}| \leq 2\sqrt{N(\mathfrak{p})}$ [Sil09, p. 143], and thus the L-series has absolute convergence in the region $\mathrm{Re}(s) > \frac{3}{2}$.

Furthermore, we also note that the recurrence given for $a_{\mathfrak{p}^r}$ resembles that of the Hecke operator $T_{\mathfrak{p}}$ for weight 2 modular forms. This hints that these L-functions $L(E/K, s)$ could be connected to modular forms and thus furthermore satisfy a functional equation, which will allow us to extend $L(E/K, s)$ to the entire complex plane.

Indeed, this has been proven for elliptic curves over the rationals, following from the modularity theorem [Wil95, TW95, BCDT01]:

Theorem 2: [LR11, p. 120] Let $E/\mathbb{Q}$ be elliptic curve over $\mathbb{Q}$. Then the L-series $L(E/\mathbb{Q}, s)$ has an analytic continuation to the entire complex plane. Furthermore, if we define the completed L-function

$$\Lambda(E/\mathbb{Q},\, s) := (N_{E/\mathbb{Q}})^{s/2}(2\pi)^{-s}\Gamma(s)L(E/\mathbb{Q},\, s)$$

where $N_{E/\mathbb{Q}}$ denotes the conductor of E, then this satisfies the functional equation

$$\Lambda(E/\mathbb{Q},\, s) = w_{E/\mathbb{Q}} \cdot \Lambda(E/\mathbb{Q},\, 2-s) \tag{2}$$

where $w_{E/\mathbb{Q}}$ is the global root number of $E/\mathbb{Q}$, which is always either -1 or $+1$. The root number $w_{E/\mathbb{Q}}$ is conjecturally determined by the parity of the rank of E, as described in Conjecture 4.

The natural next question would be to ask whether one can compute specific values of $L(E/\mathbb{Q}, s)$? This leads us to mentioning arguably one of the most famous open problems in this area: the Birch and Swinnerton-Dyer conjecture.

Conjecture 3: [Sil09, p. 452] (Birch and Swinnerton-Dyer) Let $E/\mathbb{Q}$ be an elliptic curve. Then the following two statements are conjectured to hold (often referred to as *weak BSD* and *strong BSD* respectively):

1. (*Weak BSD*) The function $L(E/\mathbb{Q}, s)$ has a zero at $s = 1$ of order $\mathrm{rank}(E/\mathbb{Q})$.
2. (*Strong BSD*) The residue of $L(E/\mathbb{Q}, s)$ at $s = 1$ satisfies the following explicit formulae in terms of invariants of $E/\mathbb{Q}$:

$$\lim_{s\to 1} \frac{L(E/\mathbb{Q}, s)}{(s-1)^{\mathrm{rank}(E/\mathbb{Q})}} = \frac{|\text{Ш}| \cdot \Omega_E \cdot \mathrm{Reg}(E/\mathbb{Q}) \cdot \prod_p c_p}{|E_{\mathrm{tors}}(\mathbb{Q})|^2} \tag{3}$$

   where Ω_E is the real period (or twice if $E(\mathbb{R})$ disconnected), Ш is the Tate-Shafarevich group of $E/\mathbb{Q}$, $\mathrm{Reg}(E/\mathbb{Q})$ is the elliptic regulator, $E_{\mathrm{tors}}(\mathbb{Q})$ is the torision subgroup of E over $\mathbb{Q}$, and where c_p is the Tamagawa number of E at p. All these invariants are defined in further details in Silverman [Sil09, p. 415].

This conjecture is one of the famous Millennium Prize problems given by the Clay Maths Institute, which is still currently unsolved in the general case.[2] We do remark that some partial progress has been made. One notable case is, if $\mathrm{ord}_{s=1}L(E/\mathbb{Q}, s) \leq 1$, then Kolyvagin [Kol89] proved that weak BSD holds and that $\text{Ш}(E/\mathbb{Q})$ is finite.

Functional equation (over number fields)

For elliptic curves E over general number fields K, many of the following results are still conjectural in most cases. We now define the L-function for arbitrary number fields K. Let E/K be an elliptic curve, then we define:

$$\Lambda(E/K,\, s) := \left(\mathrm{Norm}(N_{E/K})\, d_K^2\right)^{s/2} \left((2\pi)^{-s}\Gamma(s)\right)^{[K:\mathbb{Q}]} L(E/K,\, s)$$

[2] At this point, it also seems customary to quote the following from John Tate in 1974: "*This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined to the order of a group X which is not known to be finite!*" We do remark that $L(E/\mathbb{Q}, s)$ has been proven to be defined at $s = 1$, although the finiteness of Ш is still an open problem in general.

where, as before, $N_{E/K}$ denotes the conductor of E/K, and where d_K denotes the discriminant of K.

Now the Hasse–Weil conjecture predicts that this extends to an analytic continuation to the entire complex plane, and satisfies the functional equation

$$\Lambda(E/K,\, s) = w_{E/K} \cdot \Lambda(E/K,\, 2-s)$$

where as with the rational case, $w_{E/K} \in \{\pm 1\}$ is the global root number of E/K.

Using modularity, the above conjecture has been proven in the case where $K = \mathbb{Q}$ and where K is a real quadratic field [FLS15], with Allen–Khare–Thorne proving that this also holds for a positive proportion of elliptic curves over certain CM fields [AKT19].

We also mention that it is conjectured that $w_{E/K}$ is directly related to the rank of E/K:

Conjecture 4: (Parity conjecture) Let E/K be an elliptic curve. Then we have

$$w_{E/K} = (-1)^{\mathrm{rank}(E/K)}$$

We remark that the parity conjecture is a consequence of BSD, and whilst the parity conjecture hasn't been proved in full generality, it has been computationally verified for many examples of elliptic curves, with no counterexample yet being found. Dokchitser and Dokchitser [DD11] have shown that the parity conjecture follows from the Tate-Shafarevich conjecture, which unconditionally proves the above result in certain cases.

For completeness, we also finally state the generalised Birch and Swinnerton-Dyer conjecture for elliptic curves over arbitrary number fields:

Conjecture 5: Let E/K be an elliptic curve. Then $L(E/K, s)$ has analytic continuation to $\mathbb{C}$ and satisfies

1. The function $L(E/K, s)$ has a zero at $s = 1$ of order $\mathrm{rank}(E/K)$.
2. The residue of $L(E/K, s)$ at $s = 1$ satisfies the following explicit formulae in terms of invariants of E/K:

$$\lim_{s\to 1} \frac{L(E/K, s)}{(s-1)^{\mathrm{rank}(E/K)}} = \frac{|\text{Ш}| \cdot \Omega_E \cdot \mathrm{Reg}(E/K) \cdot \prod_p c_p}{\sqrt{d_K} \cdot |E_{\mathrm{tors}}(K)|^2}$$

   where d_K denotes the discriminant of K, with the invariants of E defined similarly as with Conjecture 3.

Twisted elliptic curves

Given an elliptic curve E over $\mathbb{Q}$ and a positive integer N, we can consider a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, and study the L-function of $E/\mathbb{Q}$ twisted by χ, given by:

$$L(E/\mathbb{Q}, \chi, s) := \sum_{\substack{n\geq 1 \\ (n,N)=1}} \frac{a_n \chi(n)}{n^s} \tag{4}$$

These twisted L-functions also satisfy a similar functional equation as given in (2), which has been proven over $\mathbb{Q}$ due to modularity.

Theorem 6: [DFK04, p. 186] Let E be an elliptic curve over $\mathbb{Q}$, and let χ be a primitive character of conductor $\mathfrak{f}_\chi$, coprime to the conductor $N_{E/\mathbb{Q}}$ of $E/\mathbb{Q}$. Then $L(E/\mathbb{Q},\chi,s)$ also has analytic continuation to the entire complex plane. Furthermore, we can define the completed L-function

$$\Lambda(E/\mathbb{Q},\chi,s) := \mathfrak{f}_\chi^s (N_{E/\mathbb{Q}})^{s/2}(2\pi)^{-s}\Gamma(s)\, L(E/\mathbb{Q},\chi,s) \tag{5}$$

which then satisfies the following functional equation:

$$\Lambda(E/\mathbb{Q},\chi,s) = \omega_{E/\mathbb{Q}}\, \chi(N_{E/\mathbb{Q}})\, \tau(\chi)^2\, \mathfrak{f}_\chi^{-1}\, \Lambda(E/\mathbb{Q},\overline{\chi},2-s)$$

where as before $\omega_{E/\mathbb{Q}}$ is the root number of E, and where $\tau(\chi)$ denotes the Gauss sum of χ [Miy89, p. 80], defined by

$$\tau(\chi) := \sum_{n=0}^{\mathfrak{f}_\chi-1} \chi(n)\zeta_{\mathfrak{f}_\chi}^n, \quad \text{where } \zeta_{\mathfrak{f}_\chi} = e^{2\pi i/\mathfrak{f}_\chi}.$$

We note that the criterion for $\mathfrak{f}_\chi$ to be coprime to $N_{E/\mathbb{Q}}$ is not essential for there to exist a functional equation relating $\Lambda(E/\mathbb{Q},\chi,s)$ to $\Lambda(E/\mathbb{Q},\overline{\chi},2-s)$, although in this case the exact definition for $\Lambda(E/\mathbb{Q},\chi,s)$ given in (5) could change.

In general, formulating a BSD-style conjecture to give the exact leading term of $L(E/\mathbb{Q},\chi,s)$ at $s=1$ seems to be out of reach. At best, Dokchitser–Evans–Wiersema [DEW21] has given a conjectural result for the norm of the residue of $L(E/\mathbb{Q},\chi,s)$ at $s=1$, up to a sign. One inherent difficulty in stating precisely the value of $L(E/\mathbb{Q},\chi,1)$ is that there are many examples of two different elliptic curves $E_1/\mathbb{Q}$ and $E_2/\mathbb{Q}$ such that all the arithmetic invariants mentioned in (3) are the same between E_1 and E_2, but such that $L(E_1/\mathbb{Q},\chi,1) \neq L(E_2/\mathbb{Q},\chi,1)$ [DEW21, p. 24].

However, it's possible to at least give the L-value at $s=1$ up to algebraic multiples. Indeed, if $L(E/\mathbb{Q},\chi,1) \neq 0$, then we have

$$L(E/\mathbb{Q},\chi,1) \in \overline{\mathbb{Q}} \cdot \Omega_E^{(\mathrm{sgn}(\chi))}$$

where the period taken $\Omega_E^{(\mathrm{sgn}(\chi))}$ depends on the sign of χ. In other words, if $\chi(-1)=1$, then we take the real period Ω_E^+ of E, whereas if $\chi(-1)=-1$, we take the imaginary period Ω_E^- of E, as defined in [DEW21, p. 6].

More generally, if the rank of $E/\mathbb{Q}$ is non-zero, then one has to also consider taking the regulator $\mathrm{Reg}(E/\mathbb{Q})$ into account for the transcendental part.[3]

In order to generalise the above results of twists of elliptic curves to number fields, we'll first need to introduce the theory of Hecke Grössencharacters.

Grössencharacters

As a further generalisation of Dirichlet characters, we shall also study **Hecke Grössencharacters**, which can be briefly described as the automorphic forms for GL_1 over a number field K. These were first studied by Hecke in the early 20th century as a generalisation of Dirichlet characters [Hec18, Hec20]. He furthermore proved a generalised functional equation for any Hecke character by generalising the proof for the Riemann-zeta function.

[3] Interestingly, it seems to be conjectured, although not proven, whether the regulator $\mathrm{Reg}(E/\mathbb{Q})$ is transcendental for any particular elliptic curve of non-zero rank.

It's worth mentioning that an elegant definition of Hecke characters can be given as a continuous group homomorphism from the adele group $\mathbb{A}_K^\times$ modulo $K^\times$ to $\mathbb{C}^\times$:

$$\psi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$$

Whilst the adelic approach gives a nice short description of Grössencharacters, it isn't always the easiest approach to work with computationally. We therefore consider also describing Hecke Grössencharacters as a map on fractional ideals in K, where an explicit procedure to pass between the two definitions can be found in [Dej17, p. 19].

We will therefore avoid using the adelic language for the remainder of this project, and will thus simply define Grössencharacters in the following explicit manner:

Let K be a number field, and let Σ_K denote the set of embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}}$ into a fixed algebraic closure $\overline{\mathbb{Q}}$. We first define an **infinity type** of K as a sequence of integers $T = (n_\sigma)_{\sigma \in \Sigma_K}$ indexed by the embeddings of K. For any given element $\alpha \in K^\times$, we define the evaluation of an infinity type T at an α as

$$T(\alpha) := \prod_{\sigma \in \Sigma_K} (\alpha^\sigma)^{n_\sigma}$$

where the product is taken over all embeddings σ of K. Given some ideal $\mathfrak{m} \lhd \mathcal{O}_K$, we also define $I(\mathfrak{m})$ as the set of fractional ideals in K coprime to $\mathfrak{m}$.

We shall now state a more explicit definition for Grössencharacters. For simplicity, we take the convention here that our Grössencharacters are unramified at the real places.[4] This means that we consider the modulus of a Grössencharacter ψ as simply an ideal $\mathfrak{m}$ of K.

Definition 7: [Sch88, p. 1] Let K be a number field, let T be an infinity type of K, and fix some modulus $\mathfrak{m} \lhd \mathcal{O}_K$. A **Grössencharacter** of K of infinity type T, modulo $\mathfrak{m}$, is a homomorphism $\psi : I(\mathfrak{m}) \to \mathbb{C}^\times$ such that for any principal ideal $\mathfrak{n} = (\lambda)$ where $\lambda \equiv 1 \bmod \mathfrak{m}$, we have

$$\psi(\mathfrak{n}) = T(\lambda) = \prod_{\sigma \in \Sigma_K} (\lambda^\sigma)^{n_\sigma}$$

By convention, we can furthermore extend the values of ψ to all ideals in $\mathcal{O}_K$ by simply setting $\psi(\mathfrak{n}) = 0$ if $\mathfrak{n}$ not coprime to $\mathfrak{m}$.

Note that, by necessity for ψ to be well-defined, we do require that $T(u) = 1$ for all units $u \in \mathcal{O}_K^\times$ where $u \equiv 1 \bmod \mathfrak{m}$. By Dirichlet's unit theorem, we also require that $n_\sigma + n_{\overline{\sigma}}$ is constant for all embeddings σ of K [Roh10, p. 9]. We call the constant value $w = n_\sigma + n_{\overline{\sigma}}$ the **weight** of T.

Note that, if K is totally real, then the constraint that $n_\sigma + n_{\overline{\sigma}}$ is constant simply implies that $T(\alpha)$ is some power of the norm $N(\alpha)^{w/2}$ where $w \in 2\mathbb{Z}$, and thus any Grössencharacter ψ of K is simply the product of some finite order character with some power of the norm map.

For fields which are non totally real, the situation becomes somewhat more interesting. For simplicity, we shall consider the specialisation to the imaginary quadratic number field case, and in practice, will often only consider class number 1 imaginary quadratic fields.

[4] We remark that this does lose generality compared to the adelic approach, however all our examples of Grössencharacters will satisfy this simplification. Some authors seem to denote these instead as *(algebraic) Hecke characters*. Although others seemingly use Hecke characters as a synonym for Grössencharacters, so the terminology isn't always consistent.

Let K be an imaginary quadratic field, let $(a,b) \in \mathbb{Z}^2$ be an infinity type, and $\mathfrak{m}$ modulus. A Grössencharacter of infinity type (a,b) mod $\mathfrak{m}$ can thus be given as a map $\psi : I(\mathfrak{m}) \to \mathbb{C}^\times$ such that for any principal ideal $\mathfrak{n} = (\lambda)$ where $\lambda \equiv 1$ mod $\mathfrak{m}$, we have

$$\psi(\mathfrak{n}) = \lambda^a \overline{\lambda}^b \tag{6}$$

where, as before, we require that $u^a \overline{u}^b = 1$ for all units $u \in \mathcal{O}_K^\times$ where $u \equiv 1$ mod $\mathfrak{m}$.

We also define the **conductor** $\mathfrak{f}_\psi$ of ψ in an analogous way to Dirichlet characters, i.e. $\mathfrak{f}_\psi$ is the largest integral ideal of K such that (6) holds for all $\lambda \equiv 1$ mod $\mathfrak{f}_\psi$.

Note that given a Grössencharacter ψ_1 of infinity type (a_1, b_1) mod $\mathfrak{m}_1$, and a character ψ_2 of infinity type (a_2, b_2) mod $\mathfrak{m}_2$, then $\psi_1\psi_2$ is a Grössencharacter of infinity type $(a_1 + a_2, b_1 + b_2)$, mod $\mathfrak{m}_1\mathfrak{m}_2$ (although this need not be the conductor of $\psi_1\psi_2$).

Examples

Surprisingly, it seemed harder than expected to find many explicit easy examples of Grössencharacters in existing literature. Therefore, we shall present some examples here:

- Let K be a number field of class number 1, and $\mathfrak{m} \lhd \mathcal{O}_K$ an ideal. One very obvious example is to simply send all ideals to the form given in (6), i.e.

  $$\psi(\mathfrak{n}) = \lambda^a \overline{\lambda}^b \tag{7}$$

  for $\mathfrak{n} \lhd I(\mathfrak{m})$, where $\mathfrak{n} = (\lambda)$, with the condition that $u^a \overline{u}^b = 1$ for all $u \in \mathcal{O}_K^\times$ where $u \equiv 1$ mod $\mathfrak{m}$.

  For example, let $K = \mathbb{Q}(\sqrt{-D})$ have class number 1 (i.e. $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$). Then (7) gives a well-defined conductor 1 Grössencharacter if and only if

  $$\begin{aligned} a &\equiv b \pmod 4 \quad \text{if } D = 1, \\ a &\equiv b \pmod 6 \quad \text{if } D = 3, \\ a &\equiv b \pmod 2 \quad \text{otherwise} \end{aligned}$$

  We'll call these characters the *canonical Grössencharacter* $\psi_{a,b}$ of infinity type (a,b).

  Another standard family of examples can be obtained by taking usual finite order Dirichlet characters and extending them to K through the norm map. We'll write down a few imaginary quadratic examples. Recall that, if $K = \mathbb{Q}(\sqrt{-d})$, then p splits if and only if D is a square mod p.

- Let $K = \mathbb{Q}(\sqrt{-2})$. We note that all degree 1 primes $\mathfrak{p}$ have $N(\mathfrak{p}) \equiv 1, 3$ mod 8. Then we have the quadratic Grössencharacter

  $$\psi(\mathfrak{p}) = \begin{cases} -1 & \text{if } N(\mathfrak{p}) \equiv 3 \pmod 8 \\ 1 & \text{otherwise} \end{cases}$$

- Let $K = \mathbb{Q}(\sqrt{-7})$. Note that the degree 1 primes here are those with norm 1, 2, 4 mod 7. Let ω be a primitive cube root of unity. Then we have the Grössencharacter

  $$\psi(\mathfrak{p}) = \begin{cases} \omega & \text{if } N(\mathfrak{p}) \equiv 2 \pmod 7 \\ \omega^2 & \text{if } N(\mathfrak{p}) \equiv 4 \pmod 7 \\ 1 & \text{otherwise} \end{cases}$$

- One can also describe many examples of finite order characters which do not factor through the norm map. Again, let $K = \mathbb{Q}(\sqrt{-7})$, and consider taking the modulus $\mathfrak{m} = (3)$. Then there are exactly 4 distinct finite order characters mod $\mathfrak{m}$, each being uniquely defined from the value of any prime ideal $\mathfrak{p} = (\lambda)$ where λ is a non-square mod 3.

  For example, if $\mathfrak{p}_2$ denotes any of the two prime ideals lying above 2, then letting $\psi(\mathfrak{p}_2) = i$ yields an order 4 Grössencharacter where $\psi(\overline{\mathfrak{p}_2}) = -i$ (and thus cannot factor through the norm map)

- [Wat11, p. 8] Let $K = \mathbb{Q}(\sqrt{-1})$, and let $\mathfrak{p}_2$ be the ramified prime above 2. As $\varphi(\mathfrak{p}_2^3) = 4$, for every λ, there exists a unique unit $u \in \{\pm 1, \pm i\}$ such that $u\lambda \equiv 1 \bmod \mathfrak{p}_2^3$.

  Using this construction, we therefore have a unique Grössencharacter mod $\mathfrak{m} = \mathfrak{p}_2^3$ with infinity type $(1, 0)$.[5]

- [Loe08, p. 8] Let $K = \mathbb{Q}(\sqrt{-D})$ have class number 1, with $D \neq 1, 2, 3$. We note that in the remaining cases -1 is not a square mod D. Then if $a + b$ is odd, we can define the character

$$\psi(\mathfrak{p}) = \left(\frac{\mathfrak{p}}{D}\right) \lambda^a \overline{\lambda}^b$$

  We shall also refer to this example as the *canonical Grössencharacter* $\psi_{a,b}$ of infinity type (a, b) when $a + b$ is odd.

L-functions of Grössencharacters

We can construct directly the L-function of a Grössencharacter ψ as

$$L(\psi, s) := \sum_{\mathfrak{n} \triangleleft \mathcal{O}_K} \frac{\psi(\mathfrak{n})}{N(\mathfrak{n})^s} = \prod_{\mathfrak{p} \triangleleft \mathcal{O}_K} \left(1 - \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

where the Euler product follows by multiplicativity of ψ.

A functional equation for $L(\psi, s)$ can be given as follows [Miy89, p. 93]: Let Σ_K^{r} and Σ_K^{c} denote the real and complex embeddings of K respectively. We also use the following shorthand notation for the gamma factors $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. Then the completed L-function for ψ is

$$\Lambda(\psi, s) = \left(|d_K| N(\mathfrak{f}_\psi)\right)^{s/2} \prod_{\sigma \in \Sigma_K^{\mathrm{r}}} \Gamma_{\mathbb{R}}(s - a_\sigma) \prod_{\sigma \in \Sigma_K^{\mathrm{c}}} \Gamma_{\mathbb{C}}(s - \min(a_\sigma, a_{\overline{\sigma}}))\, L(\psi, s)$$

where d_K denotes the discriminant of K and $\mathfrak{f}_\psi$ the conductor of ψ. $\Lambda(\psi, s)$ then satisfies the functional equation

$$\Lambda(\psi, s) = i^n \tau(\psi) N(\mathfrak{f}_\psi)^{-1/2} \Lambda(\overline{\psi}, 1 + w - s)$$

where $\tau(\psi)$ is the Gauss sum of ψ, as defined in [Miy89, p. 92], and where

$$n := \sum_{\sigma \in \Sigma_K^{\mathrm{r}}} a_\sigma + \sum_{\sigma \in \Sigma_K^{\mathrm{c}}} |a_\sigma - a_{\overline{\sigma}}|$$

These L-functions have been well-studied by Shimura [Shi75] amongst others, and it's well-known that their L-values at critical values can be expressed in terms of periods of abelian

[5] We remark that taking the modular form θ_ψ attached to this Grössencharacter (as described in a later section) matches the L-series of the congruent number curve [Wat11].

varieties of CM-type.

It's worth specialising the above formulae to the case where K is imaginary quadratic. Indeed, let ψ be a Grössencharacter over K of infinity type (a,b) and conductor $\mathfrak{f}_\psi$. Then the completed L-function is therefore

$$\Lambda(\psi,s) = (|d_K|\,\mathfrak{f}_\psi^2)^{s/2}\Gamma_{\mathbb{C}}(s-\min(a,b))\,L(\psi,s)$$

which satisfies the L-function

$$\Lambda(\psi,s) = i^{|a-b|}\tau(\psi)N(\mathfrak{f}_\psi)^{-1/2}\Lambda(\overline{\psi},1+a+b-s)$$

Without spending too much time analysing these L-functions, we shall simply summarise the known results about these critical values. For a given imaginary quadratic field K, we first define the **period of** K, denoted Ω_K, for which any CM elliptic curve over K has period an algebraic multiple of $\pi\Omega_K^2$.

Definition/Theorem 8: [CS49, p. 373] Let K be an imaginary quadratic field. We define the **period of** K as the following real number:

$$\Omega_K := \left(\prod_{a=1}^{|d|-1}\Gamma\Big(\frac{a}{d}\Big)^{(\frac{d}{a})}\right)^{w/4h}$$

where d is the discriminant of K, w is the order of the group of units, h is the class number, and where the exponents of the Gamma factors are the Kronecker symbol $(\frac{d}{a})$ (i.e. the Dirichlet character mod $|d|$ associated to K).

Then any CM elliptic curve E/K has global period Ω_E being an algebraic multiple of $\pi\Omega_K^2$ (with the L-value $L(E/K,1)$ likewise being a algebraic multiple of its period). Furthermore, if K has class number 1, then in fact there exist elliptic curves E over $\mathbb{Q}$ with CM by K, such that both their real period Ω_E^+ and imaginary period Ω_E^- are algebraic multiples of Ω_K.

We also mention that the values of $L(\psi,1)$ are similarly well-known (up to algebraic multiples):

Theorem 9: [Shi75] Let K be an imaginary quadratic field, and let ψ be a Grössencharacter over K of infinity type (a,b) in the outer critical regions. Then the transcendental part of the L-value $L(\psi,1)$ can be given as:

$$L(\psi,1)\in\overline{\mathbb{Q}}\cdot\frac{\Omega_E^{|a-b|/2}}{\pi^{\max(a,b)-1}} = \overline{\mathbb{Q}}\cdot\frac{\Omega_K^{|a-b|}}{\pi^{(a+b)/2-1}}$$

where Ω_E denotes the global period of any CM-elliptic curve E over K.

With this in mind, we therefore define the **period of** ψ, denoted $p(\psi)$, as the value $p(\psi) := \Omega_K^{|a-b|}\pi^{1-(a+b)/2}$, noting that $L(\psi,1)$ is thus an algebraic multiple of $p(\psi)$. It's also worth noting that $p(\psi_1\psi_2) = p(\psi_1)p(\psi_2)$ if both ψ_1 and ψ_2 lie in the same (outer) critical region.

This allows us to not only very explicitly write down the periods Ω_E of CM elliptic curves and their leading L-function coefficients $L(E/K,1)$, but also the values of $L(\psi,1)$. To illustrate this, we shall go through a few examples:

Examples:

- Let $K = \mathbb{Q}(\sqrt{-7})$. We therefore calculate the period of K Ω_K explicitly as:

$$\Omega_K = \sqrt{\frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{\Gamma(\frac{3}{7})\Gamma(\frac{5}{7})\Gamma(\frac{6}{7})}} = 3.31921570\ldots$$

  We now fix the elliptic curve E_1 being the 16.1-CMa1 curve over K. To be fully explicit, we can express the period of E_1 as:

$$\Omega_{E_1} = \frac{1}{2\sqrt{7}}\pi\Omega_K^2 = \frac{\pi\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{2\sqrt{7}\Gamma(\frac{3}{7})\Gamma(\frac{5}{7})\Gamma(\frac{6}{7})} = 6.54096476\ldots$$

  and furthermore the L-value $L(E_1/K, 1)$ can be checked as

$$L(E_1/K, 1) = \frac{\Omega_{E_1}}{8\sqrt{7}} = 0.30903153751765917103\ldots$$

- Let E_2 be the 64.1-CMa1 curve over K. Then the period of E_2 is

$$\Omega_{E_2} = \frac{1}{2\sqrt{14}}\pi\Omega_K^2 = 4.62516054\ldots$$

  and furthermore the L-value $L(E_2/K, 1)$ is

$$L(E_2/K, 1) = \frac{\Omega_{E_2}}{2\sqrt{7}} = 0.87407318\ldots$$

- Let $\psi_{1,-3}$ be the canonical Grössencharacter of infinity type $(1, -3)$ over K. Then

$$L(\psi_{1,-3}, 1) = \frac{1}{1176}\pi^2\Omega_K^4 = 1.01867190\ldots$$

  Similarly, let $\psi_{5,-1}$ be the canonical Grössencharacter of infinity type $(5, -1)$ over K. Then

$$L(\psi_{5,-1}, 1) = \frac{1}{70\sqrt{7}}\frac{\Omega_K^6}{\pi} = 2.29834872\ldots$$

Whilst the above results regarding L-values of Grössencharacters have been well-studied, the main focus of this project is studying the L-values of elliptic curves E twisted by Grössencharacters, which shall be the focus of the next section.

Elliptic curves twisted by Grössencharacters

In an entirely analogous manner to constructing the L-function of elliptic curves over $\mathbb{Q}$ twisted by Dirichlet character, we can also construct the twisted L-function of E/K by some Grössencharacter ψ:

$$L(E/K, \psi, s) = \sum_{\mathfrak{n} \lhd \mathcal{O}_K} a_{\mathfrak{n}}(E)\psi(\mathfrak{n})N(\mathfrak{n})^{-s}$$

By complete multiplicativity of ψ, we also have that $L(E/K, \psi, s)$ can be given by the Euler product:

$$L(E/K, \psi, s) = \prod_{\mathfrak{p} \lhd \mathcal{O}_K} L_{\mathfrak{p}}(\psi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1}$$

To simplify matters, we shall give the functional equation specialised to the case where K is an imaginary quadratic field. Again, we remark that this is still conjectural in the general case, and is only known for modular elliptic curves.

Conjecture 10: Let E be an elliptic curve over some imaginary quadratic field K, and let ψ be a primitive Grössencharacter of infinity type (a,b) over K, with conductor $\mathfrak{f}_\psi$ coprime to the conductor $N_{E/K}$ of E. Then the completed L-function is given as

$$\Lambda(E/K,\psi,s) = \begin{cases} \mathfrak{f}_\psi^s (N_{E/K})^{s/2}\Gamma_{\mathbb{C}}(s-\min(a,b))^2\, L(E/K,\psi,s) & \text{if } a=b \\ \mathfrak{f}_\psi^s (N_{E/K})^{s/2}\Gamma_{\mathbb{C}}(s-\min(a,b))\Gamma_{\mathbb{C}}(s-\min(a,b)-1)\, L(E/K,\psi,s) & \text{if } a\neq b \end{cases}$$

where we note the exact form depends on whether a equals b. This then satisfies the functional equation

$$\Lambda(E/K,\psi,s) = \omega_{E/K,\psi}\,\Lambda(E/K,\overline{\psi},2+a+b-s)$$

where $\omega_{E/K,\psi}$ is given by

$$\omega_{E/K,\psi} := \omega_{E/K}\psi(N_{E/K})\tau(\psi)^2\mathfrak{f}_\psi^{-1} \tag{8}$$

where as before $\omega_{E/K}$ is the global root number for E/K and where $\tau(\psi)$ is the Gauss sum of ψ.

Given an elliptic curve E/K, and Grössencharacter ψ, we say that a point $z\in\mathbb{Z}$ is a **critical point** (in the Deligne sense) if none of the Gamma factors on either side of the functional equation for $L(E/K,\psi,s)$ have a pole at z. [BDP13, p. 1090].

From the functional equation for $\Lambda(E/K,\psi,s)$, we therefore note that $L(E/K,\psi,1)$ is a critical value if one of the following three conditions holds:

- $a=b=0$,
- $a\geq 1$ and $b\leq -1$,
- $a\leq -1$ and $b\geq 1$.

We shall informally refer to the first condition as the *inner* critical region, and the latter two conditions as the *outer* critical region. These are depicted in Figure 1.

We note that the earlier remarks regarding the difficulties behind calculating L-values of elliptic curves $E/\mathbb{Q}$ twisted by Dirichlet characters hint that stating a precise conjecture for the value of $L(E/K,\psi,1)$ is out of reach. Therefore, we'll have to resort to results whereby the values $L(E/K,\psi,1)$ are known only up to algebraic multiples.

In the case that ψ has trivial infinity type , then its known that $L(E/K,\psi,1)$ is simply some algebraic multiple of the period Ω_E. As an example, if $K=\mathbb{Q}(\sqrt{-7})$ and E denotes the 16.1-CMa curve and ψ is the order 2 Hecke character modulo (3), then $L(E/K,\psi,1)=\frac{16}{3}L(E/K,1)$.

Not as much is known in the outer critical region, where $(a,b)\neq(0,0)$, which shall be the main focus of this project.

Modular forms associated to Grössencharacters

One of the many useful procedures one can apply to Grössencharacters is to construct a modular form θ_ψ. This allows us study Grössenchareacters using the extensive machinery that modular forms provide.

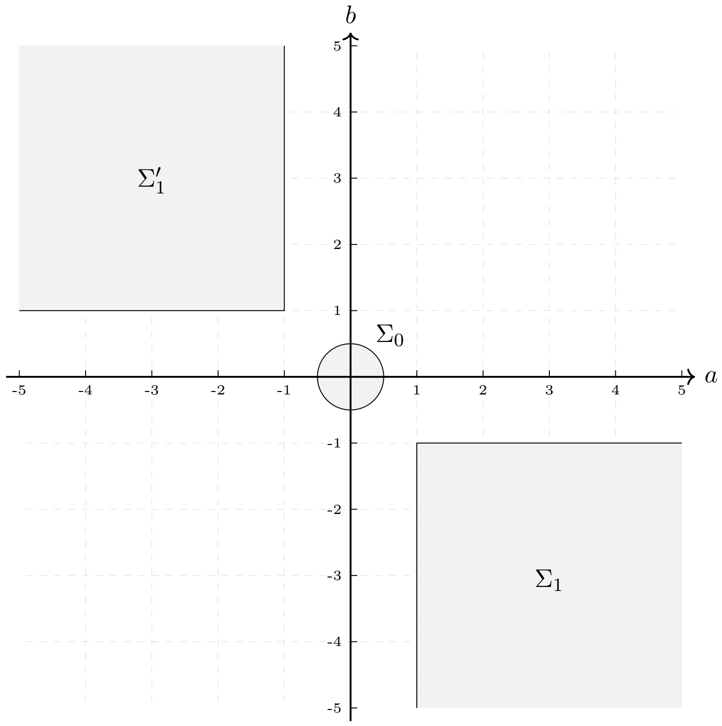

Figure 1: Regions where $L(E/K, \psi, s)$ is critical at $s = 1$ for ψ of infinity type (a, b). Region Σ_0 denotes the inner critical region, and regions Σ_1 and Σ_1' denote the outer critical region.

We first give a very brief reminder of modular forms as well as get some of the standard notation underway.

Firstly, we recall that the group $\mathrm{SL}_2(\mathbb{Z})$ of 2×2 matrices over $\mathbb{Z}$ with determinant 1 can be generated by the two matrices $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ [DS05, p. 2]. We also define Γ_∞ as the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\{\pm T\}$, i.e.

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \;\middle|\; m \in \mathbb{Z} \right\}$$

Definition 11: [DS05, p. 17] Let $\mathcal{H}$ denote the upper-half plane, and let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We say that a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a **modular form** of weight k on Γ if the following conditions hold:

- f is invariant under the slash operator for every $\gamma \in \Gamma$, i.e. we have
$$(f|_k\gamma)(z) = f(z)$$
for all $\gamma \in \Gamma$, where $(f|_k\gamma)(z) := j(\gamma, z)^{-k} f(\gamma z)$ and where $j(\gamma, z) = (cz+d)$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- f is holomorphic at all cusps. This is equivalent to the condition that $(f|_k\gamma)(z)$ is bounded as $\mathrm{Im}(z) \to \infty$, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

We shall also make frequent of the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ for some positive integer $N \geq 1$ ($\Gamma_0(N)$ is sometimes also called the *Hecke congruence subgroup* of level N). These are defined to be [DS05, p. 13] :

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0, \text{ and } a, d \equiv 1 \pmod{N} \right\}$$

We denote $\mathcal{M}_k(\Gamma)$ as the space of weight k modular forms for Γ, and similarly denote $\mathcal{S}_k(\Gamma)$ as the space of weight k cusp forms for Γ. For brevity, when $\Gamma = \Gamma_1(N)$, then we'll denote the space of modular (resp. cusp) forms of weight k for Γ simply as $\mathcal{M}_k(N)$ (resp. $\mathcal{S}_k(N)$).

Before defining the modular form θ_ψ attached to a Grössencharacter ψ, we must also introduce the notion of modular forms with character:

Definition 12: Let N be a positive integer, k an integer, and χ a Dirichlet character. Then we say that a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is **modular form of level N, weight k and character** χ, if the following conditions hold:

1. $f(\gamma z) = \chi(d)(cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.
2. f is holomorphic at all cusps.

We denote the space of all modular forms of level N, weight k and character χ as $\mathcal{M}_k(N, \chi)$, and similarly denote the subspace of cusp forms as $\mathcal{S}_k(N, \chi)$. It's worth mentioning that the spaces $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ decompose as

$$\mathcal{M}_k(N) = \sum_{\chi} \mathcal{M}_k(N, \chi), \quad \text{and} \quad \mathcal{S}_k(N) = \sum_{\chi} \mathcal{S}_k(N, \chi).$$

We now consider the construction of a modular form from a Grössencharacter ψ. It is possible to do this for both real and imaginary quadratic fields, however for our purposes, we shall simply state the imaginary quadratic case.

Definition/Theorem 13: [Iwa97, p. 213] Let K be an imaginary quadratic field, with discriminant $D < 0$. Let ψ be a Grössencharacter of K modulo $\mathfrak{m}$ of infinity type (a, b). We define the theta series θ_ψ associated to ψ as

$$\theta_\psi(z) := \sum_{\mathfrak{n} \lhd \mathcal{O}_K} \psi(\mathfrak{n}) N(\mathfrak{n})^{|a-b|/2} q^{N(\mathfrak{n})}, \quad q = e^{2\pi i z}$$

where as before $N(\mathfrak{n})$ denotes the norm of $\mathfrak{n}$. Then θ_ψ is a modular form of weight $|a-b|+1$ and of level $|D| \cdot N(\mathfrak{f}_\psi)$ with character χ, where χ is the Dirichlet character (modulo $|D| \cdot N(\mathfrak{f}_\psi)$) given by

$$\chi(n) = \left(\frac{D}{n} \right) \psi((n)). \quad \text{for } n \in \mathbb{Z}$$

where $\left(\frac{D}{n}\right)$ denotes the Kronecker symbol.

Moreover, we have that θ_ψ is primitive if ψ is primitive, and furthermore that θ_ψ is a cusp form if $|a-b| > 0$.

We shall not go into detail regarding a proof of the above, however these statements with proofs can be found in both Miyake [Miy89, p. 183] and Iwaniec [Iwa97, p. 213]. In summary, one applies the functional equation for ψ to show that θ_ψ satisfies a set of sufficient conditions given by Weil [Wei67] to prove that θ_ψ is a modular form.

By construction of θ_ψ, we also note that the L-function of θ_ψ is simply the L-function of ψ shifted by $\frac{|a-b|}{2}$, i.e. $L(\theta_\psi, s) = L(\psi, s - \frac{|a-b|}{2})$.

Computational results

To practically compute values of L-functions of elliptic curves, we use Sage's implementation of Dokchitser's L-function calculator [Dok04]. This can handle L-functions of any motivic origin and can compute values to any arbitrary precision. The calculator works by essentially doing computations with the inverse Mellin transform of $\Lambda(E/K, \psi, s)$, which allows for much faster convergence compared to constructing the Euler product naively.[6]

In practice, usage of the calculator only requires knowledge of the conductor N, the gamma factors in the completed L-function, the weight w, and the Fourier coefficients a_n for sufficiently many positive integers n (depending on the precision required).

We note that the sign $\omega_{E/K,\psi}$ in the functional equation for $\Lambda(E/K, \psi, s)$ is straightforward to compute assuming the conductor $N_{E/K}$ of E is coprime to the conductor $\mathfrak{f}_\psi$ of ψ, as shown in (8). However, in the case where $N_{E/K}$ and $\mathfrak{f}_\psi$ are not coprime, it's not always obvious what $\omega_{E/K,\psi}$ should be.

Nevertheless, as the functional equation is linear in $\omega_{E/K,\psi}$, then assuming all remaining parameters are correct, we can easily just solve for the sign using the functional equation, as done per [Dok04, p. 16]. As a remaining check, we can ensure that $|\omega_{E/K,\psi}| = 1$ and $\omega_{E/K,\psi} \in \overline{\mathbb{Q}}$.

Computing sufficiently many coefficients $a_\mathfrak{n}(E)$ and $\psi(\mathfrak{n})$ is straightforward. We do note that it's not always clear what the conductor/level N is again when $N_{E/K}$ and $\mathfrak{f}_\psi$ are not coprime. Although this isn't a practical issue, since we can always just try various guesses for N, and then simply check that the functional equations holds for various small values of s.

The only remaining computation is to calculate the gamma factors as per Dokchitser's [Dok04, p. 3] format. By simply applying Legendre's duplication formula, we note that the definition for the completed L-function $\Lambda(E/K, \psi, s)$ (in the case where $\mathfrak{f}_\psi$ and $N_{E/K}$ are coprime) can be given as

$$\Lambda(E/K, \psi, s) := \mathfrak{f}_\psi^s (N_{E/K})^{s/2} \pi^{-2s} \gamma(s) L(E/K, \psi, s)$$

where $\gamma(s) = \Gamma(\frac{s+\lambda_1}{2})\Gamma(\frac{s+\lambda_2}{2})\Gamma(\frac{s+\lambda_3}{2})\Gamma(\frac{s+\lambda_4}{2})$, and where the gamma parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ depend only on $\min(a, b)$ as:

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} (-a, -a, 1-a, 1-a) & \text{if } a = b \\ (-a, -a, -1-a, 1-a) & \text{if } a < b \\ (-b, -b, -1-b, 1-b) & \text{if } a > b \end{cases}$$

With the above in mind, we are now ready to evaluate $L(E/K, \psi, 1)$ for various different elliptic curves E and Grössencharacters ψ.

We let K be the fixed imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$, and let E_0 be the 16.1-CMa1 curve over K. We denote the period of E_0 as $\Omega_E = 6.54096\ldots$.

For each infinity type (a, b), we recall that $\psi_{a,b}$ denotes the canonical Grössencharacter of infinity type (a, b), i.e. for all $(\lambda) \lhd \mathcal{O}_K$ we define

$$\psi_{a,b}((\lambda)) := \begin{cases} \lambda^a \overline{\lambda}^b & \text{if } a+b \text{ even} \\ (\frac{\lambda}{\mathfrak{p}_7}) \lambda^a \overline{\lambda}^b & \text{if } a+b \text{ odd} \end{cases}$$

[6]We note that even in regions of absolute convergence, manually constructing the Euler product converges far too slow to be of any use in practice.

where $\mathfrak{p}_7$ is the unique prime ideal in K lying above 7. Note that this gives a well-defined conductor 1 Grössencharacter if $a+b$ is even, otherwise it has conductor $\mathfrak{p}_7$ if $a+b$ is odd.

We shall first give some values of $L(E_0/K, \psi, 1)$ for canonical Grössencharacters of various different infinity types. All calculations were done to 1000 bits of precision (about 300 decimal places).

Table 1: Table of L-values $L(E_0/K, \psi_{a,b}, 1)$ for various infinity types (a,b) in the Σ_1 outer critical region.[7] Plot is given in the usual Cartesian convention (i.e. the columns denote a in increasing order, and the rows denote b in decreasing order). For brevity, we denote $\alpha = \frac{1+\sqrt{-7}}{2}$.

	0	**1**	**2**	**3**	**4**
0	$\frac{\Omega_E}{8\sqrt{7}}$	0	0	0	0
-1	.	$\frac{i\alpha}{7\overline{\alpha}^3} \cdot \frac{\Omega_E^2}{\pi}$	$\frac{i\alpha^5\overline{\alpha}^2}{7} \cdot \frac{\Omega_E^3}{\pi^3}$	$\frac{i\alpha}{\overline{\alpha}} \cdot \frac{\Omega_E^4}{\pi^5}$	$\frac{i3(9\sqrt{-7}+106)\alpha^5\overline{\alpha}^2}{1} \cdot \frac{\Omega_E^5}{\pi^7}$
-2	.	$\frac{\alpha^7}{7^3\overline{\alpha}^2} \cdot \frac{\Omega_E^3}{\pi}$	$\frac{i\alpha^4}{7^2\overline{\alpha}^4} \cdot \frac{\Omega_E^4}{\pi^3}$	$\frac{i(2-\sqrt{-7})\overline{\alpha}^7}{7^2\alpha} \cdot \frac{\Omega_E^5}{\pi^5}$	$\frac{i(3\sqrt{-7}-2)\alpha^4}{7\overline{\alpha}^4} \cdot \frac{\Omega_E^6}{\pi^7}$
-3	.	$\frac{i\alpha^5}{3 \cdot 7^2\overline{\alpha}^9} \cdot \frac{\Omega_E^4}{\pi}$	$\frac{i(2-\sqrt{-7})\alpha^8}{3 \cdot 7^4\overline{\alpha}^4} \cdot \frac{\Omega_E^5}{\pi^3}$	$\frac{3\alpha^5}{7^3\overline{\alpha}^7} \cdot \frac{\Omega_E^6}{\pi^5}$	$\frac{3(18\sqrt{-7}-19)\alpha^8}{7^3\overline{\alpha}^4} \cdot \frac{\Omega_E^7}{\pi^7}$
-4	.	$\frac{i(-9\sqrt{-7}+106)\alpha^8}{3 \cdot 7^6\overline{\alpha}^9} \cdot \frac{\Omega_E^5}{\pi}$	$\frac{i(-2-3\sqrt{-7})\alpha^5}{3^2 7^3\overline{\alpha}^{11}} \cdot \frac{\Omega_E^6}{\pi^3}$	$\frac{i(18\sqrt{-7}+19)\alpha^8}{3 \cdot 7^5\overline{\alpha}^8} \cdot \frac{\Omega_E^7}{\pi^5}$	$\frac{i11\alpha^5}{7^4\overline{\alpha}^{11}} \cdot \frac{\Omega_E^8}{\pi^7}$

To further investigate the ratios of L-values with the same infinity type, we compute several of the L-values for the infinity types $(1,-1)$ and $(1,-3)$, compared across three different elliptic curves 16.1-CMa, 44.3a, 46.2a, (the latter two of which are not $\mathbb{Q}$-curves), and three different Grössencharacters, with finite order part having order 1, 2 and 4 respectively.

For notation, we denote χ_1 as the unique order 2 Grössencharacter mod (3), and denote χ_2 as the unique order 4 Grössencharacter mod (3) such that $\chi_2((\alpha)) = i$. We note that χ_1 does factor through the norm, however χ_2 doesn't.

Table 2: List of a few elliptic curves E over $K = \mathbb{Q}(\sqrt{-7})$. As before, $\alpha = \frac{1+\sqrt{-7}}{2}$. The ratio of L-values $\frac{L(E/K,\, \chi_i\psi_{1,-1},\, 1)}{L(E_0/K,\, \psi_{1,-1},\, 1)}$ is tabulated for E_0 the isogeny class 16.1-CMa curve and $\psi_{1,-1}$ the canonical Grössencharacter of infinity type $(1,-1)$.

Elliptic curves		**Grössencharacters** (value of ratios)		
Isog Class	**Globally minimal model**	$\psi_{1,-1}$	$\chi_1\,\psi_{1,-1}$	$\chi_2\,\psi_{1,-1}$
16.1-CMa	$y^2 + \alpha xy = x^3 + (-\alpha - 1)\,x^2 + x$	1	$\frac{8}{9}$	$\frac{4}{9}(1+\sqrt{7})\sqrt{3}$
44.3-a	$y^2 + xy = x^3 + \alpha x^2 + x + 1$	$\frac{\overline{\alpha}^4}{(2+\sqrt{-7})}$	$\frac{-\sqrt{-7}\alpha^3\overline{\alpha}^7}{(2+\sqrt{-7})3^2}$	$\frac{i\sqrt{\alpha^3\overline{\alpha}^{11}}}{3\sqrt{3}(2+\sqrt{-7})}$
46.2-a	$y^2 + xy + \alpha y = x^3 - \alpha x^2 - \alpha x$	$\frac{-\alpha\overline{\alpha}^4}{(4-\sqrt{-7})}$	$\frac{-\alpha^3\overline{\alpha}^6}{3^2(4-\sqrt{-7})}$	$\in \mathbb{Q}(i, \sqrt{7}, \sqrt{3})$

[7]These values might possibly be off by a sign and/or conjugation, however I've doubled checked that at least the transcendental parts are correct.

Table 3: List of a few elliptic curves E over $K = \mathbb{Q}(\sqrt{-7})$. As before, $\alpha = \frac{1+\sqrt{-7}}{2}$. The ratio of L-values $\frac{L(E/K,\, \chi_i \psi_{1,-3},\, 1)}{L(E_0/K,\, \psi_{1,-3},\, 1)}$ is tabulated for E_0 the isogeny class 16.1-CMa curve and $\psi_{1,-3}$ the canonical Grössencharacter of infinity type $(1, -3)$.

Elliptic curves		**Grössencharacters** (value of ratios)		
Isog Class	**Globally minimal model**	$\psi_{1,-3}$	$\chi_1\,\psi_{1,-3}$	$\chi_2\,\psi_{1,-3}$
16.1-CMa	$y^2 + \alpha xy = x^3 + (-\alpha - 1)\,x^2 + x$	1	$\frac{\alpha^2\overline{\alpha}^3(6\sqrt{7}-5)}{3^4}$	$\in \mathbb{Q}(i, \sqrt{7}, \sqrt{3})$[8]
44.3-a	$y^2 + xy = x^3 + \alpha x^2 + x + 1$	$-\frac{\overline{\alpha}^{13}}{\alpha^3(2+\sqrt{-7})^3}$	$-\frac{5\alpha^4\overline{\alpha}^{14}}{3^4(2+\sqrt{-7})^3}$	$\in \mathbb{Q}(i, \sqrt{7}, \sqrt{3})$
46.2-a	$y^2 + xy + \alpha y = x^3 - \alpha x^2 - \alpha x$	$\frac{\alpha\overline{\alpha}^{13}}{(4-\sqrt{-7})^3}$	$\frac{\sqrt{-7}(-4-7\sqrt{-7})\alpha^2\overline{\alpha}^{13}}{3^4(4-\sqrt{-7})^3}$	?

Based on the above computational evidence, we can thus make the following conjecture:

Conjecture 14: Let K be an imaginary quadratic field. Then for any elliptic curve E over K, and for any Grössencharacter ψ of infinity type (a, b) in the outer critical regions, we have

$$L(E/K, \psi, 1) \in \overline{\mathbb{Q}} \cdot \frac{\Omega_{E_0}^{|a-b|}}{\pi^{2\max(a,b)-1}} = \overline{\mathbb{Q}} \cdot \frac{\Omega_K^{2|a-b|}}{\pi^{a+b-1}}$$

where E_0/K can be any elliptic curve with CM by K.

This result has already been proven in the case where E/K is a base change curve from $\mathbb{Q}$, as discussed in a later section. However, we note that our conjecture states more generally that this occurs for any elliptic curve E/K.

Indeed, we can now show that our computational results shown above are consistent with a famous conjecture first stated by Deligne [Del79].

Deligne's period conjecture

Background

Before stating Deligne's period conjecture, we shall first go over the basics of motives, specifically over imaginary quadratic fields K. We won't go into too many specifics, but rather simply consider motives in the rather naive sense, as a black box containing a list of specified data. The following survey articles by Mazur [Maz04] and Serre [Ser91] give a nice non-technical overview of the basics of motives.

Throughout this section, we also use the notation $x \sim_L y$ to mean that $x = \ell y$ for some $\ell \in L$, and will simply use $x \sim y$ to mean that x is some algebraic multiple of y. Moreover, we will use the tensor symbol $\otimes$ without subscript to mean tensoring over $\mathbb{Q}$.

Let K be an imaginary quadratic field. A **motive** M over K with coefficients in some number field E, of dimension/rank n and weight w gives rise to the following objects:

[8] At this point, it seems to become a bit harder to give a clean short radical expression for these L-values. The minimal polynomial for this ratio is $1853020188851841x^8 - 5751771653614392x^6 + 7190515169067024x^4 - 3821319219916800x^2 + 1116308285440000$, which one can at least verify has splitting field $\mathbb{Q}(i, \sqrt{7}, \sqrt{3})$.

- *Betti realisation:* An n-dimensional E-vector space $H_B(M)$.

- *de Rham realisation:* An n-dimensional E-vector space $H_{dR}(M)$.

- *Hodge decomposition:* The complexification of $H_B(M)$ decomposes into free $E \otimes \mathbb{C}$ modules as:

  $$H_B(M) \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}(M)$$

  We shall assume that our motives M are **pure of weight** w which means that we have $H^{p,q}(M) = 0$ if $p + q \neq w$.

  We can therefore denote the **Hodge type** of M as $(p_1, q_1), \ldots, (p_n, q_n)$ assuming without loss of generality that $p_1 > \cdots > p_n$, where $p_i + q_i = w$. $H_B(M)$ then decomposes as $\oplus_{i=1}^n H^{p_i,q_i}(M)$.

- *Infinite Frobenius:* An E-linear isomorphism $F_\infty : H_B(M) \longrightarrow H_B(M^c)$, where M^c denotes the conjugated motive of M. We shall also denote the Hodge type of M^c as $(p_1^c, q_1^c), \ldots, (p_n^c, q_n^c)$, where we define $p_i^c = q_{n+1-i}$, which implies the Hodge numbers p_i^c are also in decreasing order.

  We thus have that F_∞ induces E-linear isomorphisms between the subspaces H^{p_i,q_i} of M and M^c as

  $$F_\infty : H^{p_i,q_i}(M) \xrightarrow{\sim} H^{p^c_{n+1-1},q^c_{n+1-i}}(M^c)$$

  for all $i = 1, \ldots, n$.

- *Comparison isomorphism:* An isomorphism between the complexifications of $H_B(M)$ and $H_{dR}(M)$:

  $$I_\infty : H_B(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{dR}(M) \otimes \mathbb{C}$$

  For some fixed E-basis of $H_B(M)$ and $H_{dR}(M)$, we can extend them to $E \otimes \mathbb{C}$ bases of $H_B(M) \otimes \mathbb{C}$ and $H_{dR}(M) \otimes \mathbb{C}$. We define the **determinant period** $\delta(M)$ to be the determinant of I_∞ with respect to these bases (this is well-defined up to multiplication by $E^\times$).

- *Filtration on H_{dR}:* There's a decreasing filtration $\{F^k H_{dR}(M) : k \in \mathbb{Z}\}$ on $H_{dR}(M)$ such that

  $$F^k H_{dR}(M) \otimes \mathbb{C} = I_\infty\Bigg(\bigoplus_{p_i \geq k} H^{p_i,q_i}(M)\Bigg)$$

  which implies that we have the filtration

  $$H_{dR}(M) = F^{p_n} H_{dR}(M) \supsetneq F^{p_{n-1}} H_{dR}(M) \supsetneq \cdots \supsetneq F^{p_1} H_{dR}(M) \supsetneq F^{p_1+1} H_{dR}(M) = 0$$

- *Motivic periods:* For each $i = 1, \ldots, n$, we define ω_i as a non-zero element in $H^{p_i,q_i}(M)$ such that $I_\infty(\omega_i)$ is equivalent to an element in $F^{p_i} H_{dR}(M)$ (mod $F^{p_{i-1}} H_{dR}(M) \otimes \mathbb{C}$), noting that this is well-defined by the above defined filtraton on H_{dR}.

  Thus we have that $\{\omega_1, \ldots, \omega_n\}$ is an $E \otimes \mathbb{C}$-basis for $H_B(M) \otimes \mathbb{C}$, and likewise that $\{I_\infty(\omega_1), \ldots, I_\infty(\omega_k)\}$ is an $E \otimes \mathbb{C}$ basis for $H_{dR}(M) \otimes \mathbb{C}$.

  We similarly define ω_i^c as non-zero elements in $H^{p_i^c,q_i^c}(M^c)$ in an analogous manner.

As $H^{p_i^c, q_i^c}(M^c)$ is a free rank 1 $E \otimes \mathbb{C}$-module, we thus define the numbers $Q_1(M), \dots, Q_n(M) \in E \otimes \mathbb{C}$ as the **motivic periods** of M, where $F_\infty \omega_i = Q_i(M)\omega_{n+1-i}^c$ for all $i = 1, \dots, n$ (again, these are well-defined up to $E^\times$).

One can also define L-functions of arbitrary motives, with details given in [Del79]. We shall not go through the exact details here, but simply give some examples relevant to our case:

- Let E be an elliptic curve over K. We can therefore consider the motive $M = h^1(E)$ corresponding to the first cohomology of E. M has Hodge type $(1, 0), (0, 1)$, and the L-function is simply the usual L-function, as defined in Definition 1: $L(M, s) = L(E/K, s)$.

- A relatively elementary example of a motive is the **Tate motive**[9] $\mathbb{Q}(n)$. A full summary of its properties is given in [Del79, p. 13], however we shall simply state that $\mathbb{Q}(n)$ has Hodge type $(-n, -n)$, with the determinant period $\delta(\mathbb{Q}(n))$ being $(2\pi i)^n$, with L-function $L(\mathbb{Q}(n), s) = \zeta(s + n)$.

- An often useful construction is to consider the Tate twist of a motive $M(n) = M \otimes \mathbb{Q}(n)$. For example, for any elliptic curve E/K, we can also consider the Tate twist of the motive $h^1(E)$, given by $M = h^1(E)(n)$. In a naive sense, this simply shifts all values such that the L-function is shifted by n, i.e. we have $L(M, s) = L(E/K, s + n)$, where M now has Hodge type $(1 - n, -n), (-n, 1 - n)$.

- Finally, for a given motive M over K, we also consider the **determinant motive** $\det(M)$, as defined in [Har13, p. 122]. This is a rank 1 motive over K of weight nw. For example, if $E = h^1(E)(1)$, then $\det(E)$ is simply the Tate motive $\mathbb{Q}(1)$.

In order to state Deligne's conjecture, we also introduce the notion of **Weil restriction**. Indeed, if M is a motive over an imaginary quadratic field K, then the Weil restriction of M, denoted $\mathrm{Res}_{K/\mathbb{Q}}(M)$, is a motive over $\mathbb{Q}$ with twice the dimension of M, and such that $\mathrm{Res}_{K/\mathbb{Q}}(M) = M \oplus M^c$. We also remark that, for any motive M over K, we have equality of the L-functions: $L(M, s) = L(\mathrm{Res}_{K/\mathbb{Q}}(M), s)$. For brevity, we will sometimes also denote the Weil restriction simply as $R(M)$ where it is clear from context.

With all of this in mind, we can now state Deligne's period conjecture.

Conjecture 15: [Del79] (*Deligne's period conjecture*) Let M be a motive of weight w with coefficients in E, with an embedding $\sigma : E \to \mathbb{C}$. Let $H_B(M)$ be the Betti realisation of M, and let $H_B(M)^+$ be the subspace of $H_B(M)$ fixed by F_∞. Assume that M is *critical* (i.e. that 0 is critical for M).

We also define $F^+(M)$ as the subspace $F^+(M) := F^{\lfloor w/2 \rfloor} H_{dR}(M)$, and then define $H_{dR}(M)^+$ as the quotient $H_{dR}(M)/F^+(M)$. We now have a period isomorphism $I^+ : H_B(M)^+ \xrightarrow{\sim} H_{dR}(M)^+$, induced by the comparison isomorphism:

$$I^+ : H_B(M)^+ \otimes \mathbb{C} \to H_B(M) \otimes \mathbb{C} \xrightarrow{I_\infty} H_{dR}(M) \otimes \mathbb{C} \to H_{dR}(M)^+ \otimes \mathbb{C}$$

where the first and last maps are the natural embedding and quotient maps respectively.

[9] Sometimes also denoted as $\mathbb{Z}(n)$.

Define the **period** of M as $c^+(M) = \det(I^+)$, with the determinant again calculated with respect to some E-rational basis of $H_B(M)^+$ and $H_{dR}(M)^+$ (this is well-defined upto multiplication by a number in $E^\times$).

Then if 0 is critical for M, then Deligne's period conjecture states that

$$L(M,0) \sim_E c^+(M).$$

We now proceed with showing that our results obtained computationally are compatible with Deligne's period conjecture.

First, we calculate the period of a tensor product of motives. This was computed generally for arbitrary motives in [HL16] and [Gue16], however we shall specialise to the case where our motives have rank 2 and 1 respectively. Our proof shall mostly follow the arguments given in [Lin15].

Lemma 16: Let K be an imaginary quadratic field, and let M be a motive of rank 2 over K with coefficients in some number field E. Similarly, let N be a motive of rank 1 over K with coefficients in E'. Let $w(M)$ and $w(N)$ be the weights of M and N respectively, and let $w := w(M) + w(N)$ be the weight of $M \otimes N$.

Let M have Hodge type $(p_1, q_1), (p_2, q_2)$, assuming wlog that $p_1 > p_2$, and let N have Hodge type (r, s). Define the set $A := \left\{i \in \{1, 2\} \mid p_i + r > \frac{\omega}{2}\right\}$.

The period $c^+(\mathrm{Res}_{K/\mathbb{Q}}(M \otimes N))$ of the restriction of $M \otimes N$ to $\mathbb{Q}$ can then be given as

$$c^+(\mathrm{Res}_{K/\mathbb{Q}}(M \otimes N)) \sim_{EE'K} \left(\prod_{i\in A} Q_i Q'\right)^{-1} \delta(M)\delta(N)^2$$

where $Q_1 := Q_1(M)$ and $Q_2 := Q_2(M)$ are the motivic periods of M, and $Q' := Q_1(N)$ the motivic period of N.

Proof: [Lin15, p. 61] We first note that the Betti and de Rham realizations of the tensor product $\mathrm{Res}_{K/\mathbb{Q}}(M \otimes N)$ factor as

$$H_B(\mathrm{Res}_{K/\mathbb{Q}}(M \otimes N)) = (H_B(M) \otimes H_B(N)) \oplus (H_B(M^c) \otimes H_B(N^c)) \tag{9}$$

and

$$H_{dR}(\mathrm{Res}_{K/\mathbb{Q}}(M \otimes N)) = (H_{dR}(M) \otimes H_{dR}(N)) \oplus (H_{dR}(M^c) \otimes H_{dR}(N^c)).$$

For the conjugate motives M^c and N^c, we note that the Hodge type of M^c is $((p_1^c, q_1^c), (p_2^c, q_2^c)) = ((q_2, p_2), (q_1, p_1))$, and similarly the Hodge type of N^c is (s, r).

In order to explicitly calculate periods, we must first fix bases for all the relevant vector spaces. Let $\{e_1, e_2\}$ be an E-basis of $H_B(M)$, and similarly let $\{e_1^c, e_2^c\}$ be an E-basis of $H_B(M^c)$ (where we define $e_i^c := F_\infty e_i$). Similarly, we define $\{f_1\}$ to be an E' basis of $H_B(N)$ (and $\{f_1^c\}$ a basis for $H_B(N^c)$, where $f_1^c = F_\infty f_1$).

As defined earlier, we also fix any non-zero $\omega_1 \in H^{p_1,q_1}(M)$ and $\omega_2 \in H^{p_2,q_2}(M)$ such that $\{\omega_1, \omega_2\}$ is a basis for $H_{dR}(M)$ (to simplify notation, we identity ω_i with $I_\infty(\omega_i)$). Again, we also define $\{\omega_1^c, \omega_2^c\}$ similarly. Also define $\{\mu_1\}$ and $\{\mu_1^c\}$ as a basis for $H_{dR}(N)$ and $H_{dR}(N^c)$

respectively.

We recall that the motivic periods Q_1, Q_2 and Q' are defined such that

$$F_\infty \omega_1 = Q_1 \omega_2^c, \quad F_\infty \omega_2 = Q_2 \omega_1^c, \quad \text{and} \quad F_\infty \mu_1 = Q' \mu_1^c \tag{10}$$

Now given that $H_B(R(M \otimes N))$ factors as in (9), this implies an EE'-basis for $H_B(R(M \otimes N))$ can be given as

$$\big\{(e_1 \otimes f_1 + e_1^c \otimes f_1^c),\ (e_2 \otimes f_1 + e_2^c \otimes f_1^c)\big\}.$$

Now let A be as defined above, and also similarly define $T := \{i \in \{1,2\} \mid p_i^c + s > \frac{w}{2}\}$. By definition of $H_{dR}(R(M \otimes N))^+$, an $E \otimes \mathbb{C}$ basis $\mathcal{B}$ for $H_{dR}(R(M \otimes N))^+ \otimes \mathbb{C}$ can be given as

$$\mathcal{B} := \big\{\omega_a \otimes \mu_1,\ \omega_t^c \otimes \mu_1^c \ (\mathrm{mod}\ F^+(R(M \otimes N))) \ \big|\ a \notin A, t \notin T\big\}$$

We note that in general this basis is not rational (i.e. not contained in $H_{dR}(R(M \otimes N))^+$), however noting our definition for the basis elements ω_i, one can transform $\mathcal{B}$ by a unipotent change-of-basis matrix to obtain a rational basis. As unipotent matrices have determinant one, this implies we can thus use $\mathcal{B}$ to calculate the period $c^+(R(M \otimes N))$.

Note that if $a \in A$, then $3 - a \in T$. This, along with (10), implies that we have

$$F_\infty(\omega_a \otimes \mu_1) = Q_a Q' \omega_{3-a}^c \mu_1^c \in F^+(R(M \otimes N)) \otimes \mathbb{C}$$

and similarly, since $F_\infty^2 = \mathrm{Id}$, we have

$$F_\infty(\omega_t^c \otimes \mu_1^c) = (Q_{3-a} Q')^{-1} \omega_{3-t}^c \mu_1^c \in F^+(R(M \otimes N)) \otimes \mathbb{C}$$

for all $t \notin T$. Now, by noting that the complexification passes through quotients, i.e.

$$\begin{aligned} H_{dR}(R(M \otimes N))^+ \otimes \mathbb{C} &= \big(H_{dR}(R(M \otimes N))/F^+(R(M \otimes N))\big) \otimes \mathbb{C} \\ &\cong \big(H_{dR}(R(M \otimes N)) \otimes \mathbb{C}\big) \,/\, \big(F^+(R(M \otimes N)) \otimes \mathbb{C}\big), \end{aligned}$$

we can therefore rewrite the basis $\mathcal{B}$ as

$$\mathcal{B} = \big\{(1 + F_\infty)\omega_a \otimes \mu_1,\ (1 + F_\infty)\omega_t^c \otimes \mu_1^c \ (\mathrm{mod}\ F^+(R(M \otimes N)) \otimes \mathbb{C}) \ \big|\ a \notin A, t \notin T\big\}.$$

We are now in position to finally start computing the matrix representing I_∞. As $\{e_1, e_2\}$ is an E-basis for $H_B(M)$, and $\{f_1\}$ an E' basis for $H_B(N)$, we have constants $A_{i,j}, B \in \mathbb{C}$ such that

$$\begin{aligned} &\omega_1 = A_{1,1} e_1 + A_{2,1} e_2, \quad \omega_2 = A_{1,2} e_1 + A_{2,2} e_2, \\ &\omega_1^c = A_{1,1}^c e_1^c + A_{2,1}^c e_2^c, \quad \omega_2^c = A_{1,2}^c e_1^c + A_{2,2}^c e_2^c, \\ &\mu_1 = B f_1, \quad \text{and} \quad \mu_1^c = B^c f_1^c \end{aligned}$$

Note that we therefore have, for any $a, t \in \{1, 2\}$:

$$\begin{aligned} (1 + F_\infty)\omega_a \mu_1 &= (1 + F_\infty)(A_{1,a} B\, e_1 \otimes f_1 + A_{2,a}\, B e_2 \otimes f_1) \\ &= A_{1,a} B(e_1 \otimes f_1 + e_1^c \otimes f_1^c) + A_{2,a} B(e_2 \otimes f_1 + e_2^c \otimes f_1^c), \quad \text{and} \end{aligned}$$

$$\begin{aligned} (1 + F_\infty)\omega_t^c \mu_1^c &= (1 + F_\infty)(A_{1,t}^c B^c\, e_1^c \otimes f_1^c + A_{2,1}^c B^c\, e_2^c \otimes f_1^c) \\ &= A_{1,t}^c B^c(e_1 \otimes f_1 + e_1^c \otimes f_1^c) + A_{2,t}^c B^c(e_2 \otimes f_1 + e_2^c \otimes f_1^c) \end{aligned}$$

Therefore, the Deligne period $c^+(R(M \otimes N))$ is, upto multiplication by EE', the determinant of the matrix:

$$\begin{pmatrix} A_{1,a} B & \cdots & A_{1,t}^c B^c \cdots \\ A_{2,a} B & \cdots & A_{2,t}^c B^c \cdots \end{pmatrix}$$

where a ranges over all $\{1,2\}$ not in A, and similarly where t ranges over all $\{1,2\}$ not in T. We remark that $|A|+|T|=2$, thus the above matrix will always be a square matrix.
We do have that

$$A^c_{i,j}B^c = (Q_{3-j})^{-1}(Q')^{-1}A_{i,3-j}B \tag{11}$$

Thus, the Deligne period is the determinant of

$$\begin{pmatrix} A_{1,a}B & \cdots & (Q_{3-t}Q')^{-1}A_{1,3-t}B\cdots \\ A_{2,a}B & \cdots & (Q_{3-t}Q')^{-1}A_{2,3-t}B\cdots \end{pmatrix}$$

Noting that $\det(A_{i,j}) = \delta(M)$ and $B = \delta(N)$, it's thus clear that the determinant can be given as

$$\left(\prod_{i\in A} Q_i Q'\right)^{-1} \delta(M)\delta(N)^2$$

recalling that $3-i \notin T$ if and only if $i \in A$. □

We now apply the above lemma to prove the main result for this section:

Theorem 17: Let K be an imaginary quadratic field, and E an elliptic curve over K, with ψ a Grössencharacter of infinity type (a,b) in the outer critical region. Then assuming Deligne's period conjecture, we have

$$L(E/K,\psi,1) \sim \pi p(\psi)^2$$

where $p(\psi)$ denotes the period of ψ.

Proof: We simply apply the result from Lemma 16. Let $M = h^1(E)(1)$ be the motive associated to E (tensored by the Tate motive $\mathbb{Q}(1)$), and let N be the motive associated to ψ. We note that the Hodge type of M is $(0,-1),(-1,0)$ and the Hodge type of N is $(-a,-b)$.

Note that the weight w of the tensor product $M\otimes N$ is $-1-a-b$, and thus the set A is simply

$$A = \left\{p \in \{0,-1\} \;\middle|\; p > \tfrac{a-b-1}{2}\right\}$$

As ψ is in the outer critical region, we have $a \neq b$. Let's first consider the case where $a > b$. It's therefore clear that $A = \emptyset$, and thus $c^+(R(M\otimes N)) \sim \delta(M)\delta(N)^2$.

We note from Blasius' [Bla86] results that $\delta(N) \sim_{E(\psi)} p(\psi)$, where $p(\psi)$ is the period of ψ, defined in the appendix of [HK91]. To calculate $\delta(M)$, we can simply take the determinant motive $\det(M)$ and note that this is the Tate motive $\mathbb{Q}(1)$. Therefore we have

$$\delta(M) \sim \delta(\det(M)) = \delta(\mathbb{Q}(1)) \sim \pi$$

Therefore, this implies that

$$c^+(R(M\otimes N)) \sim \pi p(\psi)^2$$

which proves the claim, assuming Deligne's period conjecture. In the case that $a < b$, we have $A = \{1,2\}$ and therefore

$$\begin{aligned} c^+(R(M\otimes N)) &\sim (Q_1(M)Q_2(M))^{-1}Q'(N)^{-2}\delta(M)\delta(N)^2 \\ &\sim \delta(M^c)\delta(N^c)^2 \end{aligned}$$

which follows directly from the proof of lemma (by keeping $A^c_{i,j}B^c$ instead of substituting with (11)). The result then follows analogously to the first case. □

Remark: For completeness, it's worth noting that in the case where $a = b = 0$, we note that $A = \{1\}$, and thus

$$\begin{aligned} c^+(R(M \otimes N)) &\sim Q_1(M)^{-1} Q'(N)^{-1} \delta(M) \delta(N)^2 \\ &\sim Q_1(M)^{-1} \delta(M) \cdot \delta(N) \delta(N^c) \end{aligned}$$

Therefore, the period $c^+(R(M \otimes N))$ contains the motivic period Q_1 of M in a rather essential way which cannot be eliminated . This is consistent with our computational results that $L(E/K, \psi, 1)$ is not an algebraic multiple of $\pi p(\psi)^2$ if the infinity type (a, b) of ψ is trivial (unless E is a curve with CM by K).

Finally, noting the definition of $p(\psi)$, this implies that Deligne's period conjecture implies our Conjecture 14. All that remains is to prove this unconditionally. Whilst we'll leave a full proof for future work, we can outline a possible approach in the next section.

Rankin-Selberg convolutions

In order to prove rationality statements in the case where E/K is an elliptic curve which has a base change from $\mathbb{Q}$, this has been dealt with using the theory of Rankin-Selberg convolutions, as shown in [Shi76]. At the core of this idea, is the fact that E arises from a modular form f_E (for $\mathrm{GL}_2(\mathbb{Q})$), as well as ψ arising from a modular form θ_ψ as defined in Definition 13. The Rankin-Selberg convolution therefore allows us to interpret the twisted L-function $L(E/K, \psi, s)$ as the integral over the product of the differential forms arising from both f_E and θ_ψ.

At the heart of this idea, lies the unfolding method, first introduced by Rankin [Ran39].

Setting the stage

Before going through the main proof of the unfolding method, we first need to get a few definitions and concepts under our belt. This includes defining nearly holomorphic modular forms as well as suitable differential operators and Eisenstein series. To begin, we first define the Petersson inner product:

Definition 18: [DS05, p. 182] Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let $f, g \in \mathcal{M}_k(N)$ such that at least one of f, g is a cusp form. We then define the **Petersson inner product** $\langle f, g \rangle$ by

$$\langle f, g \rangle := \frac{1}{m(\Phi)} \int_\Phi \overline{f(z)} g(z) y^{k-2} dx dy$$

where, as usual, x and y are the real and imaginary parts of z respectively. We let Φ denote the fundamental domain for Γ in $\mathcal{H}$, and $m(\Phi)$ is the measure of the fundamental domain Φ with respect to $y^{-2} dx dy$. Moreover, for any $f \in \mathcal{S}_k(N)$, we say the **Petersson norm** of f is $\langle f, f \rangle$.

In order to effectively apply the Rankin-Selberg method, we also need to deal with nonholomorphic functions with satisfy the modularity condition. We therefore define the notion of nearly holomorphic modular forms:

Definition 19: [Shi07, p. 57] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let t and k be non-negative integers. We say that a continuous function $f : \mathcal{H} \to \mathbb{C}$ is a **nearly holomorphic modular form** of weight k for Γ if the following two conditions hold:

1. f is invariant under the slash operator for every $\gamma \in \Gamma$, i.e. $(f|_k \gamma)(z) = f(z)$ for all $\gamma \in \Gamma$.

2. For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$(f|_k\gamma)(z) = \sum_{\nu=0}^{t} y^{-\nu} \sum_{n=0}^{\infty} c_{\nu,\gamma,n} e^{2\pi i n z/N_\gamma}$$

for some $t \geq 0$, some coefficients $c_{\nu,\gamma,n} \in \mathbb{C}$ and some positive integer $N_\gamma \in \mathbb{Z}_{>0}$.

We call t the degree of f, and denote the space of all nearly holomorphic modular forms of weight k on Γ and of degree t as $\mathcal{N}_k^t(\Gamma)$, and define $\mathcal{N}_k(\Gamma)$ as the space of all nearly holomorphic modular forms of weight k on Γ (of any degree).

As with modular forms with character χ, we can similarly define **nearly holomorphic modular forms with character χ of level N**, by replacing the first condition with the transformation property:

$$(f|_k\gamma)(z) = \chi(d)f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We denote the space of weight k nearly holomorphic modular forms with character χ of level N as $\mathcal{N}_k(N,\chi)$, and as usual, denote those with degree t as $\mathcal{N}_k^t(N,\chi)$.

We now define some differential operators which play well with nearly holomorphic modular forms:

Definition 20: [Shi07, p. 56] For a given integer k and differentiable function f on $\mathcal{H}$, we can define the differential operator δ_λ, as

$$\delta_\lambda f := \frac{1}{2\pi i}\Big(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\Big) f$$

Furthermore, we can also define the operator $\delta_\lambda^{(r)}$ as $\delta_\lambda^{(r)} := \delta_{\lambda+2r-2} \cdots \delta_{\lambda+2}\delta_\lambda$. For convention, we also define $\delta_\lambda^{(0)}$ as simply the identity operator.

One of the neat properties of these operator is that one can verify that, for any nearly holomorphic modular form $f \in \mathcal{N}_k(\Gamma)$, we have that $\delta_k(f|_k\gamma) = (\delta_k f)|_{k+2}$, hence δ_k "lifts" the weight of f by 2, with $\delta_k^{(r)}$ lifting the weight by $2r$.

Therefore, we have that $\delta_k^{(r)}$ sends elements of $\mathcal{N}_k^t(\Gamma)$ to $\mathcal{N}_{k+2r}^{t+r}(\Gamma)$ and in particular, sends elements of $\mathcal{M}_k(\Gamma)$ into $\mathcal{N}_{k+2p}^p(\Gamma)$ [Shi07, p. 58].

To obtain some explicit examples of nearly holomorphic modular forms, we can apply the above defined differential operators to the following Eisenstein series:

Definition 21: [Shi76, p. 787] Let N be a positive integer, and λ be a non-negative integer. Let ω be a Dirichlet character modulo N such that $\omega(-1) = (-1)^\lambda$. Then we define the **Eisenstein series** $E_{\lambda,N}(z,s,\omega)$ as a function where $z \in \mathcal{H}$ and $s \in \mathbb{C}$ defined by

$$E_{\lambda,N}(z,s,\omega) := \sum_{\gamma \in \mathcal{R}} \omega(d)(cz+d)^{-\lambda}|cz+d|^{-2s}, \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

and where $\mathcal{R}$ denotes a set of coset representatives for $\Gamma_\infty \backslash \Gamma_0(N)$.

Note that $E_{\lambda,N}(z,s,\omega)$ is absolutely convergent for $\mathrm{Re}(s) > 2 - \lambda$, and can be continued to a meromorphic function in s on the complex plane [Shi07, p. 64]. When $s = 0$, we shall simply omit the s for brevity, i.e. we set $E_{\lambda,N}(z,\omega) := E_{\lambda,N}(z,0,\omega)$.

We also note that the operator δ_λ acts on $E_{\lambda,N}$ by the following calculation:

$$\delta_\lambda E_{\lambda,N}(z,\omega) = -\frac{\lambda}{4\pi y} E_{\lambda+2,N}(z,-1,\omega)$$

Applying this r times, we obtain the following identity [Shi76, p. 789]:

$$E_{\lambda+2r,N}(z,-r,\omega) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+r)}(-4\pi y)^r \delta_\lambda^{(r)} E_{\lambda,N}(z,\omega) \tag{12}$$

For example, whilst $E_{2,N}(z,\omega)$ is not holomorphic, it is a nearly holomorphic modular form, with $E_{2,N}(z,\omega) \in \mathcal{N}_2^1(N,\omega)$. By therefore applying $\delta_k^{(r)}$, this gives rise to an important family of examples of nearly holomorphic modular forms, where we have that $\delta_2^{(r)} E_{2,N}(z,\omega) \in \mathcal{N}_{2+2r}^{1+r}(N,\omega)$.

Finally, before we can proceed with the unfolding method, we need to state two technical lemmas, both from Shimura [Shi76], which shall be used when proving rationality results later. We refer to these lemmas informally as the structure theorem for nearly holomorphic modular forms, and orthogonality of modular forms, respectively.

Lemma 22: [Shi76, p. 795] (Structure theorem of nearly holomorphic forms) Let $h \in \mathcal{N}_k^t(N,\chi)$ with $k > 2t$. Then h can be written uniquely in the form

$$h(z) = \sum_{\nu=0}^{t} \delta_{k-2\nu}^{(\nu)} g_\nu$$

where $g_\nu \in \mathcal{M}_{k-2\nu}(N,\chi)$.

Lemma 23: [Shi76, p. 794] (Orthogonality) Let $f \in \mathcal{S}_k(N,\chi)$ and $g \in \mathcal{M}_\ell(N,\overline{\chi})$, with $k = \ell + 2r$ for some positive integer r. Then we have $\langle f_\rho, \delta_\ell^{(r)} g\rangle = 0$.

We've finally completed setting the stage, and shall now use the above to prove the unfolding method. We'll follow Shimura [Shi76] throughout, although a nice exposition of the level 1 case can also be found in Alecu [Ale16].

The unfolding method

Theorem 24: [Shi76, p. 789]. Let N be a positive integer, and let k, ℓ be integral weights such that $k > \ell$. Let χ, ψ be Dirichlet characters modulo N. Suppose $f \in \mathcal{S}_k(N,\chi)$ and $g \in \mathcal{M}_\ell(N,\psi)$, and let r be a non-negative integer such that $\ell + 2r < k$. We define the Dirichlet series $D(s,f,g)$ as the sum

$$D(s,f,g) := \sum_{n=1}^{\infty} a_n b_n n^{-s}$$

where $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are the Fourier coefficients of f and g respectively. Then we have that

$$D(k-1-r,f,g) = c_{N,k,\ell,r}\, \pi^k \left\langle f_\rho, g \cdot E_{\lambda,N}(z,\chi\psi)\right\rangle$$

where $\lambda = k - \ell - 2r$ and where

$$c_{N,k,\ell,r} = \frac{\Gamma(k-\ell-2r)}{\Gamma(k-1-r)\Gamma(k-\ell-r)} \cdot \frac{(-1)^r 4^{k-1} N}{3} \cdot \prod_{p|N}(1+p^{-1}).$$

Proof sketch: [Shi76, p. 786] We first give Fourier expansions to the modular forms f, g as

$$f(z) := \sum_{n=1}^{\infty} a_n e^{2\pi i n z}, \quad \text{and} \quad g(z) := \sum_{n=1}^{\infty} b_n e^{2\pi i n z}.$$

We shall denote $f_\rho(z)$ as the dual modular form to f. That is, we define $f_\rho(z) = \sum_{n=1}^{\infty} \overline{a_n} e^{2\pi i n z}$, noting that $f_\rho \in \mathcal{S}_k(N, \overline{\chi})$.

Firstly, we note that Parseval's theorem gives us

$$\int_0^1 \overline{f_\rho(z)} g(z)\, dx = \sum_{n=1}^{\infty} a_n b_n e^{-4\pi n y}, \quad \text{where } z = x + iy$$

By then changing variables in the integral definition of the gamma function $\Gamma(s)$, we note that $\Gamma(s)$ can be given as

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt = (4\pi n)^s \int_0^\infty y^{s-1} e^{-4\pi n y} dy$$

where we've set $t = 4\pi n y$ for some arbitrary $n \in \mathbb{N}$. By using the above to rewrite n^{-s}, this therefore allows us to express $D(s, f, g)$ as

$$\begin{aligned} D(s, f, g) = \sum_{n=1}^{\infty} a_n b_n n^{-s} &= \frac{(4\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} a_n b_n \int_0^\infty y^{s-1} e^{-4\pi n y} dy \\ &= \frac{(4\pi)^s}{\Gamma(s)} \int_0^\infty \int_0^1 y^{s-1} \overline{f_\rho}(z) g(z) dx dy \end{aligned} \tag{13}$$

where the last line arises from switching the integral and summation, and applying Parseval's theorem, noting that this is valid at least for sufficiently large $\mathrm{Re}(s)$.

At this stage, we can apply the unfolding method, as introduced by Rankin [Ran39], noting that this makes essential use of the fact that f and g are modular forms. First, we note that, since $f \in \mathcal{S}_k(N, \chi)$ and $g \in \mathcal{M}_\ell(N, \psi)$. then by definition, $\overline{f_\rho} g$ transforms as

$$(\overline{f_\rho} g y^{s+1}) \circ \gamma = \chi(d)\psi(d)(cz + d)^{\ell - k} |cz + d|^{2k-2-2s}\, \overline{f_\rho} g y^{s+1} \tag{14}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

We let $\mathcal{R}$ be a set of coset representatives for $\Gamma_\infty \backslash \Gamma_0(N)$, and we denote Φ by a fundamental domain for $\Gamma_0(N)$ in the upper half plane. By therefore replacing the vertical strip $[0, 1] \times [0, \infty)$ with a set of translates of the fundamental domain Φ for $\Gamma_0(N) \backslash \mathcal{H}$, we can rewrite the right-hand side of (13) as follows:

$$\begin{aligned} D(s, f, g) &= \frac{(4\pi)^s}{\Gamma(s)} \sum_{\gamma \in \mathcal{R}} \int_{\gamma\Phi} y^{s-1} \overline{f_\rho} g dx dy \\ &= \frac{(4\pi)^s}{\Gamma(s)} \sum_{\gamma \in \mathcal{R}} \int_{\Phi} y^{-2} (\overline{f_\rho} g y^{s+1}) \cdot \gamma dx dy \\ &= \frac{(4\pi)^s}{\Gamma(s)} \int_{\Phi} \overline{f_\rho} g E_{k-\ell, N}(z, s + 1 - k, \chi\psi) y^{s-1} dx dy \end{aligned}$$

where we exchanged the sum and integral, used how $\overline{f_\rho} g$ transform in (14), and applied the definition of Eisenstein series. This idea is somewhat illustrated in the level 1 case in Figure 2.

Finally, by applying the identity given in (12), we arrive at the following expression for $D(s, f, g)$ at $s = k - 1 - r$:

$$\begin{aligned} D(k-1-r, f, g) &= \frac{(4\pi)^{k-1-r}}{\Gamma(k-1-r)} \int_\Phi \overline{f_\rho} g E_{k-\ell,N}(z, -r, \chi\psi) y^{k-r-2} dx dy \\ &= \frac{(4\pi)^{k-1-r}}{\Gamma(k-1-r)} \int_\Phi \overline{f_\rho} g \frac{\Gamma(k-\ell-2r)}{\Gamma(k-\ell-r)} (-4\pi y)^r \delta^{(r)}_{k-\ell-2r} E_{k-\ell-2r,N}(z, \chi\psi) y^{k-r-2} dx dy \\ &= c_{N,k,\ell,r}\, \pi^k \left\langle f_\rho,\, g \cdot \delta^{(r)}_\lambda E_{\lambda,N}(z, \chi\psi) \right\rangle \end{aligned}$$

where $\lambda = k - \ell - 2r$, and where the constant $c_{N,k,\ell,r}$ is as given in the problem statement. We also note that the measure $m(\Phi)$ of the fundamental domain Φ with respect to $y^{-2}dxdy$ can be calculated as:

$$m(\Phi) = \frac{\pi}{3}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = \frac{\pi N}{3} \prod_{p \mid N} (1 + p^{-1}),$$

the contribution of which is included in $c_{N,k,\ell,r}$, and thus this proves the theorem. □

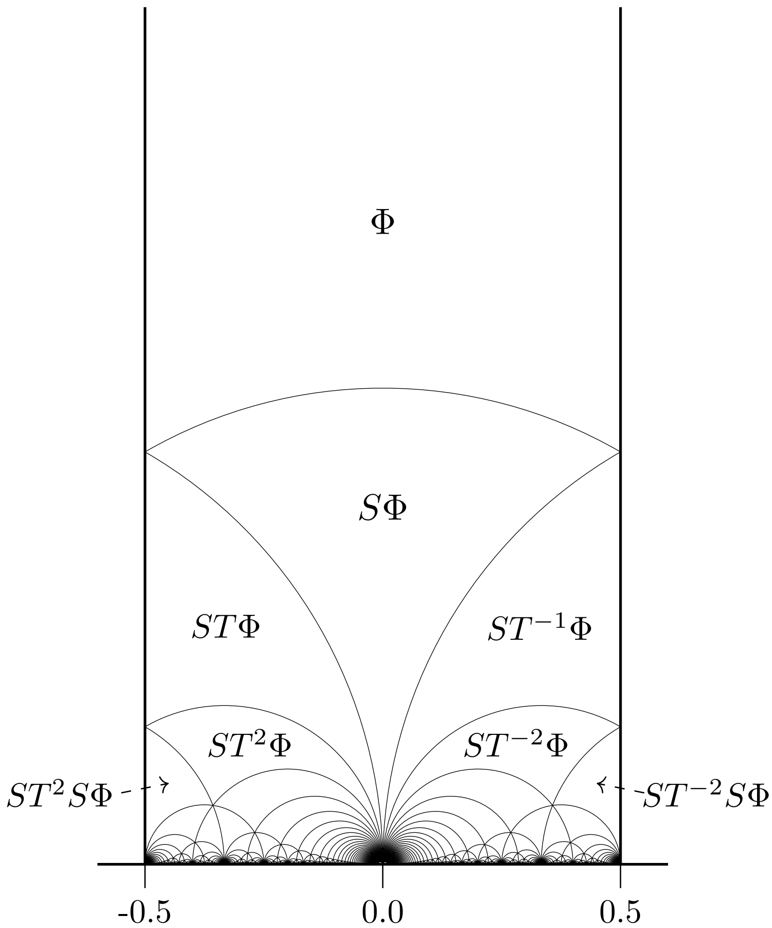

Figure 2: Partition of $[-\frac{1}{2}, \frac{1}{2}] \times [0, \infty)$ into translates $\gamma(\Phi)$ as γ ranges over $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$. This illustrates the idea of the unfolding method in the level $N = 1$ case, with the only minor difference being adjusting the translates instead over $[0, 1] \times [0, \infty)$ in our case.

Having done the unfolding method, we can now prove the main theorem which allows us to relate L-values with Petersson norms:

Theorem 25: [Shi76, p. 789] Let f be a primitive element of $\mathcal{S}_k(N)$ with coefficients in K_f, and let $g \in \mathcal{M}_\ell(N)$ with coefficients in K_g, with weights $k > \ell$. Let m be an integer such that $\frac{1}{2}(k + \ell - 2) < m < k$. Then $D(m, f, g)$ is an element of $K_f K_g \pi^k \langle f, f \rangle$.

Proof sketch: [Shi76, p. 795] The main idea is to simply use our result above from Theorem 24 and apply lemmas 22 and 23. Indeed, the unfolding method already tells us that $D(m, f, g)$ is a rational multiple of $\pi^k \big\langle f_\rho,\, g \cdot \delta_\lambda^{(r)} E_{\lambda,N}(z, \chi\psi)\big\rangle$.

All that remains is to prove that the ratio of $\big\langle f_\rho,\, g \cdot \delta_\lambda^{(r)} E_{\lambda,N}(z, \chi\psi)\big\rangle$ with the Petersson norm of f is in $K_f K_g$. Firstly, from the structure theorem, we have the existence of modular forms $g_\nu \in \mathcal{M}_{k-2\nu}(N, \overline{\chi})$ such that

$$g \cdot \delta_\lambda^{(r)} E_{\lambda,N}(z, \chi\psi) = \sum_{\nu=0}^{t} \delta_{k-2\nu}^{(\nu)} g_\nu$$

where t is either $r+1$ if $\lambda = 2$, or r otherwise. Now by applying orthogonality proven in Lemma 23, we have

$$\big\langle f_\rho,\, g \cdot \delta_\lambda^{(r)} E_{\lambda,N}(z, \chi\psi)\big\rangle = \big\langle f_\rho,\, \sum_{\nu=0}^{t} \delta_{k-2\nu}^{(\nu)} g_\nu \big\rangle = \big\langle f_\rho,\, g_0 \big\rangle$$

Since $K_{g_0} = K_g$ and $K_{f_\rho} = K_f$, the proof concludes by simply noting that for a normalised eigenform f and modular form $h \in \mathcal{M}_k(N)$, the ratio $\langle f, h\rangle / \langle f, f\rangle$ belongs to $K_f K_h$, as shown in [Shi76, p. 792]. □

Base-change elliptic curves E/K

We now apply the above approach to elliptic curves E over imaginary quadratic fields K, which are base change from $\mathbb{Q}$ [10]. This can be applied in both the inner and outer critical regions.

Inner critical region

Corollary 26: Fix K an imaginary quadratic field. Let E/K be an elliptic curve which is base change from $\mathbb{Q}$, and let ψ be a Grössencharacter over K in the inner critical region (i.e. of infinity type $(0, 0)$). Then $L(E/K, \psi, 1)$ is a $K(\psi)$-multiple of $\pi^2 \langle f_E, f_E\rangle$.

Proof: This follows almost immediately from Theorem 25, by setting $f = f_E$ being the modular form associated to E, and $g = \theta_\psi$ as defined in Definition 13, which are both modular forms of weights $k = 2$ and $\ell = 1$ respectively. By setting $m = 1$ (indeed, this is the only integer falling within the critical region), the result follows, since $L(E/K, \psi, 1) = D(1, f_E, \theta_\psi) \sim_{K(\psi)} \pi^2 \langle f_E, f_E\rangle$. □

It's worth mentioning that there are existing methods to deal with arbitrary curves E/K in the inner critical region, most notably of which involves using modular symbols, as shown in [Hid93, p. 186]. We won't go into any detail here regarding this approach, although we'll also

[10] It's possible that a similar strategy of applying the Rankin-Selberg method could also work for $\mathbb{Q}$-curves (i.e. curves E/K which are isogenous over $\overline{K}$ to each of its Galois conjugates), although we have not verified this.

just mention that far more generalised results have been obtained by extending these methods. [11]

Outer critical region

Corollary 27: Fix K an imaginary quadratic field. Let E/K be an elliptic curve which is base change from $\mathbb{Q}$, and let ψ be a Grössencharacter over K in the outer critical region. Then $L(E/K, \psi, 1)$ is a $K(\psi)$-multiple of $\pi^{|a-b|+1}\langle\theta_\psi, \theta_\psi\rangle$.

Proof: We again simply apply Theorem 25, now with $f = \theta_\psi$, and $g = f_E$, which are modular forms of weights $|a-b|+1 \geq 3$ and 2 respectively. We set $m = \frac{|a-b|}{2} + 1$, which yields the result that $L(E/K, \psi, 1) = L(f_E \times \theta_\psi, \frac{|a-b|}{2} + 1) \sim_{K(\psi)} \pi^{|a-b|+1}\langle\theta_\psi, \theta_\psi\rangle$. □

We do remark that one can furthermore prove that the Petersson norm $\langle\theta_\psi, \theta_\psi\rangle$ is an algebraic multiple of $\frac{\Omega_K^{2|a-b|}}{\pi^{2\max(a,b)}}$, as shown in [Sim17, p. 59]. This therefore proves furthermore that the transcendental part of $L(E/K, \psi, 1)$ only depends on the infinity type (a, b).

Elliptic curves over $\mathbb{Q}$

It's perhaps worth going over the extent to which the Rankin-Selberg method can be applied to elliptic curves $E/\mathbb{Q}$. Unfortunately, it doesn't work as well as the case for base change curves E/K, essentially due to property that $\mathbb{Q}$ has a real place and so one has to consider both the real period Ω_E^+ and imaginary period Ω_E^-, where the sign of χ dictates which of the two periods one sees from the L-value $L(E/\mathbb{Q}, \chi, 1)$.

Nevertheless, by again following Shimura [Shi76] we shall give a proposition which at least proves that the transcendental part of L-values $L(E/\mathbb{Q}, \chi, 1)$ only depends on E and $\mathrm{sgn}(\chi)$.

Proposition 28: Let E be an elliptic curve over $\mathbb{Q}$, and χ a Dirichlet character modulo N. Then the transcendental part of $L(E/\mathbb{Q}, \chi, 1)$ only depends on E and the sign of χ.

Proof: [Shi76, p. 798] First, we need to pick a Dirichlet character ξ such that $(\xi\chi)(-1) = -1$ and $L(E/\mathbb{Q}, \xi, 1) \neq 0$. Such a character does exist by Rohrlich [Roh89] and clearly depends only on E and $\mathrm{sgn}(\chi)$. We now express the product of the L-functions of χ and ξ as the following series:

$$L(\chi, s)L(\xi, s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

Now by a result from Hecke [Hec37, p. 334], one can define a suitable constant b_0 such that the series

$$g(z) := \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

is a weight 1 modular form, specifically lying in $\mathcal{M}_1(Nq, \chi\xi)$. One can also show simply by a formal calculation [Shi76, p. 790] that we have

$$L(\chi\xi, 2s-1)D(s, f_E, g) = D(s, f_E, \chi)D(s, f_E, \xi) \tag{15}$$

[11] Indeed, Grobner [Gro18] (building on earlier work by Harris, Raghuram, Mahnkopf [GH16, Mah05, Rag10, Rag16] and others) proved that if Π_1 (resp. Π_2) are cohomological cuspidial (resp. abelian) automorphic representations on GL_n (resp. GL_m) over some CM-field F, which satisfy a particular interlacing condition on their infinity types, then one can express the Rankin-Selberg L-function $L(\Pi_1 \times \Pi_2, s)$ at some critical point $s = \frac{1}{2} + m$ as an algebraic multiple of the Whittaker periods $p(\Pi_1), p(\Pi_2)$, a non-zero archimedean period $p(m, \Pi_{1,\infty}, \Pi_{2,\infty})$ and the Gauss sum $\mathcal{G}(\omega_{\Pi'_f})$ of the central character of Π'_f [Gro18, p. 80].

We remark that the L-values of Dirichlet characters have been well-studied, with an explicit formula given in [Mar10, p. 59]. Specifically, for any odd Dirichlet character ψ, we have that $L(\psi, 1)$ is an algebraic multiple of π.

Thus, by evaluating equation (15) at $s = 1$ and applying Theorem 25, we obtain

$$L(E/\mathbb{Q}, \chi, 1) = \frac{L(\chi\xi, 1)D(1, f_E, g)}{D(1, f_E, \xi)} \sim \frac{\pi^3 \langle f_E, f_E \rangle}{L(E/\mathbb{Q}, \xi, 1)}$$

and thus the transcendental part of $L(E/\mathbb{Q}, \chi, 1)$ only depends on E and $\mathrm{sgn}(\chi)$. $\square$

We note that there are a few shortcomings regarding the above result. Firstly, we are required to use a result from Rohrlich [Roh89] asserting that $L(E/\mathbb{Q}, \xi, 1) \neq 0$ for some auxiliary Dirichlet character ξ, which was not known at the time Shimura [Shi76] first gave the above argument.

Secondly, we observe that the Rankin-Selberg method only allows one to express $L(E/\mathbb{Q}, \chi, 1)$ in terms of $\langle f_E, f_E \rangle$, and thus we don't see the periods Ω_E^+ or Ω_E^- individually, unless one also involves $L(E/\mathbb{Q}, \xi, 1)$ for some ξ. On the flip side, for elliptic curves E over imaginary quadratic fields K, there only exists one period, hence this difficulty is eliminated in the imaginary quadratic case.

Specifically, we mention that Bouganis–Dokchitser [BD07, p. 7] has proven that, for any elliptic curve $E/\mathbb{Q}$, the Petersson norm $\langle f_E, f_E \rangle$ is a rational multiple of $i\pi^{-3}\Omega_E^+\Omega_E^-$, thus illustrating the limitations of the Rankin-Selberg method to only observe the product of the two periods $\Omega_E^+\Omega_E^-$ in the case of elliptic curves over the rationals.

Arbitrary elliptic curves E/K

Unlike the existing results for base-change elliptic curves E/K, the situation for arbitrary curves E/K is still far from complete. Firstly, modularity for curves over imaginary quadratic fields K has not been proven in general, and is a topic of current research. We do remark that partial results have been obtained by Allen–Khare–Thorne [AKT19] who have proven that a positive proportion of elliptic curves over K are modular.

Assuming modularity, in order to extend existing rationality results to the outer critical region, we consider the general theory of Langlands functoriality. In a naive sense, this involves lifting an automorphic representation Π (and hence its L-function) to some group other than $\mathrm{GL}_2(K)$.

We won't go into any further detail on these ideas for this project, although it's worth mentioning the survey article by Clozel [Clo86], which are based on results proved jointly with Arthur [AC89]. Here it discusses lifting automorphic forms from $\mathrm{GL}_2(K)$ to $\mathrm{GL}_4(\mathbb{Q})$. In order to apply this for general E, we first remark that the group $\mathrm{GL}_4(\mathbb{Q})$ on its own does not have a Shimura variety, however, the group $\mathrm{GSp}_4(\mathbb{Q})$ does have a Shimura variety, therefore allowing us to use coherent cohomology and attempt to derive rationality statements for arbitrary curves E/K in the outer critical region.

This therefore suggests functorially lifting Π to groups for which there exists a Shimura variety (examples include $\mathrm{GSp}_4(\mathbb{Q})$, $\mathrm{U}(2, 2)$, or $\mathrm{U}(3, 1)$) which might give us a shot at proving our results for arbitrary E.

We also mention, noting that both our computational results and conditional theoretical results are consistent with there being no contribution of the period for E in the L-value $L(E/K, \psi, 1)$, this suggests the existence of a coherent cohomology integral formula on a Shimura variety where the automorphic form from E is holomorphic.

Our hope is that these ideas will be implemented in future work to provide an unconditional proof of Conjecture 14, perhaps even generalised to automorphic forms of higher rank.

Conclusion

Not only have we shown extensive computational evidence for Conjecture 14, but have also shown that it follows from Deligne's period conjecture. The period conjecture has indeed been verified for numerous examples of L-functions over the last few decades, and we can therefore have a high level of confidence in the validity of Conjecture 14.

There is certainly more scope for further computations, specifically for higher degree number fields. Although as the coefficients of the minimal polynomials of the L-values grow, the precision required likewise has to increase, which will inevitably lead to some hurdles that we'll need to overcome if we are to carry out further computations.

We also remark that it's possible to furthermore prove that Deligne's period conjecture implies a more generalised version of Lemma 16 where K can be any CM-field, as shown by Harris–Lin [HL16].

In conclusion, there are certainly many further avenues to investigate laid out by this project, both on the computational on theoretical side. Indeed, our main goal would be to hopefully give an unconditional proof of Conjecture 14, which we will pursue in future work.

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