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Calculating L-values of elliptic curves twisted by Grössencharacters (draft)

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Abstract

Given an imaginary quadratic field K, let E/K be an elliptic curve, and ψ a Grössencharacter over K of conductor $\mathfrak{f}_\psi$ and of infinity type $(a,b) \neq (0,0)$. Assuming $s=1$ is a critical value for $L(E/K,\psi,s)$, we provide computational evidence to suggest that the transcendental part of $L(E/K,\psi,1)$ only depends on K and (a,b). Specifically, we conjecture that $L(E/K,\psi,1)$ is an $K(\psi).K(\mathfrak{f}_\psi)$-multiple of $\Omega_K^{2|a-b|}/\pi^{a+b-1}$, where $K(\mathfrak{f}_\psi)$ denotes the ray class field of K for $\mathfrak{f}_\psi$, and where Ω_K denotes the Chowla–Selberg period of K. In the case where ψ^2 is unramified, we furthermore conjecture a refinement of the algebraic part of $L(E/K,\psi,1)$.

1 Introduction

The study of elliptic curves and their L-functions have been at the forefront of modern number theory over the past few decades. For a given number field K and elliptic curve E/K, a formula for the leading Taylor coefficient of $L(E/K,s)$ at $s=1$ has famously been conjectured by Birch and Swinnerton-Dyer [BSD65]. Since then, various other more elaborate conjectures have been made over the last few decades regarding general motives [Bei85, BK90, Del79].

Given an elliptic curve $E/\mathbb{Q}$ and Dirichlet character χ, one can consider the twisted L-function $L(E/\mathbb{Q},\chi,s)$. Whilst these L-values at $s=1$ have also been extensively studied, they have recently been investigated by Dokchitser, Evans, and Wiersema [DEW21]. More generally, over arbitrary number fields K one can consider the L-functions of elliptic curves $L(E/K,\psi,s)$ twisted by Grössencharacters ψ. These L-values shall be the focus of our computations.

Whilst a general motivic definition for $L(E/K,\psi,s)$ can be found in [Del79], we note that in the case where the conductor of ψ is coprime to the conductor of E, then the L-function $L(E/K,\psi,s)$ can be given by the following simple definition:

$$L(E/K,\psi,s) := \sum_{\mathfrak{n} \triangleleft \mathcal{O}_K} a_\mathfrak{n}(E/K)\psi(\mathfrak{n})N(\mathfrak{n})^{-s}$$

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where $a_{\mathfrak{n}}(E/K)$ denotes the $\mathfrak{n}$-th Fourier coefficient in the L-function of the elliptic curve $L(E/K,s) := \sum_{\mathfrak{n}\lhd\mathcal{O}_K} a_{\mathfrak{n}}(E/K)N(\mathfrak{n})^{-s}$.

For our purposes, we shall restrict to the imaginary quadratic number fields K, and shall investigate the L-values of elliptic curves E/K twisted by Grössencharacters ψ in the outer critical region, as defined in Section 2. Based on extensive numerical computations, we first make the following conjecture:

Conjecture 1. Let K be an imaginary quadratic field. Then for any elliptic curve E/K, and for any Grössencharacter ψ of conductor $\mathfrak{f}_\psi$ and of infinity type (a,b) in the outer critical region, we have

$$L(E/K,\psi,1) \in K(\psi).K(\mathfrak{f}_\psi)\cdot\frac{\Omega_K^{2|a-b|}}{\pi^{a+b-1}}$$

where $K(\psi)$ is the smallest field containing K and all values of ψ, and where $K(\mathfrak{f}_\psi)$ denotes the ray class field of K for $\mathfrak{f}_\psi$. Here, Ω_K is defined as the Chowla–Selberg period of K, given by

$$\Omega_K := \left(\prod_{a=1}^{|d|-1}\Gamma\left(\frac{a}{|d|}\right)^{\left(\frac{d}{a}\right)}\right)^{w/4h}$$

where d, h and w are the discriminant, class number, and order of unit group of K respectively.

Remark: One can show that Conjecture 1 is consistent with Deligne's period conjecture [Del79], as proven by [GH16, HL16]. Indeed, if $L(\psi,1)\neq 0$, then we note that Conjecture 1 is equivalent to the statement that

$$L(E/K,\psi,1)\in K(\psi)\cdot\pi L(\psi,1)^2.$$

In particular, we observe that $L(\psi,1)\neq 0$ if $a+b\neq 1$. We also remark that this result has already been proven in the case where E/K is base-change from $\mathbb{Q}$ using Rankin-Selberg convolutions [Shi76].

The L-values $L(\psi,1)$ have been well-studied by many authors over the past several decades. Indeed, it is well-known that $L(\psi,1)/(\Omega_K^{|a-b|}\pi^{1-(a+b)/2})$ is algebraic [Dam70], and has been observed that this lies in the compositum of $K(\psi)$ and the ray class field of K for $\mathfrak{f}_\psi$ [Kat76]. It it also worth mentioning the work of Asai [Asa07] where he explicitly describes these L-values in terms of elliptic Gauss sums.

We note that, in comparison, the L-values $L(E/K,\psi,1)$ for Grössencharacters ψ in the inner critical region have their transcendental part dependent on the elliptic curve E/K. Indeed, we have that $L(E/K,\psi,1)$ is a $K(\psi)$-multiple of $L(E/K,1)$, if the infinity type of ψ is trivial.

Whilst we do not have a precise BSD-type conjecture for the exact value of $L(E/K,\psi,1)$ in the outer critical region, we can provide a partial integrality result for quadratic Grössencharacters ψ over imaginary quadratic fields of class number 1. We define a Grössencharacter of infinity type (a,b) as *quadratic* if $\psi((\lambda)) = \pm\lambda^a\overline{\lambda}^b$ for all $\lambda\in\mathcal{O}_K$ (or equivalently, if ψ^2 is unramified).

Conjecture 2. Let K be an imaginary quadratic field of class number 1 with discriminant $-D$. Let E/K be an elliptic curve with conductor $\mathfrak{f}_E$ generated by λ_E, and ψ a quadratic Grössencharacter of infinity type (a,b) in the outer critical region, of conductor $\mathfrak{f}_\psi$ generated by λ_ψ. Define $\mathcal{L}(E/K,\psi)$ as

$$\mathcal{L}(E/K,\psi) := L(E/K,\psi,1)\frac{i\pi^{a+b-1}}{\Omega_K^{2|a-b|}}(iD)^{|a-b|}\,\overline{(\lambda_E\lambda_\psi^2)}^{c}\cdot c!(c-1)!\,2^{a+b+1} \tag{1}$$

where $c := |\min(a,b)|$. Then $\mathcal{L}(E/K,\psi)\in\mathcal{O}_K$.

We remark that our construction of $\mathcal{L}(E/K,\psi)$ is only defined for class number 1 fields, and furthermore is only defined up to a unit in $\mathcal{O}_K$. However we can still pose the following additional conjecture which relates the value of $\mathcal{L}(E/K,\psi)$ for a Grössencharacter of infinity type (a,b) with the shifted character of infinity type $(-b,-a)$:

Conjecture 3. Let K be an imaginary quadratic field of class number 1. Let E/K be an elliptic curve and ψ a Grössencharacter of infinity type (a,b) in the outer critical region, with $\mathcal{L}(E/K,\psi)$ as defined above. Let χ_{Nm} denote the norm Grössencharacter of infinity type $(1,1)$. Then

$$\mathcal{L}(E/K,\psi) = \pm\overline{\mathcal{L}}(E/K,\psi{\chi_{\mathrm{Nm}}}^{-a-b}).$$

We note that when $a=-b$, then a corollary of the above two conjectures implies that $\mathcal{L}(E/K,\psi)$ is an element of either $\mathbb{Z}$ or $\sqrt{-D}\,\mathbb{Z}$.

2 Grössencharacters

Let K be an imaginary quadratic field with some fixed embedding $K\hookrightarrow\mathbb{C}$. For a given ideal $\mathfrak{m}\lhd\mathcal{O}_K$, define $I(\mathfrak{m})$ as the group of fractional ideals in K coprime to $\mathfrak{m}$. For simplicity and ease of computation, we shall consider a Grössencharacter of infinity type (a,b) modulo $\mathfrak{m}$ as a group homomorphism $\psi: I(\mathfrak{m})\to\mathbb{C}^\times$ such that, for any principal ideal $\mathfrak{n}=(\lambda)$ where $\lambda\equiv 1 \bmod \mathfrak{m}$, we have $\psi(\mathfrak{n})=\lambda^a\overline{\lambda}^b$.

We say that a Grössencharacter ψ of infinity type (a,b) is in the *inner critical region* if $a=b=0$, or we say that ψ is in the *outer critical region* if either $a\geq 1$ and $b\leq -1$, or $a\leq -1$ and $b\geq 1$.

Note that, for any imaginary quadratic field K, we can denote $\chi_{\mathrm{Nm}}:\mathcal{O}_K^\times\to\mathbb{C}^\times$ as the norm Grössencharacter of infinity type $(1,1)$. To construct examples of Grössencharacters in the outer critical region, let K be an imaginary quadratic field of class number 1. Then for any infinity type (a,b) satisfying $a\equiv b \pmod w$, where w is the order of the unit group of K, we can define the canonical Grössencharacter $\psi_{a,b}$ of infinity type (a,b) as

$$\psi_{a,b}((\lambda)) := \lambda^a\overline{\lambda}^b.$$

To obtain examples of quadratic unitary Grössencharacters of imaginary quadratic fields of class number 1, let $\mathfrak{m}\lhd\mathcal{O}_K$ be such that $(\frac{-1}{\mathfrak{m}})=1$. Here, $(\frac{\cdot}{\mathfrak{m}})$ denotes the quadratic residue symbol in $\mathcal{O}_K$. We then define the Grössencharacter $\chi_\mathfrak{m}: I(\mathfrak{m})\to\mathbb{C}^\times$ as $\chi_\mathfrak{m}((\lambda)) := (\frac{\lambda}{\mathfrak{m}})$ for all $\lambda\in\mathcal{O}_K$. By therefore considering characters of the form $\chi_\mathfrak{m}\psi_{a,b}$, this yields infinitely many examples of quadratic Grössencharacters of infinity type (a,b).

3 Computations

All our computations of $L(E/K, \psi, 1)$ were done using the Sage implementation of Dokchitser's L-function calculator [Dok04]. This requires utilising the functional equation of the completed L-function $\Lambda(E/K, \psi, s)$. Recall that given some elliptic curve E/K of conductor $\mathfrak{f}_{E/K}$ and primitive Grössencharacter ψ of infinity type (a, b) of conductor $\mathfrak{f}_\psi$ coprime to $\mathfrak{f}_{E/K}$, we have:

$$\Lambda(E/K, \psi, s) := \mathrm{Nm}(\mathfrak{f}_{E/K}\mathfrak{f}_\psi^2)^{s/2}\, \Gamma_{\mathbb{C}}(s-c)\Gamma_{\mathbb{C}}(s-c-1)\, L(E/K, \psi, s)$$

where $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ and $c := \min(a, b)$. Recall that if E/K is modular, then the L-function $L(E/K, \psi, s)$ satisfies the functional equation

$$\Lambda(E/K, \psi, s) = \omega_{E/K,\psi}\, \Lambda(E/K, \overline{\psi}, 2 + a + b - s)$$

where $\omega_{E/K,\psi}$ is given by

$$\omega_{E/K,\psi} := \omega_{E/K}\psi(\mathfrak{f}_{E/K})\tau(\psi)^2 D^{-a-b}\mathrm{Nm}(\mathfrak{f}_\psi)^{-a-b-1}\mathrm{Nm}(\mathfrak{f}_{E/K})^{-(a+b)/2}$$

where $\omega_{E/K}$ is the global root number for E/K and where $\tau(\psi)$ is the Gauss sum of ψ.

To define the Gauss sum $\tau(\psi)$, let $\mathfrak{D}$ be the different of K, and choose $\mathfrak{c} \lhd \mathcal{O}_K$ coprime to $\mathfrak{f}_\psi$ such that $\mathfrak{D}\mathfrak{f}_\psi\mathfrak{c}$ is principal. Pick an element $t \in \mathcal{O}_K$ such that $\mathfrak{D}\mathfrak{f}_\psi\mathfrak{c} = (t)$. Then the Gauss sum $\tau(\psi)$ can be given by [Miy89, p. 92]:

$$\tau(\psi) := \frac{t^a\overline{t}^b}{\psi(\mathfrak{c})} \sum_r \frac{\psi((r))}{r^a\overline{r}^b}\, e^{2\pi i\, \mathrm{tr}(r/t)}$$

where r runs over a complete set of residues for $\mathfrak{c}/\mathfrak{f}_\psi\mathfrak{c}$. With the above convention, we note that $|\tau(\psi)|^2 = D^{a+b}N(\mathfrak{f}_\psi)^{a+b+1}$.

Whilst modularity has not been proven for all elliptic curves E/K, we make note of the work by Allen–Khare–Thorne [AKT19] proving that a positive proportion of elliptic curves over K are modular, with notable recent work by Feng [Fen22] proving that at least 40% of elliptic curves over imaginary quadratic fields K are modular.

Using the above functional equation with Dokchitser's L-function calculator, we computed L-values $L(E/K, \psi, 1)$ for various elliptic curves E/K and Grössencharacters ψ over the fields $K = \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11})$ and $\mathbb{Q}(\sqrt{-19})$.

We have tabulated some of the values obtained below for $K = \mathbb{Q}(\sqrt{-7})$, where we have denoted $\alpha := \frac{1+\sqrt{-7}}{2}$ for brevity. All isogeny classes of elliptic curves are given by their LMFDB label [LMF21]. All L-value computations were done to (at least) 500 bits of precision.

Table 1: List of values $\mathcal{L}(E/K,\psi)$ for various elliptic curves E over $K=\mathbb{Q}(\sqrt{-7})$ twisted by Grössencharacters ψ of infinity type $(1,-1)$.

Elliptic curves	**Grössencharacters**				
Isogeny Class	$\psi_{1,-1}$	$\chi_{(3)}\,\psi_{1,-1}$	$\chi_{(5)}\,\psi_{1,-1}$	$\chi_{(4\alpha-1)}\,\psi_{1,-1}$	$\chi_{(4\alpha-3)}\,\psi_{1,-1}$
16.1-CMa	±1	$\pm2^3$	$\pm2^3*3$	$\pm2^6$	$\pm2^7$
16.5-CMa	±1	$\pm2^3$	$\pm2^3*3$	$\pm2^7$	$\pm2^6$
28.2-a	±2	$\pm2^3$	0	$\pm2^3*3^2$	$\pm2^3*3^2$
44.3-a	±2	$\pm\sqrt{-7}\cdot2^4$	$\pm2^4$	$\pm2^5$	$\pm\sqrt{-7}\cdot2^6$
44.4-a	±2	$\pm\sqrt{-7}\cdot2^4$	$\pm2^4$	$\pm\sqrt{-7}\cdot2^6$	$\pm2^5$
46.2-a	±2	$\pm2^3$	$\pm2^3*3$	$\pm\sqrt{-7}\cdot2^4$	$\pm2^7$

Table 2: List of values $\mathcal{L}(E/K,\psi)$ for various elliptic curves E over $K=\mathbb{Q}(\sqrt{-7})$ twisted by Grössencharacters ψ of infinity type $(2,-2)$.

Elliptic curves	**Grössencharacters**				
Isogeny Class	$\psi_{1,-1}$	$\chi_{(3)}\,\psi_{1,-1}$	$\chi_{(5)}\,\psi_{1,-1}$	$\chi_{(4\alpha-1)}\,\psi_{1,-1}$	$\chi_{(4\alpha-3)}\,\psi_{1,-1}$
16.1-CMa	$\pm2^2$	$\pm2^4*3^2*5$	$\pm2^4*3*43$	$\pm2^7*11^2$	$\pm2^9*41$
16.5-CMa	$\pm2^2$	$\pm2^4*3^2*5$	$\pm2^4*3*43$	$\pm2^9*41$	$\pm2^7*11^2$
28.2-a	0	0	$\pm\sqrt{-7}^3\cdot2^7$	$\pm2^5*3*7^2$	$\pm2^5*3*7^2$
44.3-a	$\pm2^3$	$\pm\sqrt{-7}\cdot2^6*3$	$\pm2^5*3*19$	$\pm2^8*73$	$\pm\sqrt{-7}\cdot2^6*3^3$
44.4-a	$\pm2^3$	$\pm\sqrt{-7}\cdot2^6*3$	$\pm2^5*3*19$	$\pm\sqrt{-7}\cdot2^6*3^3$	$\pm2^8*73$
46.2-a	$\pm2^3*3$	$\pm2^6*3$	$\pm2^6*283$	$\pm\sqrt{-7}\cdot2^7*17$	$\pm2^6*53$

Table 3: List of values $\mathcal{L}(E/K,\psi)$ for various elliptic curves E over $K=\mathbb{Q}(\sqrt{-7})$ twisted by Grössencharacters ψ of infinity type $(1,-3)$.

Elliptic curves	**Grössencharacters**		
Isogeny Class	$\psi_{1,-3}$	$\chi_{(3)}\,\psi_{1,-3}$	$\chi_{(5)}\,\psi_{1,-3}$
16.1-CMa	$\pm\alpha^2$	$\pm(12\alpha-11)(-\alpha+1)^3\alpha^4*3^2$	$\pm(-12\alpha-47)(-8\alpha+3)(-\alpha+1)^3\alpha^4*5$
16.5-CMa	$\pm(-\alpha+1)^2$	$\pm(-12\alpha+1)(-\alpha+1)^4\alpha^3*3^2$	$\pm(12\alpha-59)(-\alpha+1)^4\alpha^3(-8\alpha+5)5$
28.2-a	$\pm2^2$	$\pm2^7*3^3$	$\pm2^9*3^3*5$
44.3-a	$\pm(-\alpha+1)^4\alpha^2$	$\pm(-\alpha+1)^5\alpha^9*3^2*5$	$\pm(-48\alpha+13)(-\alpha+1)^5\alpha^6(-2\alpha+3)3*5$
44.4-a	$\pm(-\alpha+1)^2\alpha^4$	$\pm(-\alpha+1)^9\alpha^5*3^2*5$	$\pm(48\alpha-35)(-\alpha+1)^6\alpha^5(2\alpha+1)3*5$

Whilst we believe conjecture 1 holds for all elliptic curves E/K and Grössencharacters ψ in the outer critical region for any imaginary quadratic field K, we have yet to formulate an analogous statement to Conjecture 2 for Grössencharacters of higher order, although it's very likely that a similar integrality statement for arbitrary ψ can be made. Extending and refining the statements of our conjectures will be the focus of future work.

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References

[AKT19] Allen, P., Khare, C., Thorne, J. (2019), *Modularity of $GL_2(\mathbb{F}_p)$-representations over CM fields*, arXiv:1910.12986.

[Asa07] Asai, T. (2007) *Elliptic Gauss sums and Hecke L-values at $s=1$*. Proceedings of the Symposium on Algebraic Number Theory and Related Topics, pp.79–121, RIMS Kôkyûroku Bessatsu, B4, Res. Inst. Math. Sci. (RIMS), Kyoto.

[Bei85] Beilinson, A. (1985) *Higher regulators and values of L-functions.* Jour. Soviet. Math. Vol. 30, pp.2036-2070.

[BSD65] Birch, B., Swinnerton-Dyer, H. (1965) *Notes on elliptic curves II*, J. für die Reine und Angew. Math., Vol. 218, pp.79-108.

[BK90] Bloch, S., Kato, K. (1990) *L-functions and Tamagawa numbers of motives* , The Grothendieck Festschrift I, Progress in Mathematics, 86, Birkhäuser (1990) pp. 333–400.

[Dam70] Damerell, R.M. (1970) *L-functions of elliptic curves with complex multiplication. I.* Acta Arith., Vol. 17, pp.287-301.

[Del79] Deligne, P. (1979) *Valeurs de fonctions L et périodes d'intégrales,* Proceedings of Symposia in Pure Mathematics (AMS), Vol. 33, No. 2, pp. 313–346.

[Dok04] Dokchitser, T. (2004) *Computing Special Values of Motivic L-Functions,* Experimental Mathematics, Experiment. Math. Vol. 13, No. 2, pp.137-150.

[DEW21] Dokchitser, V., Evans, R. and Wiersema, H. (2021) *On a BSD-type formula for L-values of Artin twists of elliptic curves,* Journal für die reine und angewandte Mathematik (Crelles Journal), Vol. 2021, No. 773, pp. 199-230.

[Fen22] Feng, Z. (2022) *A lower bound on the proportion of modular elliptic curves over Galois CM fields*, arXiv:2203.00731.

[GH16] Grobner, H., Harris, M. (2016) *Whittaker periods, motivic periods, and special values of tensor product L-functions*, J. Inst. Math. Jussieu, Vol. 15, No. 4, pp.711-769.

[HL16] Harris, M., Lin, J. (2016) *Period relations and special values of Rankin-Selberg L-functions*, Contemporary Mathematics (to appear).

[Kat76] Katz, N.M. (1976) *p-adic Interpolation of Real Analytic Eisenstein Series*, Ann. Math., Vol. 104, No. 3, pp.459-571.

[LMF21] The LMFDB Collaboration (2021) *The L-functions and modular forms database*, Available at: `http://www.lmfdb.org`.

[Miy89] Miyake, T. (1989) *Modular Forms*, Springer-Verlag, Berlin, New York.

[Sag21] The Sage Developers (2021) *SageMath, the Sage Mathematics Software System (Version 9.3)*, Available at: `https://www.sagemath.org`.

[Shi76] Shimura, G. (1976) *The special values of the zeta functions associated with cusp forms.*, Comm. Pure Appl. Math. Vol. 29, No. 6, pp.783–804.

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