A survey of applications to Matomäki–Radziwiłł’s theorem

Multiplicative functions study group, Week 10

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Main theorem of Matomäki-Radziwiłł

Theorem (Matomäki-Radziwiłł, 2015)

Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function and let $2 \leq h \leq X$. Then for $(1 + o(1))X$ values of $X \leq x \leq 2X$ we have

$$\left| \frac{1}{h} \sum_{x \leq n \leq x+h} f(n) - \frac{1}{X} \sum_{X \leq n \leq 2X} f(n) \right| \leq o(1) \quad \text{as } h, X \to \infty.$$
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\]

E.g. Applying the above to the Liouville function \( \lambda(n) = (-1)^{\Omega(n)} \) with \( h = X^\delta \), we get

Theorem (Matomäki-Radziwiłł, 2015)

Let \( \delta > 0 \). For \((1 + o(1))X\) values of \( X \leq x \leq 2X \), we have

\[
\sum_{x \leq n \leq x+X^\delta} \lambda(n) = o(X^\delta) \quad \text{as } X \to \infty.
\]
Main theorem of Matomäki-Radziwiłł

Theorem (Matomäki-Radziwiłł, 2015)

Let \( f : [-1, 1] \) be a multiplicative function and let \( 10 \leq h \leq x \). Then we have

\[
\frac{1}{h \sqrt{x} \log 2} \sum_{x \leq n_1 \leq x+h \sqrt{x}} f(n_1) f(n_2) = \left( \frac{1}{\sqrt{x}} \sum_{\sqrt{x} \leq n \leq 2 \sqrt{x}} f(n) \right)^2 + o(1)
\]

as \( h, X \to \infty \)
Chowla’s conjecture

Conjecture (Chowla, 1965)

Let $h_1, \ldots, h_k$ be distinct natural numbers. Then

$$\sum_{1 \leq n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X) \quad \text{as } X \to \infty.$$
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- Compare with twin prime conjecture: $\sum_{n \leq x} \theta(n)\theta(n + 2) \to \infty$ (where $\theta(p) := \log p$ if $p$ prime, and 0 otherwise)
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- Compare with twin prime conjecture: $\sum_{n \leq x} \theta(n)\theta(n + 2) \to \infty$ (where $\theta(p) := \log p$ if $p$ prime, and 0 otherwise)
- Could generalise conjecture to

$$\sum_{1 \leq n \leq x} \lambda(a_1 n + h_1) \cdots \lambda(a_k n + h_k) = o(X)$$

such that $a_i h_j - a_j h_i \neq 0$ for all $i < j$. 


Averaged form of Chowla’s conjecture

**Theorem (Matomäki–Radziwiłł–Tao (2015))**

For any natural number $k$, and any $10 \leq H \leq X$, we have

$$\sum_{1 \leq h_1, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n + h_1) \cdots \lambda(n + h_k) \right| = o(H^k X)$$

provided that $H \to \infty$ arbitrarily slowly with $X \to \infty$. 

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- Also have slightly stronger bound:

\[
\sum_{1 \leq h_2, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n)\lambda(n + h_2) \cdots \lambda(n + h_k) \right| = o(H^{k-1} X)
\]
Sketch proof

Can use the Fourier-analytic identity:

\[
\int_{X}^{X+H} f(n) e^{i \alpha n} dn = \sum_{h} |h| \leq N(H - |h|) \int_{X}^{X+H} \bar{f}(n) f(n+h) dn = o(HX)
\]

uniformly for all \( \alpha \in \mathbb{T} \).

Using the circle method:

• Majors arcs: Use the result from Matomäki-Radziwiłł.
• Minor arcs: Use an argument of Katai and Bourgain–Sarnak–Ziegler.
Sketch proof

Can use the Fourier-analytic identity:

$$\int_{\mathbb{T}} \left( \int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+H} f(n)e(\alpha n) \right|^2 \, dx \right)^2 \, d\alpha = \sum_{|h| \leq N} (H - |h|)^2 \left| \sum_{n} f(n)\bar{f}(n + h) \right|^2$$

to reduce the proof for $k = 2$ to showing an estimate of the form:

$$\int_{0}^{X} \left| \sum_{x \leq n \leq x+H} \lambda(n)e(\alpha n) \right| \, dx = o(HX)$$

uniformly for all $\alpha \in \mathbb{T}$. 
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Elliott’s conjecture

• Can we replace $\lambda$ with arbitrary 1-bounded multiplicative functions $g_i : \mathbb{N} \to \mathbb{C}$?
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• Must be careful to avoid Dirichlet characters, e.g. $g_1 = \chi$ and $g_2 = \overline{\chi}$, then $g_1(n)g_2(n + h)$ will be positive biased.
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- Must be careful to avoid Dirichlet characters, e.g. $g_1 = \chi$ and $g_2 = \overline{\chi}$, then $g_1(n)g_2(n + h)$ will be positive biased.

Conjecture (Elliott, 1992)

Let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ be positive integers such that $a_i b_j - a_j b_i \neq 0$. Suppose there is an index $\ell$ such that $g_\ell$ satisfies

$$\sum_{p} \frac{1 - \text{Re}(g_\ell(p)\chi(p)p^{-it})}{p} = \infty$$

for all Dirichlet characters $\chi$ and $t \in \mathbb{R}$. (i.e. $g_\ell$ does not behave “like a Dirichlet character”). Then

$$\sum_{1 \leq n \leq X} g_1(a_1 n + b_1) \cdots g_k(a_k n + b_k) = o(X) \quad \text{as} \quad X \to \infty.$$
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- This is **not** true in general (if $g_j$ complex-valued).
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- Matomäki-Radziwiłł–Tao construct a counterexample, where $g_\ell$ can be arbitrarily close to a sequence of functions of the form $n \mapsto n^{it_m}$ without **globally** pretending to be $n^{it}$ for any fixed $t$. 
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Elliott’s conjecture (corrected)

Let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ be positive integers such that $a_ib_j - a_jb_i \neq 0$. Suppose there is an index $\ell$ such that $g_\ell$ satisfies

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$$\sum_{1 \leq n \leq X} g_1(a_1 n + b_1) \cdots g_k(a_k n + b_k) = o(X) \quad \text{as} \quad X \to \infty.$$
Averaged form of Elliott’s conjecture

**Theorem (Matomäki-Radziwiłł–Tao (2015))**

Let $10 \leq H \leq X$ and $A \geq 1$. Let $g_1, \ldots, g_k : \mathbb{N} \to \mathbb{C}$ be 1-bounded functions with at least one $g_{\ell}$ being multiplicative and “non-pretentious”. Then

$$
\sum_{1 \leq h_1, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} g_1(a_1 n + b_1 + h_1) \cdots g_k(a_k n + b_k + h_k) \right| = o(H^k X)
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as $H, X \to \infty$. 

• Also have slightly stronger bound:

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as $H, X \to \infty$. 
Conjecture (Chowla, 1965)
For any positive integer \( k \), and choice of signs \( \epsilon_1, \ldots, \epsilon_k \in \{-1, 1\} \), the set of positive integers \( n \) such that 
\[
\lambda(n+1) = \epsilon_1, \quad \lambda(n+2) = \epsilon_2, \ldots, \quad \lambda(n+k) = \epsilon_k,
\]
has density \( \frac{1}{2^k} \).

• For \( k = 1 \), this follows by the prime number theorem:
\[
P_n \leq X \lambda(n) = o(X).
\]

Theorem (Hildebrand, 1986)
Let \( \epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\} \). Then there exist infinitely many \( n \) such that
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\lambda(n+1) = \epsilon_1, \quad \lambda(n+2) = \epsilon_2, \quad \lambda(n+3) = \epsilon_3.
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Proof uses ad hoc elementary arguments, using the multiplicative property of \( \lambda \) with the primes 2, 3, 5.
Sign patterns of the Liouville function

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**Theorem (Matomäki-Radziwiłł–Tao, 2015)**

Let $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, +1\}$. Then there are a positive (lower) density of integers $n$ such that

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- In 2017, Klurman–Manerla proved the upper logarithmic density is at least $1/28$.
- In 2017, Tao–Teräväinen proven the logarithmic density for $k = 3$ is exactly $\frac{1}{8}$, and for $k = 4$ is at least $\frac{1}{32}$.
Sign patterns of the Liouville function

- By Matomäki-Radziwiłł, we have

\[
\limsup_{X \to \infty} \frac{1}{X} \sum_{X \leq x \leq 2X} \left| \frac{1}{h} \sum_{x \leq n \leq x+h} \lambda(n) \right| = o(1) \quad \text{as } h \to \infty.
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- By summation by parts we get

\[
\limsup_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} \lambda(n) = o(1) \quad \text{as } h \to \infty.
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- Thus, for any \( \epsilon > 0 \) and any \( h \) sufficiently large, we have

\[
\sum_{j=0}^{h} \lambda(n+j) \leq \epsilon h \quad \text{with asymptotic probability at least } 1 - \epsilon.
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Similarly with \( \lambda \) replaced by \( \mu \).
Sign patterns of the Liouville function

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with asymptotic probability at least \( 1 - \epsilon \). Similarly with \( \lambda \) replaced by \( \mu \).
Sign patterns of the Liouville function

Sketch proof for $k = 2$:

• Case $(+,-)$: Assume for contradiction $(+, -)$ occurs with density zero. Then $(-, +)$ also occurs with density zero.

• Thus $(+, +)$ and $(-, -)$ occur with combined density 1, thus $\lambda(n) = \lambda(n+1)$ a.a.s.

• By finite additivity, one obtained for any fixed $h$: 
  
  $\lambda(n) = \lambda(n+1) = \cdots = \lambda(n+h)$

  a.a.s, which contradicts our previous estimate. Similarly for $(-, +)$.

• Case $(+, +)$. Using the $k = 1$ case and inclusion-exclusion, we have that $(-, -)$ occurs with the same density.

• From the pigeonhole principle, at least one of $\lambda(2n+1) = \lambda(2n)$, $\lambda(2n+2) = \lambda(2n+1)$, and $\lambda(2n) = \lambda(2n+2)$ must hold for any $n$, hence their probabilities must sum to at least 1.

• But $P[\lambda(n+1) = \lambda(n)]$ is average of $P[\lambda(2n+1) = \lambda(2n)]$ and $P[\lambda(2n+2) = \lambda(2n+1)]$, and so the density of $(+, +)$ is at least $1/6$. 
Sign patterns of the Liouville function

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Sign patterns of the Liouville function

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• Thus $(+, +)$ and $(-, -)$ occur with combined density 1, thus $\lambda(n) = \lambda(n + 1)$ a.a.s.
Sign patterns of the Liouville function

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- Thus $(+, +)$ and $(-, -)$ occur with combined density 1, thus $\lambda(n) = \lambda(n + 1)$ a.a.s.

- By finite additivity, one obtained for any fixed $h$: $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h)$ a.a.s, which contradicts our previous estimate. Similarly for $(-, +)$. 
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- By finite additivity, one obtained for any fixed $h$: $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h)$ a.a.s, which contradicts our previous estimate. Similarly for $(-, +)$.
- Case $(+, +)$. Using the $k = 1$ case and inclusion-exclusion, we have that $(-, -)$ occurs with the same density.
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- Thus $(+, +)$ and $(-, -)$ occur with combined density 1, thus $\lambda(n) = \lambda(n + 1)$ a.a.s.
- By finite additivity, one obtained for any fixed $h$: $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h)$ a.a.s, which contradicts our previous estimate. Similarly for $(-, +)$.
- Case $(+, +)$. Using the $k = 1$ case and inclusion-exclusion, we have that $(−, −)$ occurs with the same density.
- From the pigeonhole principle, at least one of $\lambda(2n + 1) = \lambda(2n)$, $\lambda(2n + 2) = \lambda(2n + 1)$, and $\lambda(2n) = \lambda(2n + 2)$ must hold for any $n$, hence their probabilities must sum to at least 1.
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Sketch proof for $k = 2$:

- Case $(+, -)$: Assume for contradiction $(+, -)$ occurs with density zero. Then $(-, +)$ also occurs with density zero.
- Thus $(+, +)$ and $(-, -)$ occur with combined density 1, thus $\lambda(n) = \lambda(n + 1)$ a.a.s.
- By finite additivity, one obtained for any fixed $h$: $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h)$ a.a.s, which contradicts our previous estimate. Similarly for $(-, +)$.
- Case $(+, +)$. Using the $k = 1$ case and inclusion-exclusion, we have that $(-, -)$ occurs with the same density.
- From the pigeonhole principle, at least one of $\lambda(2n + 1) = \lambda(2n)$, $\lambda(2n + 2) = \lambda(2n + 1)$, and $\lambda(2n) = \lambda(2n + 2)$ must hold for any $n$, hence their probabilities must sum to at least 1.
- But $P[\lambda(n + 1) = \lambda(n)]$ is average of $P[\lambda(2n + 1) = \lambda(2n)]$ and $P[\lambda(2n + 2) = \lambda(2n + 1)]$, and so the density of $(+, +)$ is at least $1/6$. 

Case $k = 3$:

Assume for contradiction $(+, +, +)$ occurs with density zero. This means that if $\lambda(n) = \lambda(n + 1) = +1$, then $\lambda(n + 2) = +1$ a.a.s.

Iterating this, for any fixed $h \geq 1$, we have that $\lambda(n) = \lambda(n + 1) = +1$ implies $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h) = +1$ a.a.s.

As $(+, +)$ occurs with density $1/6$, thus $\lambda(n) = \lambda(n + 1) = \cdots = \lambda(n + h) = +1$ occurs with probability at least $c > 0$ independent of $h$. But this contradicts Matomäki-Radziwiłł.

Same argument works for $(-, +, +)$, $(+, -, -)$, $(-, -, +)$.
Case $k = 3$:

- Assume for contradiction $(+, +, -)$ occurs with density zero. This means that if $\lambda(n) = \lambda(n + 1) = +1$, then $\lambda(n + 2) = +1$ a.a.s.
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Sign patterns of the Liouville function

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Sign patterns of the Liouville function

Remaining cases: (+, +, +), (−, −, −), (+, −, +), (−, +, −)

• Case (+, +, +): Assume for contradiction (+, +, +) occurs with density zero.

• Use the multiplicative nature of $\lambda$ (using the primes 2, 3, 5), one can show $\lambda(3^n + 1) = −\lambda(3^n + 2)$ a.a.s, and $\lambda(3^n − 1) = −\lambda(3^n + 1)$ a.a.s.

• Use this to show that $\lambda$ “behaves like” the Dirichlet character $\chi_3$, i.e. $\lambda\chi_3$ is almost always constant away from multiples of 3.

• Use Matom¨ aki-Radziwi l l on $\lambda\chi_3$, i.e. for any $\epsilon > 0$ and sufficiently large $h$

\[\sum_{j=0}^{X} \lambda(n+j)\chi_3(n+j) \leq \epsilon h\] with asymptotic probability at least $1 − \epsilon$.

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with asymptotic probability at least $1 - \epsilon$. 

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Note that an analogous conjecture for $\mu$ is not true. E.g. the sign pattern $(1, 1, 1, 1)$ never occurs.
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**Theorem (Matomäki-Radziwiłł–Tao, 2015)**

Let $\epsilon_1, \epsilon_2 \in \{-1, 0, +1\}$. Then there are a positive (lower) density of integers $n$ such that $\mu(n) = \epsilon_1$ and $\mu(n + 1) = \epsilon_2$. 

• $\mu(n) = 1$ occurs with density $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.
• Thus, the events $\mu(n) = 1$, $\mu(n) = 0$, $\mu(n) = -1$ occur with asymptotic probability $\frac{1}{2}\zeta(2)$, $\frac{1}{2}\zeta(2)$, $\frac{1}{2}\zeta(2)$ respectively.
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Sign patterns of the Möbius function

Sketch proof for $k = 2$:

- Can use standard sieve theory arguments to deduce that the probability that $\mu_2(n) = \mu_2(n+1) = 1$ is $c := \prod_{p} 1 - 2p^2 = 0$.
- The case $k = 1$ and the Chinese remainder theorem proves that density for $(0, +1)$ and $(0, -1)$ are the same. Similarly, the density for $(+1, 0)$ and $(-1, 0)$ are the same.
- Thus, by inclusion-exclusion, the case $(0, 0)$ has density $1 - 2\zeta(2) + c = 0$.
- Further inclusion-exclusion shows that each of the cases $(+1, 0)$, $(-1, 0)$, $(0, +1)$, $(0, -1)$ have density $\frac{1}{2}(1 - \zeta(2) - c) = 0$. 

Sign patterns of the Möbius function

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- Further inclusion-exclusion shows that each of the cases \((+1, 0)\), \((-1, 0)\), \((0, +1)\), \((0, -1)\) have density \( \frac{1}{2} \left( \frac{1}{\zeta(2)} - c \right) = 0.1426 \ldots \)
Sign patterns of the Möbius function

Remaining cases: \((+1, +1), (+1, -1), (-1, +1), (-1, -1)\).
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- Can use this to create long “chains” \(n + a_1, n + a_2\) on which \(\mu\chi\) is constant. This is incompatible with Matomäki-Radziwiłł!
Theorem (Tao, 2015)

Let $a_1, a_2$ be positive integers and $b_1, b_2$ be integers such that $a_1 b_2 - a_2 b_1 \neq 0$. Let $1 \leq \omega(x) \leq x$ be a function that goes to infinity as $x \to \infty$. Then one has

$$X_{x/\omega(x)} < n \leq x \lambda(a_1 n + b_1) \lambda(a_2 n + b_2) n = o(\log \omega(x))$$

as $x \to \infty$.

Theorem (Tao, 2015)

Let $a_1, a_2$ be natural, $b_1, b_2$ integers such that $a_1 b_2 - a_2 b_1 \neq 0$. Let $g_1, g_2 : \mathbb{N} \to \mathbb{C}$ be multiplicative 1-bounded functions, with $g_1$ "non-pretentious". Then

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Log averaged Chowla and Elliott conjecture
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Log averaged Chowla

Case: $\omega(x) = x, a_1 = a_2 = 1, b_1 = 0, b_2 = 1$
Log averaged Chowla

Case: \( \omega(x) = x, \ a_1 = a_2 = 1, \ b_1 = 0, \ b_2 = 1 \)

**Theorem (Tao, 2015)**

\[
\sum_{1 < n \leq x} \frac{\lambda(n)\lambda(n + 1)}{n} = o(\log x) \text{ as } x \to \infty.
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Log averaged Chowla

Case: $\omega(x) = x$, $a_1 = a_2 = 1$, $b_1 = 0$, $b_2 = 1$

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This implies the following corollary

**Theorem (Tao, 2015)**

Let $\epsilon_1, \epsilon_2 \in \{-1, +1\}$. Then the set of positive integers $n$ such that $\lambda(n) = \epsilon_1$ and $\lambda(n+1) = \epsilon_2$ has logarithmic density $1/4$, i.e.

$$\frac{1}{\log x} \sum_{n \leq x} \frac{1}{n} = \frac{1}{4} + o(1)$$

Note, if natural density exists, then it is equal to the log density. But can have log density existing, but natural density doesn't exits.
Log density of Möbius function

Log averaged Elliott conjectured, applied to $\mu$ and $\mu^2$ provides the following estimates

\[
\sum_{n \leq x} \frac{\mu(n)\mu(n + 1)}{n}, \quad \sum_{n \leq x} \frac{\mu^2(n)\mu(n + 1)}{n}, \quad \sum_{n \leq x} \frac{\mu(n)\mu^2(n + 1)}{n} = o(\log x)
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Theorem (Tao, 2015)

Let $c := \prod_p (1 - \frac{2}{p^2}) = 0.3226 \ldots$. Let $\epsilon_1, \epsilon_2 \in \{-1, 0, +1\}$. Then the set \{n : $\mu(n) = \epsilon_1, \mu(n+1) = \epsilon_2$\} has logarithmic density:

- $1 - \frac{2}{\zeta(2)} + c = 0.1067 \ldots$, if $(\epsilon_1, \epsilon_2) = (0, 0)$.
- $\frac{1}{2} \left( \frac{1}{\zeta(2)} - c \right) = 0.1426 \ldots$, if $(\epsilon_1, \epsilon_2) = (1, 0), (-1, 0), (0, 1), (0, -1)$.
- $\frac{c}{4} = 0.0806 \ldots$ otherwise.
Erdős discrepancy problem

Conjecture (Erdős, 1932)

Let \( f : \mathbb{N} \rightarrow \{-1, +1\} \). Then

\[
\sup_{n, d \in \mathbb{N}} |f(d) + f(2d) + \cdots + f(nd)| = \infty
\]

(equivalently, for all \( C > 0 \), there exists \( n, d \geq 1 \) such that

\[
|f(d) + \cdots + f(nd)| \geq C
\]

• Van der Waerden's theorem (1927) implies that

\[
\sup_{a, n, d \geq 1} |f(a) + f(a+d) + \cdots + f(a+(n-1)d)| = \infty
\]

• Roth showed \( \sup_{n \leq N, a, d} \geq \frac{1}{2} \frac{N}{4} \). This is best possible bound (Matousek, Spencer, 1996)
**Erdős discrepancy problem**

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- Roth showed $\sup_{n \leq N, a, d}$ is at least $\geq \frac{1}{20} N^{1/4}$. This is best possible bound (Matousek, Spencer, 1996)
Erdös discrepancy problem

“A precipice lies two paces to your left and a pit of vipers two paces to your right. Can you devise a series of steps that will avoid the hazards, even if you are forced to take every second, third or Nth step in your series?” - Quanta Magazine, 2015
Erdős discrepancy problem

Erdős:  For all $C > 0$, there exists some $n, d \geq 1$ such that $|f(d) + \cdots + f(nd)| \geq C$. 

• Case $C = 1$: Clearly true for $n = 1$.

• Case $C = 2$: The finite sequence $+1, -1, -1, +1, -1, +1, +1, -1, -1, +1, -1$ has discrepancy 1. A brute force search (or by an elementary deduction argument) proves any sequence of length $N \geq 12$ has discrepancy at least 2.

• Case $C = 3$: Konev–Lisitsa (2014) obtained a sequence of length $N = 1160$ with discrepancy $2^{22}$.
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$$- + + - + - - + - + + - + - - + - + - + - - - + -$$

$$+ - + - + + - - + + - - + - - + - + - + - + - + + - \ldots$$

This was proven optimal using a SAT solver.
Erdös discrepancy problem

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- Can construct examples where $\sup_{n \leq N, d} |f(1) + \cdots + f(nd)| \ll \log N$. 

<table>
<thead>
<tr>
<th>$C$</th>
<th>Minimum length $N$ required for any $\pm 1$ sequence to have discrepancy $\geq C$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$10$</td>
</tr>
<tr>
<td>$3$</td>
<td>$247$</td>
</tr>
<tr>
<td>$4$</td>
<td>$127,646$</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
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<th>Completely multiplicative</th>
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</tr>
</thead>
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<td>12</td>
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<tr>
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<td>247</td>
<td>345</td>
<td>1161</td>
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Table: Minimum length $N$ required for any $\pm 1$ sequence to have discrepancy $\geq C$. 
**Examples**

- Conjecture is false if any positive density of zeros are allowed: E.g. Let $\chi: \mathbb{N} \to \mathbb{C}$ be a non-principal Dirichlet character of period $q$. Then $|\chi(d) + \chi(2d) + \cdots + \chi(nd)| \leq q$.

- (Borwein-Choi-Coons): Let $\tilde{\chi}_3: \mathbb{N} \to \mathbb{C}$ such that $\tilde{\chi}_3(n) := +1$ if $n = 3k(3m + 1)$ $-1$ if $n = 3k(3m + 2)$. Then $\chi(1) + \cdots + \chi(n)$ is the number of 1s in the base 3 expansion of $n$, thus this grows as $O(\log N)$.

- (Vector-valued BCC): Let $H$ be a real Hilbert space with orthonormal basis $e_0, e_1, e_2, \ldots$. Let $f: \mathbb{N} \to H$ be the function $f(n) := +e_k$ if $n = 3k(3m + 1)$ $-e_k$ if $n = 3k(3m + 2)$. Using the Pythagorean theorem, we get $\|f(1) + \cdots + f(n)\|_H = O(\sqrt{\log n})$. 


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Erdős discrepancy problem

**Theorem (Tao, 2015)**

Let \( f : \mathbb{N} \rightarrow H \) where \( H \) is some real or complex Hilbert space, such that \( \| f(n) \|_H = 1 \) for all \( n \). Then

\[
\sup_{n,d \in \mathbb{N}} \| f(d) + f(2d) + \cdots + f(nd) \|_H = \infty.
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*In the case of \( H = \mathbb{R} \), this proves the Erdős discrepancy problem!*
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In the case of $H = \mathbb{R}$, this proves the Erdős discrepancy problem!

- Can in principle give an effective (albeit weak) lower bound on

  $$\sup_{n \leq N, d} \|f(d) + \cdots + f(nd)\|_H.$$
Sketch proof

- Can use a Fourier analytic argument to reduce the theorem to showing that $\sup_{n \in \mathbb{N}} E|g(1) + g(2) + \cdots + g(n)|^2 = \infty$ where $g: \mathbb{N} \to S_1$ is a (stochastic) completely multiplicative function.

- Since $|g(1) + \cdots + g(n)|^2 = P|g(n)|^2 + P_{i \neq j} g(i)g(j)$, it suffices to study the covariances $P_n \leq x g(n)g(n+h)$.

- Use the log averaged Elliott theorem to show that any counterexample to the above must behave like a Dirichlet character $n \mapsto \chi(n)$.

- Do a similar (but more intricate) analysis of the Borwein-Choi-Coons example to establish a lower bound for $E|g(1) + \cdots + g(n)|^2$.

- Proof can give an effective lower bound on the growth rate, but will be weaker than $\sqrt{\log N}$. 
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- Can use a Fourier analytic argument to reduce the theorem to showing that

\[ \sup_{n \in \mathbb{N}} \mathbb{E} |g(1) + g(2) + \cdots + g(n)|^2 = \infty \]

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"In number theory's land of yore,
Where mysteries and conjectures soar,
Matomäki and Radziwiłł's paper stands,
A testament to their brilliant hands.
Short and long averages, they relate,
A powerful tool to analyze
f's fate,
With cancellations for Möbius shown,
And Chowla's conjecture slowly grown.
A masterpiece of theory and proof,
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References

A note on the Liouville function in short intervals

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Tao, T. (2016)
The logarithmically averaged Chowla and Elliott conjectures for two-point correlations

Tao, T. (2016)
The Erdős discrepancy problem
Thank you!