Computing genus 2 curves over $\mathbb{Q}$ whose Jacobian has good reduction away from 2

The Mordell conjecture 100 years later, Massachusetts Institute of Technology

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Genus 2 curves

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List all genus 2 curves $C/\mathbb{Q}$ whose Jacobians have good reduction away from 2.

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So far, we've found 512 examples of genus 2 curves $C/\mathbb{Q}$ such that $\mathrm{Jac}(C)$ is good outside 2.

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Easier Problem

Fix a small set of primes S. Find all genus 2 curves $C/\mathbb{Q}$ with good reduction outside S and whose Jacobian has good reduction away from 2.

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Rough Idea:

- Let $C/\mathbb{Q} : y^2 = c(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)(x - \alpha_5)(x - \alpha_6)$ be such a curve, where $\alpha_i \in \mathbb{Q}(J[2])$.

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- Let $\psi_1, \psi_2, \ldots, \psi_t$ be a set of S-unit generators over $\mathbb{Q}(J[2])$.
  For each pair $1 \le i < j \le 6$, let $\alpha_i - \alpha_j = \psi_1^{a_{1,i,j}} \psi_2^{a_{2,i,j}} \cdots \psi_t^{a_{t,i,j}}$.

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- Solve the linear system! Brute force, integer programming, closest vector problem...

Summary

Theorem (V. WIP)

There are at least 512 $\mathbb{Q}$-isomorphism classes of genus 2 curves $C/\mathbb{Q}$ whose Jacobian has good reduction away from 2. In particular,

1. *There are exactly 78 such genus 2 curves $C/\mathbb{Q}$ with $rad(\Delta_{min}) = 6$.*
2. *There are exactly 28 such genus 2 curves $C/\mathbb{Q}$ with $rad(\Delta_{min}) = 10$.*
3. *There are exactly 24 such genus 2 curves $C/\mathbb{Q}$ with $rad(\Delta_{min}) = 14$.*

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- Are there any curves $C/\mathbb{Q}$ with $\mathrm{Jac}(C)$ good outside 2 not in our list?
- If you can find any more curves, you'll win £100.[1]

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[1]Terms and conditions apply!