Computing genus 2 curves over $\mathbb{Q}$ whose Jacobians have good reduction away from 2

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Genus 2 curves

Problem (Poonen 1996)

List all genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2.

Smart (1997) computed all 366 genus 2 curves with good reduction outside 2. But there are more! Some examples of other curves $C/\mathbb{Q}$ where $\mathrm{Jac}(C)$ good outside 2:

- $C : y^2 = x^5 - 14x^3 + 81x$ has bad reduction at $\{2, 3\}$.
- $C : y^2 = 2x^5 - 9x^4 - 24x^3 + 22x^2 + 78x - 41$ has bad reduction at $\{2, 5\}$.
- $C : y^2 = 2x^5 + x^4 - 16x^3 - 72x^2 + 240x + 136$ has bad reduction at $\{2, 7\}$.
- $C : y^2 = x^5 + 478x^3 + 57122x$ has bad reduction at $\{2, 13\}$.
- $C : y^2 = x^5 + 28x^4 - 868x^3 - 6160x^2 + 43076x - 149072$ has bad reduction at $\{2, 3, 11\}$.

So far, we've found 512 examples of genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2.

Genus 2 curves good outside S

Easier Problem

Fix a small set of primes S. List all genus 2 curves $C/\mathbb{Q}$ with good reduction outside S and whose Jacobians have good reduction away from 2.

Smart's S-unit strategy:

- Let $C/\mathbb{Q} : y^2 = c(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)(x-\alpha_6)$ be such a curve, where $\alpha_i \in \mathbb{Q}(J[2])$.
- Use Siegel's identity: $\dfrac{(\alpha_i-\alpha_j)(\alpha_k-\alpha_\ell)}{(\alpha_i-\alpha_k)(\alpha_j-\alpha_\ell)} + \dfrac{(\alpha_i-\alpha_\ell)(\alpha_j-\alpha_k)}{(\alpha_i-\alpha_k)(\alpha_j-\alpha_\ell)} = 1$
- Compute all possible 2-torsion fields $\mathbb{Q}(J[2])$.
- Solve the T-unit equations $x+y=1$ for $x, y \in O_T^\times$ over $\mathbb{Q}(J[2])$, where T is the primes in $\mathbb{Q}(J[2])$ lying above S.

Solving a linear system

Let $\psi_1, \psi_2, \ldots, \psi_t$ be a set of T-unit generators over $\mathbb{Q}(J[2])$. Let $a_{k,i,j} \in \mathbb{Z}$ be given by

$$\alpha_i - \alpha_j = \psi_1^{a_{1,i,j}} \psi_2^{a_{2,i,j}} \cdots \psi_t^{a_{t,i,j}}.$$

We can compute the integers $a_{k,i,j}$ as solutions to a big linear system of equations!

Examples of linear constraints on $a_{k,i,j}$:

- Galois constraints: For all $\sigma \in \mathrm{Gal}(\mathbb{Q}(J[2])/\mathbb{Q})$, $a_{f_\sigma(k),g_\sigma(i),g_\sigma(j)} = a_{k,i,j}$.
- Cluster pictures (using that J has good reduction at all odd primes).
- Solving simple T-unit equations (i.e. $\tau + \sigma(\tau) = 1$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(J[2])/\mathbb{Q})$).

How to solve the linear system:

- Brute force.
- Closest vector problem (CVP).
- Integer linear programming (ILP).

Summary - bit.ly/genus2

Theorem (V. 2025)

There are at least 512 $\mathbb{Q}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2. This list includes all such curves $C/\mathbb{Q}$ which have good reduction away from S and where $\mathbb{Q}(J[2]) \subseteq M$, where (S, M) can be any of the following 18 pairs:

$$\big(\{2,3\}, \mathbb{Q}(\zeta_{16})\big),\ \big(\{2,3\}, \mathbb{Q}(\zeta_8, \sqrt[4]{2})\big),\ \big(\{2,3\}, \mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),\ \big(\{2,5\}, \mathbb{Q}(\zeta_{16})\big),$$
$$\big(\{2,5\}, \mathbb{Q}(\zeta_8, \sqrt[4]{2})\big),\ \big(\{2,5\}, \mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big), \big(\{2,7\}, \mathbb{Q}(\zeta_{16})\big),\ \big(\{2,7\}, \mathbb{Q}(\zeta_8, \sqrt[4]{2})\big),$$
$$\big(\{2,7\}, \mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),\ \big(\{2,3,5\}, \mathbb{Q}(\zeta_{16})\big),\ \big(\{2,3,7\}, \mathbb{Q}(\zeta_{16})\big),\ \big(\{2,3,7\}, \mathbb{Q}(\sqrt[4]{2\sqrt{2}-3})\big),$$
$$\big(\{2,5,7\}, \mathbb{Q}(\zeta_{16})\big),\ \big(\{2,5,7\}, \mathbb{Q}(\zeta_8, \sqrt[4]{2})\big),\ \big(\{2,3,5,7\}, \mathbb{Q}(\zeta_4)\big),\ \big(\{2,3,5,7\}, \mathbb{Q}(\sqrt{-2})\big),$$
$$\big(\{2,3,5,7\}, \mathbb{Q}(\sqrt{2})\big),\ \big(\{2,3,5,7,11,13\}, \mathbb{Q}\big).$$

- If you can find any more curves, you'll win ~~£100~~ \$300. (T&Cs apply)