Analysis

IMC 2024 Training

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Overview

IMC analysis can (very roughly) be broken into the following (non-disjoint) topics:

1. **Real Analysis**
2. **Integrals and Series**
3. **Functional equations/inequalities**
4. **Complex Analysis**

Real Analysis

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Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements

(a) If f is continuous and $\text{range}(f) = \mathbb{R}$ then f is monotonic.

(b) If f is monotonic and $\text{range}(f) = \mathbb{R}$ then f is continuous.

(c) If f is monotonic and f is continuous then $\text{range}(f) = \mathbb{R}$.

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Solution:

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(a) False. E.g. $f(x) = x^3 - x$.

(b) True. Assume f nondecreasing and let $a \in \mathbb{R}$. Consider the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

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(b) True. Assume f nondecreasing and let $a \in \mathbb{R}$. Consider the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

(c) False. E.g. $f(x) = \arctan x$.

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Let $f : [a, b] \to \mathbb{R}$ be continuous. Let m and M be the minimum and maximum of f on $[a, b]$ respectively. Then for any $u \in (m, M)$, there exists $c \in (a, b)$ such that $f(c) = u$.

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Theorem (Mean Value Theorem ("turn Rolle's on its side"))

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and differentiable on (a, b), for some $a < b$. Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Examples

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- all the points of the open interval $I = (a, b)$ are shadow points;
- a and b are not shadow points.

Prove that

(a) $f(x) \leq f(b)$ for all $a < x < b$;

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(a) $f(x) \leq f(b)$ for all $a < x < b$;

(b) $f(a) = f(b)$.

Hint: Suppose for contradiction $\exists c \in (a, b)$ such that $f(c) > f(b)$. Apply EVT on $[c, b]$.

Examples

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Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi)$$

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Hint: Use Rolle's theorem on $g(x) = f(x)\cos x$, and then again to $h(x) = g'(x)/\cos^2 x$.

Topology/Density

Example

Let S be an infinite set of real numbers such that $|s_1 + s_2 + \cdots + s_k| < 1$ for every finite subset $\{s_1, s_2, \ldots, s_k\} \subset S$. Show that S is countable.

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Hint: Let $I = (x - \alpha, x + \alpha) \subset [0,1]$ be some arbitrary interval. Apply EVT in some subinterval of I.

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Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(r) \leq g(r)$ for all rational r. Does this imply that $f(x) \leq g(x)$ for every real x if

(a) f and g are strictly increasing?

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Two useful facts:

- If $f, g : \mathbb{R} \to \mathbb{R}$ both continuous and coincide on a dense subset of $\mathbb{R}$, then $f = g$!
- $\mathbb{Q}$ is dense in $\mathbb{R}$ (and $\mathbb{R} \backslash \mathbb{Q}$ dense in $\mathbb{R}$).

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Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all reals x and y such that $x - y$ is rational.

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Hint: Consider the function $g_q(x) := f(x + q) - f(x)$ for some rational q.

Darboux's theorem

- Note that a function $f : [a, b] \to \mathbb{R}$ can be differentiable, but f' need *not* be continuous. E.g

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

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Theorem (Darboux's theorem)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Then for any u between $f'(a)$ and $f'(b)$, there exists a $c \in (a, b)$ such that $f'(c) = u$.

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Example

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

Integrals

Example

Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}dx \geq e^{-a^2} - e^{-b^2}$$

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Example

Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geq |x - y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

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Hint: Apply the condition for special (or for all) pairs (x, y) and integrate it.

Integrals

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Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that

$$\left| \int_0^1 f^3(x)dx - f^2(0) \int_0^1 f(x)dx \right| \le \max_{0 \le x \le 1} |f'(x)| \left(\int_0^1 f(x)dx \right)^2$$

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Hint: Let $M = \max_{0 \leq t \leq 1} |f'(t)|$. Integrate the inequality $-Mf(t) \leq f(t)f'(t) \leq Mf(t)$ first over $[0, x]$, and then over $[0, 1]$.

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Example

Compute

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{1/x} dx$$

Hint: Show a lower bound of 1. For the upper bound, split the interval into three parts.

Series/Convergence

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Example

(a) Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all n. Prove that the sequence

$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$

has a finite limit or tends to infinity.

(b) Prove that for all $\alpha > 1$ there exists a sequence $a_1, a_2, \ldots$ with the same properties such that

$$\lim_{n\to\infty} \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha.$$

Convergence

Theorem (Bolzano-Weierstrass)

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Theorem (Monotone convergence theorem)

Let (a_n) be a bounded monotone sequence of real numbers (i.e. either nondecreasing or nonincreasing). Then $\lim a_n$ exists.

L'Hôpital's rule

A very useful theorem for evaluating limits!

Theorem (L'Hôpital's rule)

Let $f, g : I \to \mathbb{R}$ be differentiable functions. Let $c \in I$ (could have $c = \pm\infty$ if I open-ended) and $g'(x) = 0$ for $x \in I \backslash \{c\}$.
Then if either $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} |g(x)| = \infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if the right hand side exists.

- Several IMC problems require applying L'Hôpital (possibly several times)

Examples

Example

Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n \to \infty} \int_0^1 f(nx)\,\mathrm{d}x = L.$$

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Hint: Let $F(x) = \int_0^x f$. Apply L'Hôpital to $\frac{F(t)}{t}$.

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Example

Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \leq 1$$

for all x. Prove that $\lim_{x\to\infty} f(x) = 0$.

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Hint: Apply L'Hôpital twice to $f(x)e^{x^2/2}/e^{x^2/2}$.

Limits

Tips to evaluate limits:

- Guess an explicit formula and prove it (maybe using induction?)
- Use the sandwich/squeeze theorem

Theorem (Sandwich theorem)

Let $f, g, h : I \to \mathbb{R}$ *be functions such that* $g(x) \leq f(x) \leq h(x)$ *for all* $x \in I$ *and* $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$*. Then* $\lim_{x\to a} f(x) = L$*.*

- Use L'Hôspital!
- Use monotone convergence theorem
- Use Riemann sums (not really done in recent IMCs?)

Most likely, you'll have to try some combination of the above along with some creative constructions.

Convergence/Divergence

To show **convergence/divergence** of a series $\sum a_n$:

- Ratio test
- Comparison with geometric series
- Integral test
- Alternating series test
- Root test

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Example

Let $C = \{4, 6, 8, 9, 10, \dots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n} \right)^n$$

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Hint: Show that $\frac{a_n}{n} \leq \frac{2}{3}$ for $n > 4$.

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(a) A sequence $x_1, x_2, \ldots$ of real numbers satisfies $x_{n+1} = x_n \cos x_n$ for all $n \geq 1$. Does it follow that this sequence converges for all initial values x_1?

(b) A sequence $y_1, y_2, \ldots$ of real numbers satisfies $y_{n+1} = y_n \sin y_n$ for all $n \geq 1$. Does it follow that this sequence converges for all initial values y_1?

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Example

Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1, a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n} \quad \text{for } n \geq 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

Convergence/Divergence

Example

Let $(a_n)_{n=0}^{\infty}$ be a sequence with $\frac{1}{2} < a_n < 1$ for all $n \geq 0$. Define the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_0 = a_0, \quad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1} x_n} \quad (n \geq 0)$$

What are the possible values of $\lim_{n \to \infty} x_n$? Can such a sequence diverge?

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Example

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and $a_{n+1}^3 = a_n^2 - 8$ for $n = 0, 1, 2, \ldots$. Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

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Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$a_0 = 1, \quad a_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} \frac{a_k}{n-k+2}$$

Find the limit $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k}$.

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Hint: Differentiate the generating function for (a_n).

Functional equations/inequalities

Example

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f(0) = 0$, and for all $x \in [0, \infty)$,

$$f''(x) - 5f'(x) + 6f(x) \geq 0$$

Prove that for all $x \in [0, \infty)$, $f(x) \geq 3e^{2x} - 2e^{3x}$.

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Hint: The expression $f'' \cdot f - 2(f')^2$ is a part of the second derivative of some fraction.

Functional equations/inequalities

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Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

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Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies $f'(t) > f(f(t))$ for all $t \in \mathbb{R}$. Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

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Example

Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x+y) \geq f(x) + yf(f(x)) \quad \text{for all } x, y \in \mathbb{R}.$$

Complex analysis

- The complex numbers $\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$ where $i^2 = -1$.

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Holomorphic functions

Let $D \subset \mathbb{C}$ be an open set. A function $f : D \to \mathbb{C}$ is **holomorphic** on D if the derivative $f'(z)$ exists for all $z \in D$. I.e. if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

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- Holomorphicity is a strong condition!

Theorem

Let $f : D \to \mathbb{C}$ be a holomorphic function. Then f is infinitely differentiable and analytic (i.e. locally given by a convergent power series).

Complex analysis

Some main results:

Theorem (Maximum modulus principle)

Let D be a closed and bounded nonempty subset of $\mathbb{C}$. Let $f : D \to \mathbb{C}$ be a holomorphic function. Then $|f(x)|$ attains its maximum on some point on the boundary of D.

Theorem (Cauchy's integral theorem)

Let D be a simply connected subset, and let $f : D \to \mathbb{C}$ be a holomorphic function. Let $\mathcal{C}$ be a closed contour in D. Then

$$\int_{\mathcal{C}} f(z)\, dz = 0$$

Example

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Let $D \subset \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \leq 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$|f(0)| \leq \max_{|z|=1} |f(z)p(z)|.$$

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Hint: Apply the maximum principle to $z^n \overline{p(1/\overline{z})} f(z)$.

Complex Analysis

Theorem (Cauchy's integral formula)

Let D be a simply connected subset, and let $f : D \to \mathbb{C}$ be a holomorphic function. Let $z \in D$ and L be a contour in a counterclokwise direction around z with interior contained inside D. Then

$$f(a) = \frac{1}{2\pi i} \oint_L \frac{f(z)}{z-a} dz$$

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$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z - a)^{n-1}} dz$$

Complex Analysis

Some further useful results:

Theorem (Fundamental theorem of algebra)

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Theorem (Fundamental theorem of algebra)

Every non-constant polynomial with complex coefficients has a complex root.

Theorem (Liouville's theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded holomorphic function. Then f is constant.

Theorem (Picard's Little Theorem)

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Then $Im(f)$ is either the whole of $\mathbb{C}$ or $\mathbb{C}$ minus a single point.

Example

Example (Challenge)

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \geq c_1 \geq \cdots \geq c_n \geq 0$ is a sequence of real numbers which is convex (i.e. $2c_k \leq c_{k-1} + c_{k+1}$ for every $k = 1, 2, \ldots, n-1$), and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n$$

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$$\max_{|z|\leq 1} |q(z)| \leq \max_{|z|\leq 1} |p(z)|.$$

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Prove that

$$\max_{|z| \leq 1} |q(z)| \leq \max_{|z| \leq 1} |p(z)|.$$

Hint: Use the Maximum Principle, and apply Cauchy differentiation formulas to express a_j as an integral of $p(z)/z^j$ over the unit circle.