Murmurations in Arithmetic
Murmurations study group, Introductory talk

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Murmurations

Figure: A murmuration of starlings at Gretna - Walter Baxter (cc-by-sa/2.0)
Motivation

Let $E/\mathbb{Q}$ be an elliptic curve. Recall its $L$-function

$$L(E, s) = \prod_{p \text{ prime}} L_p(E, s)^{-1} = \sum_{n \geq 1} a_n(E)n^{-s}$$

where for primes $p$ of good reduction, we have $L_p(E, s) = 1 - a_p(E)p^{-s} + p^{1-2s}$ where $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$. 

1. For a fixed elliptic curve $E/\mathbb{Q}$, how is $a_p(E)/\sqrt{p}$ distributed over all primes $p$?

2. For a fixed prime $p$, how is $a_p(E)$ distributed over all elliptic curves $E/\mathbb{F}_p$?

3. What if we restrict to elliptic curves $E/\mathbb{Q}$ of given rank and conductor, and investigate $a_p(E)$ as $p$ grows linearly with the conductor?
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Motivation

1. This was a famous conjecture of Mikio Sato and John Tate. e.g. for an elliptic curve $E / \mathbb{Q}$ without CM, the probability measure of $\theta := \arccos\left(\frac{a_p(E)}{2\sqrt{p}}\right)$ is proportional to $\sin^2 \theta d\theta$. Now a theorem (by many authors)!
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2. This is the same as the Sato-Tate distribution, i.e. for a fixed $p$, the distribution of $\theta := \arccos \left( \frac{a_p(E)}{2\sqrt{p}} \right)$ over all $E/\mathbb{F}_p$ is proportional to $\sin^2 \theta d\theta$ for large $p$ (Birch, 1968).
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3. By restricting to elliptic curves $E/\mathbb{Q}$ with given rank $r$ and conductor $N \in [N_1, N_2]$, and investigating the average of $a_p(E)$ as $p \sim N$, this gives rise to the murmurations phenomenon!
Machine-learning experiments

During 2019-2022, Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov conducted some machine-learning experiments on datasets of arithmetic curves.

In one of their experiments, they represented an elliptic curve $E/\mathbb{Q}$ as a vector of its first 1000 values of $a_p(E)$:

$$v_L(E) := (a_2(E), a_3(E), a_5(E), \ldots, a_{7919}(E)) \in \mathbb{Z}^{1000}.$$ 

Using logistic regression, they were able to predict the rank of $E$ from $v_L(E)$ with very high accuracy, e.g. to distinguish between rank 0 and rank 1 curves, the goal is to find $w \in \mathbb{R}^{1000}$ and $b \in \mathbb{R}$ such that $\sigma(v_L(E) \cdot w + b)$, where $\sigma(x) = \frac{1}{1 + e^{-x}}$ is hopefully close to either 0 or 1. The results of their experiments successfully predicted the ranks all with accuracies above 96%.
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$$\sigma(v_L(E) \cdot \mathbf{w} + b), \quad \text{where} \quad \sigma(x) = \frac{1}{1 + e^{-x}}$$

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To interpret their model, they plotted the average values of $a_p(E)$ over some conductor interval $[N_1, N_2]$ for elliptic curves with fixed rank:
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Fix some \( r \geq 0 \), and some positive integers \( N_2 > N_1 \geq 1 \). Let \( \mathcal{E}_r[N_1, N_2] \) be a set of isogeny class representatives of all rank \( r \) elliptic curves of conductor \( N \in [N_1, N_2] \). Define the following function:

\[
f_r(n) := \frac{1}{\#\mathcal{E}_r[N_1, N_2]} \sum_{E \in \mathcal{E}_r[N_1, N_2]} a_{p_n}(E)
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where \( p_n \) is the \( n \)-th prime number.
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Murmurations of elliptic curves

Figure: Scatter plot of \((n, f_r(n))\) for ranks \(r = 0\) (blue) and \(r = 1\) (red) with conductor \(N\) between \(N_1 = 7500\) and \(N_2 = 10000\) (He–Lee–Oliver–Pozdnyakov 2022).
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Figure: Scatter plot of \((n, f_r(n))\) for ranks \(r = 0\) (blue) and \(r = 2\) (green) with conductor \(N\) between \(N_1 = 5000\) and \(N_2 = 10000\) (He–Lee–Oliver–Pozdnyakov 2022).
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Murmurations of elliptic curves

Figure: Murmurations - Alain Delorme
More murmurations

Do we see murmurations in larger conductor intervals?
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Figure: Scatter plot of $(n, f_r(n))$ for ranks $r = 0$ (blue) and $r = 1$ (red) with conductor $N$ between $N_1 = 2^{14}$ and $N_2 = 2^{15}$ for all $p_n \leq 2^{15}$. 
More murmurations

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Figure: Scatter plot of \((n, f_r(n))\) for ranks \(r = 0\) (blue) and \(r = 1\) (red) with conductor \(N\) between \(N_1 = 2^{15}\) and \(N_2 = 2^{16}\) for all \(p_n \leq 2^{16}\).
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In August 2022, Drew Sutherland wrote a letter to Mike Rubinstein and Peter Sarnak, where he made the following observations:

• Murmurations occur over a wide range of conductor intervals. For a fixed $c$, plotting the averages $E_{a_p}$ over the conductor interval $[X, cX]$ seems to give the same shape ($X \to \infty$).

• Ordering by conductor is important! Ordering by absolute discriminant, naive height, Faltings height, or almost anything else won't clearly give oscillations.

• This phenomenon is not specific to elliptic curves, and can be seen for many families of arithmetic $L$-functions. E.g. Dirichlet characters, higher dimension abelian varieties, newforms for $\Gamma_0(N)$, higher genus curves, etc.

• This phenomenon appears to only occur in primitive arithmetic $L$-function, e.g. no oscillations are visible when plotting $L$-functions of products of elliptic curves.
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Drew’s Letter

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Murmurations for Dirichlet characters

Theorem (Lee–Oliver–Pozdnyakov 2023)

Assume RH. Let \( D^+ (N) \) (resp. \( D^- (N) \)) denote the set of primitive even (resp. odd) Dirichlet characters mod \( N \). Fix some \( \delta \in (\frac{1}{2}, 1) \), and let \( y := \frac{P}{X} \). Then

\[
\lim_{X \to \infty} \frac{\log X}{X^\delta} \sum_{N \in [X, X+X^{\delta}]} \sum_{\chi \in D^\pm (N)} \chi(P) G(\chi) = \begin{cases} 
\cos (2\pi y), & \text{if } + \\
-i \sin (2\pi y), & \text{if } -\end{cases}
\]

where \( G(\chi) := \sum_{a=1}^{m} \chi(a) e^{2\pi i a / m} \) is the Gauss sum of \( \chi \).

Proof uses the Fourier expansion of additive characters in terms of Dirichlet characters, the prime number theorem, and elementary analysis on \( \mathbb{R} \) (Pozdnyakov 2023).
Assume RH. Let $\mathcal{D}_+(N)$ (resp. $\mathcal{D}_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters mod $N$. Fix some $\delta \in (\frac{1}{2}, 1)$, and let $y := P/X$. Then

$$\lim_{X \to \infty} \frac{\log X}{X^\delta} \sum_{N \in [X, X+X^\delta]} \sum_{N \text{ prime}} \frac{\chi(P)}{G(\chi)} = \begin{cases} \cos(2\pi y), & \text{if } +, \\ -i \sin(2\pi y), & \text{if } -, \end{cases}$$

where $G(\chi) := \sum_{a=1}^m \chi(a)e^{2\pi ia/m}$ is the Gauss sum of $\chi$. 

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Murmurations for Dirichlet characters

Theorem (Lee–Oliver–Pozdnyakov 2023)

Assume RH. Let $D_+(N)$ (resp. $D_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters mod $N$. Fix some $\delta \in (\frac{1}{2}, 1)$, and let $y := P/X$. Then

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Murmurations for Dirichlet characters

To obtain averages over some geometric interval \([X, cX]\), integrate over the interval \([1, c]\):

**Theorem (Lee–Oliver–Pozdnyakov 2023)**

*Fix some \(c > 1\). Let \(y := P/X\). Then*

\[
\lim_{X \to \infty} \frac{\log X}{X} \sum_{\substack{N \in [X, cX] \text{ prime} \atop N \in \mathcal{D}_\pm(N)}} \frac{\chi(P)}{G(\chi)} = \begin{cases} 
\int_1^c \cos \left( \frac{2\pi y}{u} \right) du, & \text{if } + , \\
-i \int_1^c \sin \left( \frac{2\pi y}{u} \right) du, & \text{if } - .
\end{cases}
\]
Murmurations for Dirichlet characters

Figure: Scatter plot of $(n, \mathbb{E}_N \mathbb{E}_\chi(p_n)/G(\chi))$ for even (blue) and odd (red) primitive Dirichlet characters $\chi$ with level $N$ between $N_1 = 2^6$ and $N_2 = 2^7$ for all $p_n \leq 2^8$. 
Murmurations for Dirichlet characters

Figure: Scatter plot of \( (n, \mathbb{E}_N \mathbb{E}_\chi \chi(p_n) / G(\chi)) \) for even (blue) and odd (red) primitive Dirichlet characters \( \chi \) with level \( N \) between \( N_1 = 2^7 \) and \( N_2 = 2^8 \) for all \( p_n \leq 2^9 \).
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Figure: Scatter plot of \((n, \mathbb{E}_N \mathbb{E}_\chi(p_n)/G(\chi))\) for even (blue) and odd (red) primitive Dirichlet characters \(\chi\) with level \(N\) between \(N_1 = 2^8\) and \(N_2 = 2^9\) for all \(p_n \leq 2^{10}\).
Murmurations of Dirichlet characters

**Theorem (Lee–Oliver–Pozdnyakov 2023)**

Let $\mathcal{D}_\pm(N)$ be as before and let $\mathcal{I}_\pm(N)$ (resp. $\mathcal{I}_-(N)$) denote the set of imprimitive even (resp. odd) nontrivial Dirichlet characters mod $N$. Fix $\delta \in (0, 1)$ and $y := P/X$. Then

$$\lim_{X \to \infty} \frac{1}{X^\delta} \sum_{N \in [X, X+X^\delta]} \left( \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(P)}{G(\chi)} \pm \frac{1}{N} \sum_{\chi \in \mathcal{I}_\pm(N)} G(\overline{\chi})\chi(P) \right) = \begin{cases} \frac{5}{\pi^2} \cos(2\pi y), & \text{if } +, \\ -i \frac{5}{\pi^2} \sin(2\pi y), & \text{if } -, \end{cases}$$

where $G(\chi)$ is the Gauss sum of $\chi$. Similarly, for some fixed $c > 1$,

$$\lim_{X \to \infty} \frac{1}{X} \sum_{N \in [X, cX]} \left( \sum_{\chi \in \mathcal{D}_\pm(N)} \frac{\chi(P)}{G(\chi)} \pm \frac{1}{N} \sum_{\chi \in \mathcal{I}_\pm(N)} G(\overline{\chi})\chi(P) \right) = \begin{cases} \frac{5}{\pi^2} \int_1^c \cos \left( \frac{2\pi y}{u} \right) du, & \text{if } +, \\ -i \frac{5}{\pi^2} \int_1^c \sin \left( \frac{2\pi y}{u} \right) du, & \text{if } -. \end{cases}$$
Murmurations of newforms for $\Gamma_0(N)$
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Figure: Scatter plot of $(n, \mathbb{E}_N \mathbb{E}_f a_p(f))$ over all newforms $f \in H_{k}^{\text{new}}(N)$ with root number $\varepsilon$ and level $N \in [2^8, 2^9]$ for all $p_n \leq 2^9$. Top plot is weight $k = 2$ and bottom plot is weight $k = 4$. 
Murmurations of newforms for $\Gamma_0(N)$

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Figure: Scatter plot of $(n, \mathbb{E}_N E_f a_{p_n}(f))$ over all newforms $f \in H_k^{\text{new}}(N)$ with root number $\varepsilon$ and level $N \in [2^{10}, 2^{11}]$ for all $p_n \leq 2^{11}$. Top plot is weight $k = 2$ and bottom plot is weight $k = 4$. 
Let $H_{k}^{new}(N)$ be a basis of trivial character weight $k$ newforms for $\Gamma_0(N)$. Let $X, Y, P \to \infty$ with $P$ prime, and assume that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some $\delta_1, \delta_2 > 0$ with $2\delta_1 < \delta_2 < 1$. Let $y := P/X$. Then

$$\sum_{\square \text{-free}}^{\square \text{-free}} \sum_{N \in [X, X+Y]} \sum_{f \in H_{k}^{new}(N)} \varepsilon(f) a_f(P) P^{1-k/2} \frac{1}{\sum_{N \in [X, X+Y]} \sum_{f \in H_{k}^{new}(N)} 1} = M_k(y) + O_\varepsilon \left(X^{-\delta'+\epsilon} + \frac{1}{P}\right)$$
Theorem (Zubrilina 2023)

Let $H^\text{new}_k(N)$ be a basis of trivial character weight $k$ newforms for $\Gamma_0(N)$. Let $X, Y, P \to \infty$ with $P$ prime, and assume that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some $\delta_1, \delta_2 > 0$ with $2\delta_1 < \delta_2 < 1$. Let $y := P/X$. Then

$$\sum_{\square-\text{free}} \sum_{f \in H^\text{new}_k(N)} \varepsilon(f) a_f(P) P^{1-k/2} = M_k(y) + O_\varepsilon \left( X^{-\delta'+\epsilon} + \frac{1}{P} \right)$$

where $M_k(y)$ is the weight $k$ murmuration density function:

$$M_k(y) := D_k \left( A \sqrt{y} + (-1)^{k/2-1} B \sum_{1 \leq r \leq 2\sqrt{y}} c(r) \sqrt{4y - r^2} U_{k-2} \left( \frac{r}{2\sqrt{y}} \right) - \delta_{k=2}\pi y \right)$$

$$A = \prod_p \left( 1 + \frac{p}{(p+1)^2(p-1)} \right), \quad B = \prod_p \frac{p^4 - 2p^2 - p + 1}{(p^2 - 1)^2}, \quad c(r) = \prod_{p|r} \left( 1 + \frac{p^2}{p^4 - 2p^2 - p + 1} \right), \quad D_k = \frac{12}{(k-1)\pi \prod_p \left( \frac{1}{p^2 + p} \right)}$$
Murmuration density function

Figure: Murmuration density function $M_k(y)$ for weights $k = 2$ and $k = 4$. 
Murmuration density function

Figure: Murmuration density function $M_k(y)$ for weights $k = 6$ and $k = 8$. 
Zubrilina’s breakthrough

To obtain averages over some geometric interval \([X, cX]\), integrate \(uM_k(y/u)\) over the interval \([1, c]\):

**Theorem (Zubrilina 2023)**

Let \(P \ll X^{6/5}\), let \(c > 1\) be constant and \(y := P/X\) Then as \(X \to \infty\):

\[
\frac{\sum_{N \in [X, cX]} \sum_{f \in H_k^{\text{new}}(N)} \varepsilon(f) a_f(p) p^{1-k/2}}{\sum_{N \in [X, cX]} \sum_{f \in H_k^{\text{new}}(N)} 1} = \frac{2}{(c^2 - 1)} \int_1^c uM_k(y/u) du + o_y(1)
\]
Zubrilina’s breakthrough

• Case $k = c = 2$: 

\[
\begin{align*}
\text{Theorem (Zubrilina 2023)} \\
\text{Let } P &\ll X^6/5 \text{ and } y := P/X. \text{ Then as } X \to \infty, \\
\sum_{\square \text{-free} \ N \in [X, \, 2X]} \sum_{f \in H_{\text{new}}^2(N)} &\varepsilon(f) a_f(P) \\
\sum_{\square \text{-free} \ N \in [X, \, 2X]} \sum_{f \in H_{\text{new}}^2(N)} &1 \text{ converges to the function} \\
&\begin{cases} \\
\alpha \sqrt{y} - \beta y & \text{if } y \in [0, \, 1/4], \\
\alpha \sqrt{y} - \beta y + \gamma \pi y^2 - \gamma (1 - 2y)^{1/4} - 2 \gamma y^2 \arcsin \left(\frac{1}{2} y - \frac{1}{2}\right) & \text{if } y \in [1/4, \, 1/2], \\
\alpha \sqrt{y} - \beta y + 2 \gamma y^2 \left(\arcsin \left(\frac{1}{y} - 1\right) - \arcsin \left(\frac{1}{2} y - 1\right)\right) - \gamma (1 - 2y)^{1/4} + 2 \gamma (1 - y) \sqrt{2y - 1} & \text{if } y \in [1/2, \, 1],
\end{cases}
\end{align*}
\]

where \(\alpha \approx 6.38936\), \(\beta \approx 11.3536\), and \(\gamma \approx 2.6436\).
Zubrilina’s breakthrough

• Case $k = c = 2$:

**Theorem (Zubrilina 2023)**

Let $P \ll X^{6/5}$ and $y := P/X$. Then as $X \to \infty$, the dyadic average

$$
\frac{\sum_{N \in [X,2X]} \sum_{f \in H^*_2(N)} \epsilon(f) a_f(P)}{\sum_{N \in [X,2X]} \sum_{f \in H^*_2(N)} 1}
$$

converges to the function

$$
\begin{cases}
\alpha \sqrt{y} - \beta y & \text{if } y \in [0, 1/4], \\
\alpha \sqrt{y} - \beta y + \gamma \pi y^2 - \gamma (1 - 2y) \sqrt{y - 1/4} - 2\gamma y^2 \arcsin(1/2y - 1) & \text{if } y \in [1/4, 1/2], \\
\alpha \sqrt{y} - \beta y + 2\gamma y^2 (\arcsin(1/y - 1) - \arcsin(1/2y - 1)) - \gamma (1 - 2y) \sqrt{y - 1/4} + 2\gamma (1 - y) \sqrt{2y - 1} & \text{if } y \in [1/2, 1],
\end{cases}
$$

where $\alpha \approx 6.38936$, $\beta \approx 11.3536$, and $\gamma \approx 2.6436$. 
Zubrilina’s breakthrough

Figure: Plots of $\pm \frac{2}{3} \int_1^2 uM_k(y/u)du$ for weights $k = 2$ and $k = 4.$
Zubrilina’s breakthrough

Figure: Plots of $\pm \frac{2}{3} \int_1^2 uM_k(y/u)du$ for weights $k = 6$ and $k = 8$. 
Zubrilina’s breakthrough

Idea of proof for weight $k = 2$:
Zubrilina’s breakthrough

Idea of proof for weight $k = 2$:

- \[ \sum_{f \in H_2^{\text{new}}(N)} a_f(p) \varepsilon(f) = \text{Tr}(T_p \circ W_N) \]
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Idea of proof for weight \( k = 2 \):

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**Theorem (Yamauchi 1973, Skoruppa-Zagier 1988)**

For weight \( k = 2 \), \( N \) squarefree, and a prime \( P \nmid N \),

\[
\text{Tr}(T_p \circ W_N) = \frac{H_1(-4PN)}{2} + \sum_{0 < r \leq 2 \sqrt{P/N}} H_1(r^2 N^2 - 4PN) - P - 1
\]

where \( H_1(-d) \) is the Hurwitz class number.
Zubrilina’s breakthrough

Idea of proof for weight $k = 2$:

- \[ \sum_{f \in H^\text{new}^2(N)} a_f(p)\varepsilon(f) = \text{Tr}(T_p \circ W_N) \]

Theorem (Yamauchi 1973, Skoruppa-Zagier 1988)

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- Can express $H_1(-d) = \sum_{f \in \mathbb{N} : f^2 | d} h(-d/f^2) + O(1)$. 


Zubrilina’s breakthrough

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- Can express $H_1(-d) = \sum_{f \in \mathbb{N} : f^2 \mid d} h(-d/f^2) + O(1)$.
- Apply the class number formula!
In August 2023, Peter Sarnak wrote a letter to Drew Sutherland and Nina Zubrilina giving some deeper theoretical observations about murmurations in general families of $L$-functions.

In general, given some suitable family $F$ of $L$-functions with a natural ordering (usually by conductor), and a constant $\theta > 0$, we can study the double averages:

$$\sum_{P \sim N^\theta} \sum_{\pi \in F} \Phi\left(\frac{N_\pi}{N}\right) \sum_{P \sim N^\theta} \sum_{\pi \in F} \Phi\left(\frac{N_\pi}{N}\right),$$

where $\Phi : (0, \infty) \to \mathbb{R}$ is a smooth nonnegative weight function.

Sarnak remarked that these double averages are related to the 1-level densities of the zeros of $L(s, \pi)$. Using random matrix theory, Katz and Sarnak predicted that these averages for $\theta < 1$ behave differently to $\theta > 1$. The murmurations phenomenon arises at the sharp phase transition when $\theta = 1$. 

\[ \frac{1}{2} \]
Peter’s Letter

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In general, given some suitable family \( \mathcal{F} \) of \( L \)-functions with a natural ordering (usually by conductor), and a constant \( \theta > 0 \), we can study the double averages:

\[
\frac{\sum_{P \sim N^\theta} \sum_{\pi \in \mathcal{F}} \Phi(N\pi/N) a_{\pi}(P)}{\sum_{P \sim N^\theta} \sum_{\pi \in \mathcal{F}} \Phi(N\pi/N)},
\]

where \( \Phi : (0, \infty) \to \mathbb{R} \) is a smooth nonnegative weight function.
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Murmurations in the weight aspect

Another breakthrough by Bober–Booker–Lee–Lowry-Duda:

\[ \text{Theorem (Bober–Booker–Lee–Lowry-Duda 2023)} \]

Assume GRH. Fix \( \varepsilon > 0 \) small and \( \delta \in \{0, 1\} \). Fix a compact interval \( E \subset \mathbb{R} > 0 \) with \( |E| > 0 \). Let \( K, H \in \mathbb{R} > 0 \) with \( K + \varepsilon < H < K - \varepsilon \). As \( K \to \infty \):

\[ \sum_{p/N \in E} \log p \sum_{k \equiv 2 \delta \mod 4} |k - K| \leq H \sum_{f \in H_{\text{new}}} k(1) \lambda_f(p) \sum_{p/N \in E} \log p \sum_{k \equiv 2 \delta \mod 4} |k - K| \leq H \sum_{f \in H_{\text{new}}} k(1) = (\frac{-1}{2} \delta) \sqrt{N} (\nu(E) |E| + o_E(\epsilon)) , \]

where \( H_{\text{new}} k(1) \) is a basis of level 1 weight \( k \) newforms and where

\[ \nu(E) = \frac{1}{2} \zeta(2) \sum_{a, q \in \mathbb{Z} > 0 \gcd(a, q) = 1} \left( \frac{a}{q} \right) - 2 \in E \mu(q) \frac{2}{\phi(q)} \psi(q) \sigma(q) \left( q^a \right) ^{3/2} = \frac{1}{2} \sum_{t = -\infty}^{\infty} \prod_{p \nmid t} p^2 - p - 1 \cdot \prod_{p \nmid t} p^2 - p - 1 \cdot \int_{E} \cos \left( \frac{2\pi t}{\sqrt{y}} \right) dy. \]
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Another breakthrough by Bober–Booker–Lee–Lowry-Duda:

Theorem (Bober–Booker–Lee–Lowry-Duda 2023)

Assume GRH. Fix \( \epsilon > 0 \) small and \( \delta \in \{0, 1\} \). Fix a compact interval \( E \subset \mathbb{R}_{>0} \) with \( |E| > 0 \). Let \( K, H \in \mathbb{R}_{>0} \) with \( K^{5/6} + \epsilon < H < K^{1-\epsilon} \). As \( K \to \infty \):

\[
\frac{\sum_{p \mid N \in E} \log p \sum_{k \equiv 2\delta \mod 4 \mid k-K \leq H} \sum_{f \in H_k^{\text{new}}(1)} \lambda_f(p)}{\sum_{p \mid N \in E} \log p \sum_{k \equiv 2\delta \mod 4 \mid k-K \leq H} \sum_{f \in H_k^{\text{new}}(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}} \left( \frac{\nu(E)}{|E|} + o_{E, \epsilon}(1) \right),
\]

where \( H_k^{\text{new}}(1) \) is a basis of level 1 weight \( k \) newforms and where

\[
\nu(E) = \frac{1}{\zeta(2)} \sum_{a, q \in \mathbb{Z}_{>0}} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left( \frac{q}{a} \right)^3 = \frac{1}{2} \sum_{t=-\infty}^{\infty} \prod_{\rho \mid t} \frac{p^2 - p - 1}{p^2 - p} \cdot \int_E \cos \left( \frac{2\pi t}{\sqrt{y}} \right) dy.
\]
Murmurations in the weight aspect

Figure: A comparison of $(-1)^\delta \nu([0, t])$ and the left-hand side of the main theorem, scaled by $t \sqrt{N}$, for $K = 3830, H = 100$, and $t \in [0, 2]$. (Bober–Booker–Lee–Lowry–Duda 2023)
References

Murmurations of modular forms in the weight aspect

He, Y.-H., Lee, K.-H., Oliver, T., Pozdnyakov, A. (2022)
Murmurations of elliptic curves

Lee, K.H., Oliver, T., Pozdnyakov, A. (2023)
Murmurations of Dirichlet characters
Preprint, Available at: arXiv:2307.00256
References

Sarnak, P. (2023)
Letter to Drew Sutherland and Nina Zubrilina

Sutherland, A. (2023)
Murmurations of arithmetic L-functions
Talk at *Arithmetic statistics* conference, CIRM.

Sutherland, A. (2022)
Letter to Michael Rubinstein and Peter Sarnak

Zubrilina, N. (2023)
Murmurations
**Suggested talk schedule**

- **Week 3** (26 Jan): Work through He–Lee–Oliver–Pozdnyakov machine learning paper. Predicting ranks of elliptic curves using logistic regression. Background on other machine learning strategies.

- **Week 4** (02 Feb): Work through Drew Sutherland’s and Peter Sarnak’s letters. Give some background on existing conjectures and theorems on horizontal/vertical trace distributions of $a_p(f)$ (Sato-Tate conjecture, Katz-Sarnak philosophy, Birch, Serre, etc.)

- **Week 5 - 6** (09, 16 Feb): Murmurations of Dirichlet characters (Lee–Oliver–Pozdnyakov)

- **Week 7 - 9** (23 Feb; 01, 08 Mar): Murmurations of weight $k$ newforms (Nina Zubrilina)

- **Week 10** (15 Mar): Murmurations of modular forms in the weight aspect (Bober–Booker–Lee–Lowry-Duda)