A survey of applications to Matomäki–Radziwiłł's theorem

Multiplicative functions study group, Week 10

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Main theorem of Matomäki-Radziwiłł

Theorem (Matomäki-Radziwiłł, 2015)

Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function andd let $2 \leq h \leq X$. Then for $(1 + o(1))X$ values of $X \leq x \leq 2X$ we have

$$\left| \frac{1}{h} \sum_{x \leq n \leq x+h} f(n) - \frac{1}{X} \sum_{X \leq n \leq 2X} f(n) \right| \leq o(1) \qquad \text{as } h, X \to \infty.$$

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E.g. Applying the above to the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ with $h = X^{\delta}$, we get

Theorem (Matomäki-Radziwiłł, 2015)

Let $\delta > 0$. For $(1 + o(1))X$ values of $X \le x \le 2X$, we have

$$\sum_{x \le n \le x+X^{\delta}} \lambda(n) = o(X^{\delta}) \qquad \text{as } X \to \infty.$$

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Let $f : [-1,1]$ *be a multiplicative function and let* $10 \leq h \leq x$. *Then we have*

$$\frac{1}{h\sqrt{x}\log 2} \sum_{\substack{x \leq n_1 n_2 \leq x + h\sqrt{x} \\ \sqrt{x} \leq n_1 \leq 2\sqrt{x}}} f(n_1) f(n_2) = \left(\frac{1}{\sqrt{x}} \sum_{\sqrt{x} \leq n \leq 2\sqrt{x}} f(n) \right)^2 + o(1)$$

as $h, X \to \infty$

Chowla's conjecture

Conjecture (Chowla, 1965)

Let $h_1, \ldots, h_k$ be distinct natural numbers. Then

$$\sum_{1 \le n \le x} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X) \qquad \text{as } X \to \infty.$$

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- Compare with twin prime conjecture: $\sum_{n \le x} \theta(n)\theta(n+2) \to \infty$ (where $\theta(p) := \log p$ if p prime, and 0 otherwise)
- Could generalise conjecture to

$$\sum_{1 \le n \le x} \lambda(a_1 n + h_1) \cdots \lambda(a_k n + h_k) = o(X)$$

such that $a_i h_j - a_j h_i \neq 0$ for all $i < j$.

Averaged form of Chowla's conjecture

Theorem (Matomäki-Radziwiłł–Tao (2015))

For any natural number k, and any $10 \leq H \leq X$, we have

$$\sum_{1 \leq h_1, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n+h_1) \cdots \lambda(n+h_k) \right| = o(H^k X)$$

provided that $H \to \infty$ arbitrarily slowly with $X \to \infty$.

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provided that $H \to \infty$ arbitrarily slowly with $X \to \infty$.

- Also have slightly stronger bound:

$$\sum_{1 \le h_2, \dots, h_k \le H} \left| \sum_{1 \le n \le X} \lambda(n)\lambda(n+h_2) \cdots \lambda(n+h_k) \right| = o(H^{k-1} X)$$

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Can use the Fourier-analytic identity:

$$\int_{\mathbb{T}} \left(\int_{\mathbb{R}} \Big| \sum_{x \le n \le x+H} f(n) e(\alpha n) \Big|^2 dx \right)^2 d\alpha = \sum_{|h| \le N} (H - |h|)^2 \left| \sum_n f(n) \bar{f}(n+h) \right|^2$$

to reduce the proof for $k = 2$ to showing an estimate of the form:

$$\int_0^X \Big| \sum_{x \le n \le x+H} \lambda(n) e(\alpha n) \Big| dx = o(HX)$$

uniformly for all $\alpha \in \mathbb{T}$.

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uniformly for all $\alpha \in \mathbb{T}$. Using the circle method:

- Majors arcs: Use the result from Matomäki-Radziwiłł.
- Minor arcs: Use an argument of Katai and Bourgain–Sarnak–Ziegler.

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Conjecture (Elliott, 1992)

Let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ be positive integers such that $a_i b_j - a_j b_i \neq 0$. Suppose there is an index ℓ such that g_ℓ satisfies

$$\sum_p \frac{1 - \mathrm{Re}(g_\ell(p)\overline{\chi(p)p^{-it}})}{p} = \infty$$

for all Dirichlet characters χ and $t \in \mathbb{R}$. (i.e. g_ℓ does not behave "like a Dirichlet character"). Then

$$\sum_{1 \leq n \leq X} g_1(a_1 n + b_1) \cdots g_k(a_k n + b_k) = o(X) \qquad \text{as } X \to \infty.$$

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Elliott's conjecture (corrected)

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Theorem (Matomäki-Radziwiłł–Tao (2015))

Let $10 \leq H \leq X$ *and* $A \geq 1$. *Let* $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ *be 1-bounded functions with at least one* g_ℓ *being multiplicative and "non-pretentious". Then*

$$\sum_{1 \leq h_1, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} g_1(a_1 n + b_1 + h_1) \ldots g_k(a_k n + b_k + h_k) \right| = o(H^k X)$$

as $H, X \to \infty$.

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Sign patterns of the Liouville function

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Conjecture (Chowla, 1965)

For any positive integer k, and choice of signs $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$, the set of positive integers n such that

$$\lambda(n+1) = \epsilon_1, \quad \lambda(n+2) = \epsilon_2, \quad \ldots, \quad \lambda(n+k) = \epsilon_k$$

has density $1/2^k$.

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Theorem (Hildebrand, 1986)

Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, +1\}$. Then there exist infinitely many n such that

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Proof uses ad hoc elementary arguments, using the multiplicative property of λ with the primes 2, 3, 5.

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Theorem (Matomäki-Radziwiłł–Tao, 2015)

Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, +1\}$. *Then there are a positive (lower) density of integers n such that*

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- In 2017, Klurman–Manerla proved the upper logarithmic density is at least $1/28$.
- In 2017, Tao–Teräväinen proven the logarithmic density for $k = 3$ is exactly $\frac{1}{8}$, and for $k = 4$ is at least $\frac{1}{32}$.

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- By Matomäki-Radziwiłł, we have

$$\limsup_{X\to\infty} \frac{1}{X} \sum_{X\le x\le 2X} \left| \frac{1}{h} \sum_{x\le n\le x+h} \lambda(n) \right| = o(1) \qquad \text{as } h \to \infty.$$

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- By summation by parts we get

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n} \left| \frac{1}{h} \sum_{j=0}^{h} \lambda(n+j) \right| = o(1) \qquad \text{as } h \to \infty.$$

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- Thus, for any $\epsilon > 0$ and any h sufficiently large, we have

$$\left| \sum_{j=0}^{h} \lambda(n+j) \right| \le \epsilon h$$

with asymptotic probability at least $1-\epsilon$. Similarly with λ replaced by μ.

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Sketch proof for $k = 2$*:*

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- Case $(+,-)$: Assume for contradiction $(+,-)$ occurs with density zero. Then $(-,+)$ also occurs with density zero.

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- Thus $(+,+)$ and $(-,-)$ occur with combined density 1, thus $\lambda(n) = \lambda(n+1)$ a.a.s.

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- Thus $(+,+)$ and $(-,-)$ occur with combined density 1, thus $\lambda(n) = \lambda(n+1)$ a.a.s.
- By finite additivity, one obtained for any fixed h: $\lambda(n) = \lambda(n+1) = \cdots = \lambda(n+h)$ a.a.s, which contradicts our previous estimate. Similarly for $(-,+)$.

Sign patterns of the Liouville function

Sketch proof for $k = 2$:

- Case $(+, -)$: Assume for contradiction $(+, -)$ occurs with density zero. Then $(-, +)$ also occurs with density zero.
- Thus $(+, +)$ and $(-, -)$ occur with combined density 1, thus $\lambda(n) = \lambda(n+1)$ a.a.s.
- By finite additivity, one obtained for any fixed h: $\lambda(n) = \lambda(n+1) = \cdots = \lambda(n+h)$ a.a.s, which contradicts our previous estimate. Similarly for $(-, +)$.
- Case $(+, +)$. Using the $k = 1$ case and inclusion-exclusion, we have that $(-, -)$ occurs with the same density.

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- Case $(+,+)$. Using the $k = 1$ case and inclusion-exclusion, we have that $(-,-)$ occurs with the same density.
- From the pigeonhole principle, at least one of $\lambda(2n+1) = \lambda(2n)$, $\lambda(2n+2) = \lambda(2n+1)$, and $\lambda(2n) = \lambda(2n+2)$ must hold for any n, hence their probabilities must sum to at least 1.

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- But $\mathbb{P}[\lambda(n+1) = \lambda(n)]$ is average of $\mathbb{P}[\lambda(2n+1) = \lambda(2n)]$ and $\mathbb{P}[\lambda(2n+2) = \lambda(2n+1)]$, and so the density of $(+,+)$ is at least $1/6$. □

Sign patterns of the Liouville function

Case $k = 3$:

Sign patterns of the Liouville function

Case $k = 3$*:*

- Assume for contradiction $(+, +, -)$ occurs with density zero. This means that if $\lambda(n) = \lambda(n+1) = +1$, then $\lambda(n+2) = +1$ a.a.s.

Sign patterns of the Liouville function

Case $k = 3$*:*

- Assume for contradiction $(+,+,-)$ occurs with density zero. This means that if $\lambda(n) = \lambda(n+1) = +1$, then $\lambda(n+2) = +1$ a.a.s.
- Iterating this, for any fixed $h \geq 1$, we have that $\lambda(n) = \lambda(n+1) = +1$ implies $\lambda(n) = \lambda(n+1) = \cdots = \lambda(n+h) = +1$ a.a.s.

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- As $(+, +)$ occurs with density $1/6$, thus $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h) = +1$ occurs with probability at least $c > 0$ independent of h. But this contradicts Matomäki-Radziwiłł.

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- Iterating this, for any fixed $h \geq 1$, we have that $\lambda(n) = \lambda(n + 1) = +1$ implies $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h) = +1$ a.a.s.
- As $(+, +)$ occurs with density $1/6$, thus $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h) = +1$ occurs with probability at least $c > 0$ independent of h. But this contradicts Matomäki-Radziwiłł.
- Same argument works for $(-, +, +)$, $(+, -, -)$, $(-, -, +)$.

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Remaining cases: $(+,+,+),(-,-,-),(+,-,+),(-,+,-)$

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- Case $(+,+,+)$: Assume for contradiction $(+,+,+)$ occurs with density zero.
- Use the multiplicative nature of λ (using the primes $2,3,5$), one can show $\lambda(3n+1) = -\lambda(3n+2)$ a.a.s, and $\lambda(3n-1) = -\lambda(3n+1)$ a.a.s.

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- Use this to show that λ "behaves like" the Dirichlet character χ_3, i.e. $\lambda\chi_3$ is almost always constant away from multiples of 3.

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$$\left|\sum_{j=0}^{h} \lambda(n+j)\chi_3(n+j)\right| \leq \epsilon h$$

  with asymptotic probability at least $1-\epsilon$.

Sign patterns of the Liouville function

Remaining cases: $(+,+,+),(-,-,-),(+,-,+),(-,+,-)$

- Case $(+,+,+)$: Assume for contradiction $(+,+,+)$ occurs with density zero.
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Note that an analogous conjecture for μ is not true. E.g. the sign pattern $(1, 1, 1, 1)$ never occurs.

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- $\mu^2(n) = 1$ occurs with density $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.
- Thus, the events $\mu(n) = 1, \mu(n) = 0, \mu(n) = -1$ occur with asymptotic probability $\frac{1}{2\zeta(2)}, 1 - \frac{1}{\zeta(2)}, \frac{1}{2\zeta(2)}$ respectively.

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Sketch proof for $k = 2$*:*

Sign patterns of the Möbius function

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- Thus, by inclusion-exclusion, the case $(0, 0)$ has density $1 - \frac{2}{\zeta(2)} + c = 0.1067\ldots$.
- Further inclusion-exclusion shows that each of the cases $(+1, 0)$, $(-1, 0)$, $(0, +1)$, $(0, -1)$ have density $\frac{1}{2}(\frac{1}{\zeta(2)} - c) = 0.1426\ldots$

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- Can use this to create long "chains" $n+a_1, n+a_2$ on which $\mu\chi$ is constant. This is incompatible with Matomäki-Radziwiłł! □

Log averaged Chowla and Elliott conjecture

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Theorem (Tao, 2015)

Let a_1, a_2 be positive integers and b_1, b_2 be integers such that $a_1 b_2 - a_2 b_1 \neq 0$. Let $1 \leq \omega(x) \leq x$ be a function that goes to infinity as $x \to \infty$. Then one has

$$\sum_{x/\omega(x) < n \leq x} \frac{\lambda(a_1 n + b_1)\lambda(a_2 n + b_2)}{n} = o(\log \omega(x)) \qquad \text{as } x \to \infty.$$

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Theorem (Tao, 2015)

Let a_1, a_2 be natural, b_1, b_2 integers such that $a_1 b_2 - a_2 b_1 \neq 0$. Let $g_1, g_2 : \mathbb{N} \to \mathbb{C}$ be multiplicative 1-bounded functions, with g_1 "non-pretentious". Then

$$\sum_{x/\omega(x)<n\leq x} \frac{g_1(a_1 n + b_1)g_2(a_2 n + b_2)}{n} = o(\log \omega(x)) \qquad \text{as } x \to \infty.$$

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Case: $\omega(x) = x,\ a_1 = a_2 = 1,\ b_1 = 0,\ b_2 = 1$

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This implies the following corollary

Theorem (Tao, 2015)

Let $\epsilon_1, \epsilon_2 \in \{-1, +1\}$. Then the set of positive integers n such that $\lambda(n) = \epsilon_1$ and $\lambda(n+1) = \epsilon_2$ has logarithmic density $1/4$, i.e.

$$\frac{1}{\log x} \sum_{\substack{n\leq x \\ \lambda(n)=\epsilon_1, \lambda(n+1)=\epsilon_2}} \frac{1}{n} = \frac{1}{4} + o(1)$$

Log density of Möbius function

Log averaged Elliott conjectured, applied to μ and μ^2 provides the following estimates

$$\sum_{n\le x}\frac{\mu(n)\mu(n+1)}{n},\quad \sum_{n\le x}\frac{\mu^2(n)\mu(n+1)}{n},\quad \sum_{n\le x}\frac{\mu(n)\mu^2(n+1)}{n}=o(\log x)$$

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Theorem (Tao, 2015)

Let $c := \prod_p (1-\frac{2}{p^2}) = 0.3226\ldots$. *Let* $\epsilon_1, \epsilon_2 \in \{-1, 0, +1\}$.
Then the set $\{n : \mu(n) = \epsilon_1, \mu(n+1) = \epsilon_2\}$ *has logarithmic density:*

- $1 - \frac{2}{\zeta(2)} + c = 0.1067\ldots,$ *if* $(\epsilon_1, \epsilon_2) = (0, 0)$.
- $\frac{1}{2}\left(\frac{1}{\zeta(2)} - c\right) = 0.1426\ldots,$ *if* $(\epsilon_1, \epsilon_2) = (1, 0), (-1, 0), (0, 1), (0, -1)$.
- $\frac{c}{4} = 0.0806\ldots$ *otherwise.*

Erdös discrepancy problem

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Conjecture (Erdös, 1932)

Let $f : \mathbb{N} \to \{-1, +1\}$. Then

$$\sup_{n,d \in \mathbb{N}} |f(d) + f(2d) + \cdots + f(nd)| = \infty$$

(equivalently, for all $C > 0$, there exists $n, d \geq 1$ such that $|f(d) + \cdots + f(nd)| \geq C$)

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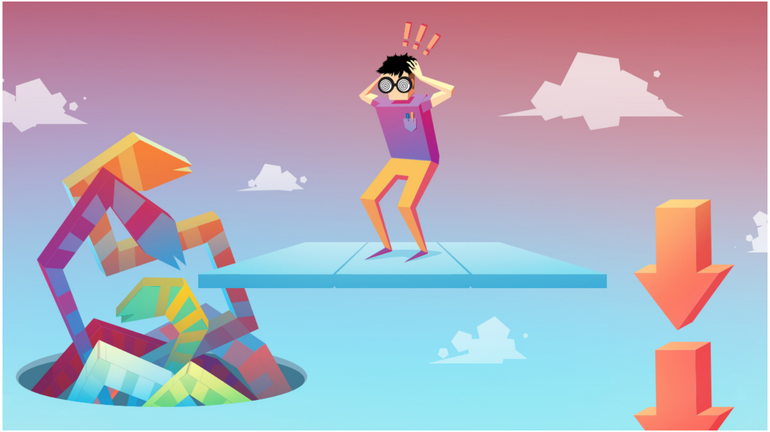
- Roth showed $\sup_{n\leq N,a,d}$ is at least $\geq \frac{1}{20}N^{1/4}$. This is best possible bound (Matousek, Spencer, 1996)

Erdös discrepancy problem



"A precipice lies two paces to your left and a pit of vipers two paces to your right. Can you devise a series of steps that will avoid the hazards, even if you are forced to take every second, third or Nth step in your series?" - Quanta Magazine, 2015

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- Case $C = 2$: The finite sequence

$$+1,\ -1,\ -1,\ +1,\ -1,\ +1,\ +1,\ -1,\ -1,\ +1,\ -1$$

has discrepancy 1. A brute force search (or by an elementary deduction argument) proves any sequence of length $N \geq 12$ has discrepancy at least 2.

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- Case $C = 3$: Konev–Lisitsa (2014) obtained a sequence of length $N = 1160$ with discrepancy 2

$$- + + - + - - + + - + + - + - - + - - + + - + - - + - - +$$
$$+ - + - - + + - + + - + - + + - - + + - + - - - + - + + - \ldots$$

  This was proven optimal using a SAT solver.

Erdös discrepancy problem

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Discrepancy bound	Completely multiplicative	Multiplicative	Unconstrained
$C = 1$	1	1	1
$C = 2$	10	12	12
$C = 3$	247	345	1161
$C = 4$	127 646	127 646	$> 130\,000$

Table: Minimum length N required for any ± 1 sequence to have discrepancy $\ge C$.

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- (Borwein-Choi-Coons): Let $\tilde{\chi}_3 : \mathbb{N} \to \mathbb{C}$ such that

$$\tilde{\chi}_3(n) := \begin{cases} +1 & \text{if } n = 3^k(3m+1) \\ -1 & \text{if } n = 3^k(3m+2) \end{cases}$$

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  Then $\chi(1) + \cdots + \chi(n)$ is the number of 1s in the base 3 expansion of n, thus this grows as $O(\log N)$.
- (Vector-valued BCC): Let H be real Hilbert space with orthonormal basis $e_0, e_1, e_2, \ldots$. Let $f : \mathbb{N} \to H$ be the function

$$f(n) := \begin{cases} +e_k & \text{if } n = 3^k(3m+1) \\ -e_k & \text{if } n = 3^k(3m+2) \end{cases}$$

  Using the Pythagorean theorem, we get $\|f(1) + \cdots + f(n)\|_H = O(\sqrt{\log n})$.

Erdös discrepancy problem

Theorem (Tao, 2015)

Let $f : \mathbb{N} \to H$ where H is some real or complex Hilbert space, such that $\|f(n)\|_H = 1$ for all n. Then

$$\sup_{n,d\in\mathbb{N}} \|f(d) + f(2d) + \cdots + f(nd)\|_H = \infty.$$

In the case of $H = \mathbb{R}$, this proves the Erdös discrepancy problem!

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- Can in principle give an effective (albeit weak) lower bound on $\sup_{n \leq N, d} \|f(d) + \cdots + f(nd)\|_H$.

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- Proof can give an effective lower bound on the growth rate, but will be weaker than $\sqrt{\log N}$.

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Awe and wonder, they shall acquire."

- ChatGPT, 2023

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Thank you!