

# Sums of two units in number fields

33<sup>èmes</sup> Journées Arithmétiques, University of Luxembourg

Robin Visser

(based on joint work with Magdaléna Tinková (Charles), Pavlo Yatsyna (Charles), and Volker Ziegler (Salzburg))

Faculty of Mathematics and Physics  
Charles University

3 July 2025

# Motivation

---

Let  $K$  be a number field. In this talk, we aim to study the equation

$$\varepsilon + \delta = n,$$

for some positive integer  $n$  and units  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

# Motivation

---

Let  $K$  be a number field. In this talk, we aim to study the equation

$$\varepsilon + \delta = n,$$

for some positive integer  $n$  and units  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

1. For a fixed  $n$ , classify the number fields  $K$  such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

# Motivation

---

Let  $K$  be a number field. In this talk, we aim to study the equation

$$\varepsilon + \delta = n,$$

for some positive integer  $n$  and units  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

1. For a fixed  $n$ , classify the number fields  $K$  such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .
2. For a fixed  $K$ , how many positive integers  $n$  are there such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ ?

# Motivation

---

Let  $K$  be a number field. In this talk, we aim to study the equation

$$\varepsilon + \delta = n,$$

for some positive integer  $n$  and units  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

1. For a fixed  $n$ , classify the number fields  $K$  such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .
2. For a fixed  $K$ , how many positive integers  $n$  are there such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ ?
3. Classify all positive integers  $n$  such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

# Motivation

---

Let  $K$  be a number field. In this talk, we aim to study the equation

$$\varepsilon + \delta = n,$$

for some positive integer  $n$  and units  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .

1. For a fixed  $n$ , classify the number fields  $K$  such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .
2. For a fixed  $K$ , how many positive integers  $n$  are there such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ ?
3. Classify all positive integers  $n$  such that  $\varepsilon + \delta = n$  for some  $\varepsilon, \delta \in \mathcal{O}_K^\times$ .
4. Classify all units  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta = n$  for some positive integer  $n$ .

# Brief history of the unit equation

---

Let  $a_1, a_2, \dots, a_n \in K^\times$  be fixed nonzero constants in  $K$ .

# Brief history of the unit equation

---

Let  $a_1, a_2, \dots, a_n \in K^\times$  be fixed nonzero constants in  $K$ .

- Siegel (1921) proved there are only finitely many  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .

# Brief history of the unit equation

---

Let  $a_1, a_2, \dots, a_n \in K^\times$  be fixed nonzero constants in  $K$ .

- Siegel (1921) proved there are only finitely many  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .
- Baker (1966/67) gave effective lower bounds on linear combinations of logarithms of algebraic numbers.

# Brief history of the unit equation

---

Let  $a_1, a_2, \dots, a_n \in K^\times$  be fixed nonzero constants in  $K$ .

- Siegel (1921) proved there are only finitely many  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .
- Baker (1966/67) gave effective lower bounds on linear combinations of logarithms of algebraic numbers.
- Györy (1972) applied Baker's method to give a completely effective algorithm to compute all  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .

# Brief history of the unit equation

---

Let  $a_1, a_2, \dots, a_n \in K^\times$  be fixed nonzero constants in  $K$ .

- Siegel (1921) proved there are only finitely many  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .
- Baker (1966/67) gave effective lower bounds on linear combinations of logarithms of algebraic numbers.
- Györy (1972) applied Baker's method to give a completely effective algorithm to compute all  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .
- Schmidt (1972) proved his celebrated subspace theorem (further generalised by Schlickewei (1977)).

# Brief history of the unit equation

---

Let  $a_1, a_2, \dots, a_n \in K^\times$  be fixed nonzero constants in  $K$ .

- Siegel (1921) proved there are only finitely many  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .
- Baker (1966/67) gave effective lower bounds on linear combinations of logarithms of algebraic numbers.
- Györy (1972) applied Baker's method to give a completely effective algorithm to compute all  $x_1, x_2 \in \mathcal{O}_K^\times$  such that  $a_1x_1 + a_2x_2 = 1$ .
- Schmidt (1972) proved his celebrated subspace theorem (further generalised by Schlickewei (1977)).
- Using Schmidt's subspace theorem, Evertse (1984) and van der Poorten and Schlickewei (1982) independently proved that there are only finitely many  $x_1, \dots, x_n \in \mathcal{O}_K^\times$  such that  $a_1x_1 + \dots + a_nx_n = 1$  and such that  $\sum_{i \in I} a_ix_i \neq 0$  for all non-empty subsets  $I$  of  $\{1, \dots, n\}$ .

# Density zero results for sums of units

---

## Theorem (Jarden–Narkiewicz 2007)

*Let  $K$  be a number field and  $m$  a positive integer. Let  $\mathcal{A}_m$  be the set of positive rational integers which can be expressed as the sum of at most  $m$  units in  $K$ . Then  $\mathcal{A}_m$  has density 0 in  $\mathbb{N}$ .*

# Density zero results for sums of units

---

## Theorem (Jarden–Narkiewicz 2007)

*Let  $K$  be a number field and  $m$  a positive integer. Let  $\mathcal{A}_m$  be the set of positive rational integers which can be expressed as the sum of at most  $m$  units in  $K$ . Then  $\mathcal{A}_m$  has density 0 in  $\mathbb{N}$ .*

*Proof sketch for  $m = 2$ :*

- Assume for contradiction  $\mathcal{A}_2$  has positive upper density. Then Szemerédi's theorem implies  $\mathcal{A}_2$  contains arbitrarily long arithmetic progressions.

# Density zero results for sums of units

---

## Theorem (Jarden–Narkiewicz 2007)

*Let  $K$  be a number field and  $m$  a positive integer. Let  $\mathcal{A}_m$  be the set of positive rational integers which can be expressed as the sum of at most  $m$  units in  $K$ . Then  $\mathcal{A}_m$  has density 0 in  $\mathbb{N}$ .*

*Proof sketch for  $m = 2$ :*

- Assume for contradiction  $\mathcal{A}_2$  has positive upper density. Then Szemerédi's theorem implies  $\mathcal{A}_2$  contains arbitrarily long arithmetic progressions.
- For any  $k$ , let  $a_1, a_2, \dots, a_k \in \mathcal{A}_2$  be an arithmetic progression with common difference  $d_k$

# Density zero results for sums of units

## Theorem (Jarden–Narkiewicz 2007)

Let  $K$  be a number field and  $m$  a positive integer. Let  $\mathcal{A}_m$  be the set of positive rational integers which can be expressed as the sum of at most  $m$  units in  $K$ . Then  $\mathcal{A}_m$  has density 0 in  $\mathbb{N}$ .

*Proof sketch for  $m = 2$ :*

- Assume for contradiction  $\mathcal{A}_2$  has positive upper density. Then Szemerédi's theorem implies  $\mathcal{A}_2$  contains arbitrarily long arithmetic progressions.
- For any  $k$ , let  $a_1, a_2, \dots, a_k \in \mathcal{A}_2$  be an arithmetic progression with common difference  $d_k$
- There exist units  $\varepsilon_i, \delta_i$  such that  $\varepsilon_1 + \delta_1 = a_1, \dots, \varepsilon_k + \delta_k = a_k$  thus  $\varepsilon_{i+1} + \delta_{i+1} - \varepsilon_i - \delta_i = d_k$  for all  $i = 1, \dots, k - 1$ , contradicting finiteness of four-term unit equations. □

# On the quantitative unit sum problem

---

Let  $\alpha \sim \beta$  iff  $\alpha = u\beta$  for some  $u \in \mathcal{O}_K^\times$ . Let  $N_m(x)$  be the number of equivalence classes  $[\alpha]_\sim$  such that  $\alpha$  is a non-degenerate sum of  $m$  units and  $N_{K/\mathbb{Q}}(\alpha) \leq x$ .

# On the quantitative unit sum problem

Let  $\alpha \sim \beta$  iff  $\alpha = u\beta$  for some  $u \in \mathcal{O}_K^\times$ . Let  $N_m(x)$  be the number of equivalence classes  $[\alpha]_\sim$  such that  $\alpha$  is a non-degenerate sum of  $m$  units and  $N_{K/\mathbb{Q}}(\alpha) \leq x$ .

**Theorem (Fuchs–Tichy–Ziegler 2009; stated for units)**

*Let  $K$  be a number field of unit rank  $r$ , and let  $\varepsilon > 0$ . Then*

$$N_m(x) = \frac{c_{m-1,r}}{m!} \left( \frac{\omega_K (\log x)^r}{\text{Reg}_K} \right)^{m-1} + o\left( (\log x)^{(m-1)r-1+\varepsilon} \right)$$

*as  $x \rightarrow \infty$ , for some explicit constant  $c_{m-1,s}$ .*

# On the quantitative unit sum problem

Let  $\alpha \sim \beta$  iff  $\alpha = u\beta$  for some  $u \in \mathcal{O}_K^\times$ . Let  $N_m(x)$  be the number of equivalence classes  $[\alpha]_\sim$  such that  $\alpha$  is a non-degenerate sum of  $m$  units and  $N_{K/\mathbb{Q}}(\alpha) \leq x$ .

**Theorem (Fuchs–Tichy–Ziegler 2009; stated for units)**

*Let  $K$  be a number field of unit rank  $r$ , and let  $\varepsilon > 0$ . Then*

$$N_m(x) = \frac{c_{m-1,r}}{m!} \left( \frac{\omega_K (\log x)^r}{\text{Reg}_K} \right)^{m-1} + o\left( (\log x)^{(m-1)r-1+\varepsilon} \right)$$

*as  $x \rightarrow \infty$ , for some explicit constant  $c_{m-1,s}$ .*

In particular, for sums of two units, we let  $\mathcal{N}_K$  denote the set of positive integers  $n$  which are the sum of two units in  $K$ . Then this gives the asymptotic polylogarithmic upper bound for  $\mathcal{N}_K$ :

$$\#\{n \in \mathcal{N}_K : n \leq x\} = O((\log x)^r)$$

# Finiteness of $\mathcal{N}_K$

---

Theorem (Tinková–V–Yatsyna 2025)

*Let  $K$  be a degree  $d$  number field not containing any real quadratic subfield. Then  $\mathcal{N}_K$  is a finite set. In particular, there are only finitely many  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta \in \mathbb{Z} \setminus \{0\}$ .*

# Finiteness of $\mathcal{N}_K$

## Theorem (Tinková–V–Yatsyna 2025)

Let  $K$  be a degree  $d$  number field not containing any real quadratic subfield. Then  $\mathcal{N}_K$  is a finite set. In particular, there are only finitely many  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta \in \mathbb{Z} \setminus \{0\}$ .

*Proof (with much thanks to Volker Ziegler):*

- Let  $\varepsilon + \delta = n$  be a solution. For any  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , we have  $\sigma(\varepsilon) + \sigma(\delta) = n$ , and thus

$$\frac{\varepsilon}{\sigma(\delta)} + \frac{\delta}{\sigma(\delta)} - \frac{\sigma(\varepsilon)}{\sigma(\delta)} = 1.$$

# Finiteness of $\mathcal{N}_K$

## Theorem (Tinková–V–Yatsyna 2025)

Let  $K$  be a degree  $d$  number field not containing any real quadratic subfield. Then  $\mathcal{N}_K$  is a finite set. In particular, there are only finitely many  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta \in \mathbb{Z} \setminus \{0\}$ .

*Proof (with much thanks to Volker Ziegler):*

- Let  $\varepsilon + \delta = n$  be a solution. For any  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , we have  $\sigma(\varepsilon) + \sigma(\delta) = n$ , and thus

$$\frac{\varepsilon}{\sigma(\delta)} + \frac{\delta}{\sigma(\delta)} - \frac{\sigma(\varepsilon)}{\sigma(\delta)} = 1.$$

- We consider two cases: If this is *non-degenerate* solution for some  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , then by finiteness of three-term unit equations, only finitely many possible  $n$ .

# Finiteness of $\mathcal{N}_K$

## Theorem (Tinková–V–Yatsyna 2025)

Let  $K$  be a degree  $d$  number field not containing any real quadratic subfield. Then  $\mathcal{N}_K$  is a finite set. In particular, there are only finitely many  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta \in \mathbb{Z} \setminus \{0\}$ .

*Proof (with much thanks to Volker Ziegler):*

- Let  $\varepsilon + \delta = n$  be a solution. For any  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , we have  $\sigma(\varepsilon) + \sigma(\delta) = n$ , and thus

$$\frac{\varepsilon}{\sigma(\delta)} + \frac{\delta}{\sigma(\delta)} - \frac{\sigma(\varepsilon)}{\sigma(\delta)} = 1.$$

- We consider two cases: If this is *non-degenerate* solution for some  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , then by finiteness of three-term unit equations, only finitely many possible  $n$ .
- If this is a *degenerate* solution for all  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , then  $\sigma(\delta) \in \{\varepsilon, \delta\}$ , and thus  $\mathbb{Q}(\delta)$  is either  $\mathbb{Q}$  or imaginary quadratic. A quick check shows that  $n \in \{1, 2\}$ .  $\square$

# Quadratic fields

---

It's easy to explicitly compute  $\mathcal{N}_K$  for all quadratic fields. As imaginary quadratic fields have only finitely many units, we easily observe that for imaginary quadratic fields  $K$ , we have

$$\mathcal{N}_K = \begin{cases} \{1, 2\} & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \{2\} & \text{otherwise} \end{cases}$$

# Quadratic fields

---

It's easy to explicitly compute  $\mathcal{N}_K$  for all quadratic fields. As imaginary quadratic fields have only finitely many units, we easily observe that for imaginary quadratic fields  $K$ , we have

$$\mathcal{N}_K = \begin{cases} \{1, 2\} & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \{2\} & \text{otherwise} \end{cases}$$

But for real quadratic fields  $K$ ,  $\mathcal{N}_K$  can be infinite!

# Quadratic fields

---

It's easy to explicitly compute  $\mathcal{N}_K$  for all quadratic fields. As imaginary quadratic fields have only finitely many units, we easily observe that for imaginary quadratic fields  $K$ , we have

$$\mathcal{N}_K = \begin{cases} \{1, 2\} & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \{2\} & \text{otherwise} \end{cases}$$

But for real quadratic fields  $K$ ,  $\mathcal{N}_K$  can be infinite!

## Theorem

*Let  $K$  be a real quadratic field and let  $\varepsilon_K$  be a fundamental unit of  $K$ . Then*

$$\mathcal{N}_K = \{|\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon_K^i)| : i \in \mathbb{Z}\}.$$

*In particular, if  $n \neq 2$  and  $n \in \mathcal{N}_K$ , then  $K = \mathbb{Q}(\sqrt{n^2 \pm 4})$ .*

# Quadratic fields

---

*Sketch proof:*

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .
- Thus  $f_\varepsilon(x) = x^2 - ax \pm 1$  and  $f_\varepsilon(x) = (x - \varepsilon)(x - \bar{\varepsilon})$ .

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .
- Thus  $f_\varepsilon(x) = x^2 - ax \pm 1$  and  $f_\varepsilon(x) = (x - \varepsilon)(x - \bar{\varepsilon})$ .
- Note  $n^2 - an \pm 1 = f_\varepsilon(n) = (n - \varepsilon)(n - \bar{\varepsilon}) = \delta\bar{\delta} = \pm 1$ .

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .
- Thus  $f_\varepsilon(x) = x^2 - ax \pm 1$  and  $f_\varepsilon(x) = (x - \varepsilon)(x - \bar{\varepsilon})$ .
- Note  $n^2 - an \pm 1 = f_\varepsilon(n) = (n - \varepsilon)(n - \bar{\varepsilon}) = \delta\bar{\delta} = \pm 1$ .
- Thus,  $n^2 - an$  is either  $-2$ ,  $0$ , or  $2$ .
  - If  $n^2 - an = \pm 2$ , then  $n = 1$  or  $n = 2$ , and so  $a = \pm 1$  or  $a = 3$ .

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .
- Thus  $f_\varepsilon(x) = x^2 - ax \pm 1$  and  $f_\varepsilon(x) = (x - \varepsilon)(x - \bar{\varepsilon})$ .
- Note  $n^2 - an \pm 1 = f_\varepsilon(n) = (n - \varepsilon)(n - \bar{\varepsilon}) = \delta\bar{\delta} = \pm 1$ .
- Thus,  $n^2 - an$  is either  $-2$ ,  $0$ , or  $2$ .
  - If  $n^2 - an = \pm 2$ , then  $n = 1$  or  $n = 2$ , and so  $a = \pm 1$  or  $a = 3$ .

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .
- Thus  $f_\varepsilon(x) = x^2 - ax \pm 1$  and  $f_\varepsilon(x) = (x - \varepsilon)(x - \bar{\varepsilon})$ .
- Note  $n^2 - an \pm 1 = f_\varepsilon(n) = (n - \varepsilon)(n - \bar{\varepsilon}) = \delta\bar{\delta} = \pm 1$ .
- Thus,  $n^2 - an$  is either  $-2$ ,  $0$ , or  $2$ .
  - If  $n^2 - an = \pm 2$ , then  $n = 1$  or  $n = 2$ , and so  $a = \pm 1$  or  $a = 3$ .
  - If  $n^2 - an = 0$ , then  $a = n$ , and therefore  $n = \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$ . □

# Quadratic fields

---

*Sketch proof:*

- Let  $\varepsilon + \delta = n$ . Let  $a := \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$  and let  $f_\varepsilon(x)$  be the minimal polynomial of  $\varepsilon$ .
- Thus  $f_\varepsilon(x) = x^2 - ax \pm 1$  and  $f_\varepsilon(x) = (x - \varepsilon)(x - \bar{\varepsilon})$ .
- Note  $n^2 - an \pm 1 = f_\varepsilon(n) = (n - \varepsilon)(n - \bar{\varepsilon}) = \delta\bar{\delta} = \pm 1$ .
- Thus,  $n^2 - an$  is either  $-2$ ,  $0$ , or  $2$ .
  - If  $n^2 - an = \pm 2$ , then  $n = 1$  or  $n = 2$ , and so  $a = \pm 1$  or  $a = 3$ .
  - If  $n^2 - an = 0$ , then  $a = n$ , and therefore  $n = \text{Tr}_{K/\mathbb{Q}}(\varepsilon)$ . □

Examples of  $\mathcal{N}_K$ :

$$\mathcal{N}_{\mathbb{Q}(\sqrt{2})} = \{2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, \dots\}$$

$$\mathcal{N}_{\mathbb{Q}(\sqrt{3})} = \{2, 4, 14, 52, 194, 724, 2702, 10084, 37634, 140452, 524174, \dots, \}$$

$$\mathcal{N}_{\mathbb{Q}(\sqrt{5})} = \{1, 2, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \dots\}$$

# Simplest cubic fields

---

# Simplest cubic fields

---

Let's recall the simplest cubic fields: For each  $a \geq -1$ , define  $K_a = \mathbb{Q}(\rho_a)$  where  $\rho_a$  is a root of

$$f_a(x) := x^3 - ax^2 - (a+3)x - 1.$$

# Simplest cubic fields

---

Let's recall the simplest cubic fields: For each  $a \geq -1$ , define  $K_a = \mathbb{Q}(\rho_a)$  where  $\rho_a$  is a root of

$$f_a(x) := x^3 - ax^2 - (a+3)x - 1.$$

Both  $\rho_a$  and  $\rho_a + 1$  are units, so we always have the trivial solution to the unit equation:  
 $-\rho_a + (\rho_a + 1) = 1.$

# Simplest cubic fields

Let's recall the simplest cubic fields: For each  $a \geq -1$ , define  $K_a = \mathbb{Q}(\rho_a)$  where  $\rho_a$  is a root of

$$f_a(x) := x^3 - ax^2 - (a+3)x - 1.$$

Both  $\rho_a$  and  $\rho_a + 1$  are units, so we always have the trivial solution to the unit equation:  
 $-\rho_a + (\rho_a + 1) = 1$ .

## Theorem (Vukusic–Ziegler 2022)

Let  $a \geq -1$ , and let  $\varepsilon, \delta \in \mathbb{Z}[\rho_a]^\times$  and  $|n| \leq \max(|a|^{1/3}, 1)$  satisfy

$$\varepsilon + \delta = n.$$

Then  $(\varepsilon, \delta)$  is either equivalent to the trivial solution  $(\rho_a + 1, -\rho_a)$  or equivalent to an explicit finite list of sporadic solutions.

# Cubic number fields

---

Let's now consider general cubic fields! We first classify the possible minimal polynomials of  $\varepsilon$ :

# Cubic number fields

---

Let's now consider general cubic fields! We first classify the possible minimal polynomials of  $\varepsilon$ :

## Lemma (Minemura 1998)

Let  $K$  be a cubic number field, and let  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta = n$ . Then either  $n \in \{1, 2\}$  and the minimal polynomial  $f_\varepsilon(x)$  of  $\varepsilon$  is

$$\begin{aligned} &x^3 - (a+3)x^2 + (a+4)x - 1, \quad x^3 - (a+3)x^2 + ax + 1, \\ &x^3 - (a+6)x^2 + (2a+9)x - 1, \quad x^3 - (a+6)x^2 + (2a+7)x + 1 \end{aligned}$$

for some  $a \in \mathbb{Z}$ , or

$$f_\varepsilon(x) = x^3 - (n+b)x^2 + nbx - 1$$

for some  $b \in \mathbb{Z}$ .

# Cubic number fields

---

We can argue in a similar way to the real quadratic case.

*Proof:*

# Cubic number fields

---

We can argue in a similar way to the real quadratic case.

*Proof:*

- Let  $K$  be a cubic number field, and let  $\sigma_1, \sigma_2, \sigma_3$  be the three embeddings of  $K$  in  $\mathbb{C}$ .

# Cubic number fields

---

We can argue in a similar way to the real quadratic case.

*Proof:*

- Let  $K$  be a cubic number field, and let  $\sigma_1, \sigma_2, \sigma_3$  be the three embeddings of  $K$  in  $\mathbb{C}$ .
- Let  $\varepsilon + \delta = n$ . Let  $f_\varepsilon(x) := x^3 + ax^2 + bx \pm 1 = (x - \sigma_1(\varepsilon))(x - \sigma_2(\varepsilon))(x - \sigma_3(\varepsilon))$  be the minimal polynomial of  $\varepsilon$ .

# Cubic number fields

---

We can argue in a similar way to the real quadratic case.

*Proof:*

- Let  $K$  be a cubic number field, and let  $\sigma_1, \sigma_2, \sigma_3$  be the three embeddings of  $K$  in  $\mathbb{C}$ .
- Let  $\varepsilon + \delta = n$ . Let  $f_\varepsilon(x) := x^3 + ax^2 + bx \pm 1 = (x - \sigma_1(\varepsilon))(x - \sigma_2(\varepsilon))(x - \sigma_3(\varepsilon))$  be the minimal polynomial of  $\varepsilon$ .
- We have that
$$n^3 + an^2 + bn \pm 1 = f_\varepsilon(n) = (n - \sigma_1(\varepsilon))(n - \sigma_2(\varepsilon))(n - \sigma_3(\varepsilon)) = \sigma_1(\delta)\sigma_2(\delta)\sigma_3(\delta) = \pm 1.$$

# Cubic number fields

---

We can argue in a similar way to the real quadratic case.

*Proof:*

- Let  $K$  be a cubic number field, and let  $\sigma_1, \sigma_2, \sigma_3$  be the three embeddings of  $K$  in  $\mathbb{C}$ .
- Let  $\varepsilon + \delta = n$ . Let  $f_\varepsilon(x) := x^3 + ax^2 + bx \pm 1 = (x - \sigma_1(\varepsilon))(x - \sigma_2(\varepsilon))(x - \sigma_3(\varepsilon))$  be the minimal polynomial of  $\varepsilon$ .
- We have that
$$n^3 + an^2 + bn \pm 1 = f_\varepsilon(n) = (n - \sigma_1(\varepsilon))(n - \sigma_2(\varepsilon))(n - \sigma_3(\varepsilon)) = \sigma_1(\delta)\sigma_2(\delta)\sigma_3(\delta) = \pm 1.$$
- Thus  $n^3 + an^2 + bn$  is either  $-2$ ,  $0$ , or  $2$ .

# Cubic number fields

---

We can argue in a similar way to the real quadratic case.

*Proof:*

- Let  $K$  be a cubic number field, and let  $\sigma_1, \sigma_2, \sigma_3$  be the three embeddings of  $K$  in  $\mathbb{C}$ .
- Let  $\varepsilon + \delta = n$ . Let  $f_\varepsilon(x) := x^3 + ax^2 + bx \pm 1 = (x - \sigma_1(\varepsilon))(x - \sigma_2(\varepsilon))(x - \sigma_3(\varepsilon))$  be the minimal polynomial of  $\varepsilon$ .
- We have that
$$n^3 + an^2 + bn \pm 1 = f_\varepsilon(n) = (n - \sigma_1(\varepsilon))(n - \sigma_2(\varepsilon))(n - \sigma_3(\varepsilon)) = \sigma_1(\delta)\sigma_2(\delta)\sigma_3(\delta) = \pm 1.$$
- Thus  $n^3 + an^2 + bn$  is either  $-2$ ,  $0$ , or  $2$ .
  - Case  $n^3 + an^2 + bn = \pm 2$ . Then  $n = 1$  or  $n = 2$ . If  $n = 1$ , then  $b = 1 - a$  or  $b = -3 - a$ . If  $n = 2$ , then  $b = -2a - 3$  or  $b = -2a - 5$ .
  - Case  $n^3 + an^2 + bn = 0$ . Then  $n$  is a root of  $x^2 + ax + b$ , and so  $a = -n - c$  and  $b = nc$  for some  $c \in \mathbb{Z}$ . □

# Cyclic cubic fields

Theorem (Komatsu 2025, Tinková–V–Yatsyna 2025)

Let  $K$  be a cyclic cubic number field, and let  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta = n$ . Then either  $n = 0$ , or  $(\varepsilon, \delta) = (\pm 1, \pm 1)$ ,  $(\varepsilon, \delta)$  equivalent to  $(-\rho_a, \rho_a + 1)$  for some  $a \geq -1$ , or  $K \in \{K_{-1}, K_0, K_1\}$  and  $(\varepsilon, \delta)$  is equivalent to

$$\begin{aligned} &(\rho_{-1}, -\rho_{-1} + 1), (\rho_{-1}^2, -\rho_{-1}^2 + 1), (\rho_{-1}^2 - \rho_{-1}, -\rho_{-1}^2 + \rho_{-1} + 1), \\ &(-\rho_{-1}, \rho_{-1} + 2), (\rho_{-1} + 1, -\rho_{-1} + 1), (3\rho_{-1}^2 + \rho_{-1} - 6, -2\rho_{-1}^2 - \rho_{-1} + 8), \\ &(\rho_{-1}^2, -\rho_{-1}^2 + 3), (4\rho_{-1}^2 - 5\rho_{-1}, -4\rho_{-1}^2 + 5\rho_{-1} + 3), (\rho_{-1}^2, +2\rho_{-1}, -\rho_{-1}^2 - 2\rho_{-1} + 4), \\ &(\rho_{-1}^2 - \rho_{-1}, -\rho_{-1}^2 + \rho_{-1} + 5), (5\rho_{-1}^2 + 9\rho_{-1}, -5\rho_{-1}^2 - 9\rho_{-1} + 19), \\ &(4\rho_{-1}^2 - 5\rho_{-1}, -4\rho_{-1}^2 + 5\rho_{-1} + 22), (\rho_0, -\rho_0 + 2), (\rho_0^2, -\rho_0^2 + 3), (\rho_1^2, -\rho_1^2 + 2), \\ &(\rho_1^2 - 2, -\rho_1^2 + 7), (\rho_1^2, -\rho_1^2 + 7). \end{aligned}$$

# Cyclic cubic fields

---

*Sketch proof:*

# Cyclic cubic fields

---

*Sketch proof:*

- If  $K$  is cyclic cubic, then  $\Delta_K$  is a square in  $\mathbb{Z}$ . Thus  $\text{disc}(f_\varepsilon)$  is a square in  $\mathbb{Z}$ .

# Cyclic cubic fields

---

*Sketch proof:*

- If  $K$  is cyclic cubic, then  $\Delta_K$  is a square in  $\mathbb{Z}$ . Thus  $\text{disc}(f_\epsilon)$  is a square in  $\mathbb{Z}$ .
- By the previous lemma, we have that the possible discriminants of  $f_\epsilon(x)$  are

$$a^4 + 6a^3 + 7a^2 - 6a - 31, (a^2 + 3a + 9)^2, (4a^2 + 24a + 9)(a + 3)^2, \\ 4x^4 + 48a^3 + 229a^2 + 510a + 473, (UV)^2(U + V)^2 - 2(U - V)^3 + 6(U^3 - V^3) - 27$$

# Cyclic cubic fields

---

*Sketch proof:*

- If  $K$  is cyclic cubic, then  $\Delta_K$  is a square in  $\mathbb{Z}$ . Thus  $\text{disc}(f_\varepsilon)$  is a square in  $\mathbb{Z}$ .
- By the previous lemma, we have that the possible discriminants of  $f_\varepsilon(x)$  are

$$a^4 + 6a^3 + 7a^2 - 6a - 31, (a^2 + 3a + 9)^2, (4a^2 + 24a + 9)(a + 3)^2, \\ 4x^4 + 48a^3 + 229a^2 + 510a + 473, (UV)^2(U + V)^2 - 2(U - V)^3 + 6(U^3 - V^3) - 27$$

- If any of the first four polynomials are squares, then either  $\varepsilon = \rho_a$ , or  $a \in \{-10, -6, -5, -4, -2, 0, 2, 4\}$ , by completing the square.

# Cyclic cubic fields

---

*Sketch proof:*

- If  $K$  is cyclic cubic, then  $\Delta_K$  is a square in  $\mathbb{Z}$ . Thus  $\text{disc}(f_\varepsilon)$  is a square in  $\mathbb{Z}$ .
- By the previous lemma, we have that the possible discriminants of  $f_\varepsilon(x)$  are

$$a^4 + 6a^3 + 7a^2 - 6a - 31, (a^2 + 3a + 9)^2, (4a^2 + 24a + 9)(a + 3)^2, \\ 4x^4 + 48a^3 + 229a^2 + 510a + 473, (UV)^2(U + V)^2 - 2(U - V)^3 + 6(U^3 - V^3) - 27$$

- If any of the first four polynomials are squares, then either  $\varepsilon = \rho_a$ , or  $a \in \{-10, -6, -5, -4, -2, 0, 2, 4\}$ , by completing the square.
- If the fifth polynomial is a square, then we get there exists  $W \in \mathbb{Z}$  such that

$$U^3 - WU^2V - (W + 3)UV^2 - V^3 = W^2 + 3W + 9.$$

# Cyclic cubic fields

---

*Sketch proof:*

- If  $K$  is cyclic cubic, then  $\Delta_K$  is a square in  $\mathbb{Z}$ . Thus  $\text{disc}(f_\varepsilon)$  is a square in  $\mathbb{Z}$ .
- By the previous lemma, we have that the possible discriminants of  $f_\varepsilon(x)$  are

$$a^4 + 6a^3 + 7a^2 - 6a - 31, (a^2 + 3a + 9)^2, (4a^2 + 24a + 9)(a + 3)^2, \\ 4x^4 + 48a^3 + 229a^2 + 510a + 473, (UV)^2(U + V)^2 - 2(U - V)^3 + 6(U^3 - V^3) - 27$$

- If any of the first four polynomials are squares, then either  $\varepsilon = \rho_a$ , or  $a \in \{-10, -6, -5, -4, -2, 0, 2, 4\}$ , by completing the square.
- If the fifth polynomial is a square, then we get there exists  $W \in \mathbb{Z}$  such that

$$U^3 - WU^2V - (W + 3)UV^2 - V^3 = W^2 + 3W + 9.$$

- Thus, it suffices to find all integer solutions  $(U, V)$  to an infinite family of Thue equations, parametrised by  $W$ .

# Hoshi to the rescue!

---

## Theorem (Hoshi 2011)

Let  $U, V, W$  be integers with  $U > 0$  such that

$$U^3 - WU^2V - (W + 3)UV^2 - V^3 = W^2 + 3W + 9.$$

Then  $(U, V)$  is one of the following 13 solutions:

$$(1, -3), (1, 4), (2, 1), (2, 5), (3, -22), (3, -1), (3, 0), \\ (4, -5), (5, -7), (5, -4), (7, -5), (19, 3), (22, -3).$$

# Hoshi to the rescue!

## Theorem (Hoshi 2011)

Let  $U, V, W$  be integers with  $U > 0$  such that

$$U^3 - WU^2V - (W + 3)UV^2 - V^3 = W^2 + 3W + 9.$$

Then  $(U, V)$  is one of the following 13 solutions:

$$(1, -3), (1, 4), (2, 1), (2, 5), (3, -22), (3, -1), (3, 0), \\ (4, -5), (5, -7), (5, -4), (7, -5), (19, 3), (22, -3).$$

- Hoshi shows the above problem is equivalent to finding all pairs  $(n, m)$  such that  $K_n = K_m$ , where  $K_a$  is the simplest cubic field. Hoshi (and Okazaki) show that the only such non-trivial isomorphic fields are:

$$K_{-1} = K_5 = K_{12} = K_{1259}, \quad K_0 = K_3 = K_{54}, \quad K_1 = K_{66}, \quad K_2 = K_{2389}$$

# Complex cubic fields

Let  $L_a := \mathbb{Q}(\omega_a)$  where  $\omega_a$  root of  $x^3 - ax^2 - 1$ .

## Theorem (Tinková–V–Yatsyna 2025)

Let  $K$  be a complex cubic number field, and let  $\varepsilon, \delta \in \mathcal{O}_K^\times$  such that  $\varepsilon + \delta = n$ . Then either  $n = 0$ , or  $(\varepsilon, \delta) = (\pm 1, \pm 1)$ ,  $(\varepsilon, \delta)$  equivalent to  $(\omega_a, -\omega_a + a)$  for some  $a \geq -1$ , or  $(\varepsilon, \delta)$  is equivalent to

$$\begin{aligned} &(\omega_{-1}, -\omega_{-1} + 1), (\omega_{-1}^2, -\omega_{-1}^2 + 1), (\omega_{-1}^2 + 2, -\omega_{-1}^2 - 2 + 1), \\ &(-\omega_{-1}^2 - \omega_{-1}, \omega_{-1}^2 + \omega_{-1} + 1), (\omega_{-1}^2 + 2\omega_{-1} + 2, -\omega_{-1}^2 - 2\omega_{-1} - 1), \\ &(\omega_{-1} + 1, -\omega_{-1} + 1), (\omega_{-1}^2 + \omega_{-1} + 1, -\omega_{-1}^2 - \omega_{-1} + 1), \\ &(\omega_{-1}^2 + 2\omega_{-1} + 1, -\omega_{-1}^2 - 2\omega_{-1} + 2), (\omega_{-1}^2 + 2\omega_{-1} + 2, -\omega_{-1}^2 - 2\omega_{-1} + 2) \\ &(-\omega_1^2, \omega_1^2 + 1), (\omega_1^2 - \omega_1, -\omega_1^2 + \omega_1 + 1), (\omega_1^2, -\omega_1^2 + 2), (\omega_1^2 + 1, -\omega_1^2 + 2), \end{aligned}$$

or  $(\omega, -\omega + 2)$  where  $\omega$  is a root of  $x^2 - x^2 - x - 1$ .

# Some effective results

---

## Theorem (V-Ziegler)

*Let  $K$  be an odd degree number field. Then there exists an effective algorithm to explicitly compute  $\mathcal{N}_K$ . (i.e. there exists an effectively computable constant  $C_K$  such that  $n \leq C_K$  for all  $n \in \mathcal{N}_K$ ).*

# Some effective results

---

## Theorem (V-Ziegler)

*Let  $K$  be an odd degree number field. Then there exists an effective algorithm to explicitly compute  $\mathcal{N}_K$ . (i.e. there exists an effectively computable constant  $C_K$  such that  $n \leq C_K$  for all  $n \in \mathcal{N}_K$ ).*

*Proof:* See Volker Ziegler's talk on Monday (30 June 2025)!

**Thank you!**