Abelian surfaces with good reduction away from 2

LMFDB, Computation, and Number Theory (LuCaNT) workshop

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(Hopefully easier) subproblem

Classify all isogeny classes of abelian surfaces $A/\mathbb{Q}$ with good reduction away from 2 and with full rational 2-torsion (i.e. $\mathbb{Q}(A[2]) = \mathbb{Q}$).

Faltings–Serre–Livné method

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$$L(A/K, s) = \prod_{p \text{ prime}} L_p(A/K, s).$$

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Theorem (Faltings–Serre–Livné)

Let A/K and B/K be two abelian varieties. If $L_p(A/K, s) = L_p(B/K, s)$ for some effectively computable finite set of primes p, then $L(A/K, s) = L(B/K, s)$.

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Theorem (Faltings–Serre–Livné (effective))

Let $A/\mathbb{Q}$ and $B/\mathbb{Q}$ be two abelian varieties with good reduction away from 2 and with full rational 2-torsion. Then if $L_p(A/\mathbb{Q}, s) = L_p(B/\mathbb{Q}, s)$ for each $p \in \{3, 5, 7\}$, then A and B are isogenous over $\mathbb{Q}$.

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4	?	$C_2^2 \rtimes C_8$, $D_4 \rtimes C_8$, $C_2^2.C_4 \wr C_2$	1	4	2
5	?	(many)	1	3	1

Results

Theorem

There are exactly 3 isogeny classes of abelian surfaces $A/\mathbb{Q}$ with good reduction away from 2 which contain surfaces with full rational 2-torsion. These are given by $E_1 \times E_1$, $E_1 \times E_2$ and $E_2 \times E_2$, where E_1, E_2 are the elliptic curves $E_1 : y^2 = x^3 - x$ and $E_2 : y^2 = x^3 - 4x$.

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Doing a similar (albeit longer) computation also gives the following result:

Theorem

There are exactly 19 isogeny classes of abelian surfaces $A/\mathbb{Q}$ with good reduction away from 2 which contain surfaces such that either $A[2](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^4$ or $A[2](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$.