Murmurations in Arithmetic

Murmurations study group, Introductory talk

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organised with Sam Chow and Simon Rydin Myerson

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University of Warwick

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Murmurations

Figure: *A murmuration of starlings at Gretna* - Walter Baxter (cc-by-sa/2.0)

Motivation

Let $E/\mathbb{Q}$ be an elliptic curve. Recall its L-function

$$L(E,s) = \prod_{p \text{ prime}} L_p(E,s)^{-1} = \sum_{n \geq 1} a_n(E) n^{-s}$$

where for primes p of good reduction, we have $L_p(E,s) = 1 - a_p(E)p^{-s} + p^{1-2s}$ where $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$.

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2. For a fixed prime p, how is $a_p(E)$ distributed over all elliptic curves $E/\mathbb{F}_p$?
3. What if we restrict to elliptic curves $E/\mathbb{Q}$ of given rank and conductor, and investigate $a_p(E)$ as p grows linearly with the conductor?

Motivation

1. This was a famous conjecture of **Mikio Sato** and **John Tate**. e.g. for an elliptic curve $E/\mathbb{Q}$ without CM, the probability measure of $\theta := \arccos(\frac{a_p(E)}{2\sqrt{p}})$ is proportional to $\sin^2\theta d\theta$. Now a theorem (by many authors)!

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2. This is the same as the Sato-Tate distribution, i.e. for a fixed p, the distribution of $\theta := \arccos(\frac{a_p(E)}{2\sqrt{p}})$ over all $E/\mathbb{F}_p$ is proportional to $\sin^2\theta d\theta$ for large p (Birch, 1968).

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3. By restricting to elliptic curves $E/\mathbb{Q}$ with given rank r and conductor $N \in [N_1, N_2]$, and investigating the average of $a_p(E)$ as $p \sim N$, this gives rise to the **murmurations** phenomenon!

Machine-learning experiments

During 2019-2022, Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver, and Alexey Pozdnyakov conducted some machine-learning experiments on datasets of arithmetic curves.

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In one of their experiments, they represented an elliptic curve $E/\mathbb{Q}$ as a vector of its first 1000 values of $a_p(E)$:

$$v_L(E) := \big(a_2(E), a_3(E), a_5(E), \ldots, a_{7919}(E)\big) \in \mathbb{Z}^{1000}.$$

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Using logistic regression, they were able to predict the rank of E from $v_L(E)$ with very high accuracy, e.g. to distinguish between rank 0 and rank 1 curves, the goal is to find $\mathbf{w} \in \mathbb{R}^{1000}$ and $b \in \mathbb{R}$ such that

$$\sigma(v_L(E) \cdot \mathbf{w} + b), \quad \text{where } \sigma(x) = \frac{1}{1+e^{-x}}$$

is hopefully close to either 0 or 1. The results of their experiments successfully predicted the ranks all with accuracies above 96%.

Murmurations

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$$f_r(n) := \frac{1}{\#\mathcal{E}_r[N_1, N_2]} \sum_{E \in \mathcal{E}_r[N_1, N_2]} a_{p_n}(E)$$

where p_n is the n-th prime number.

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Murmurations of elliptic curves

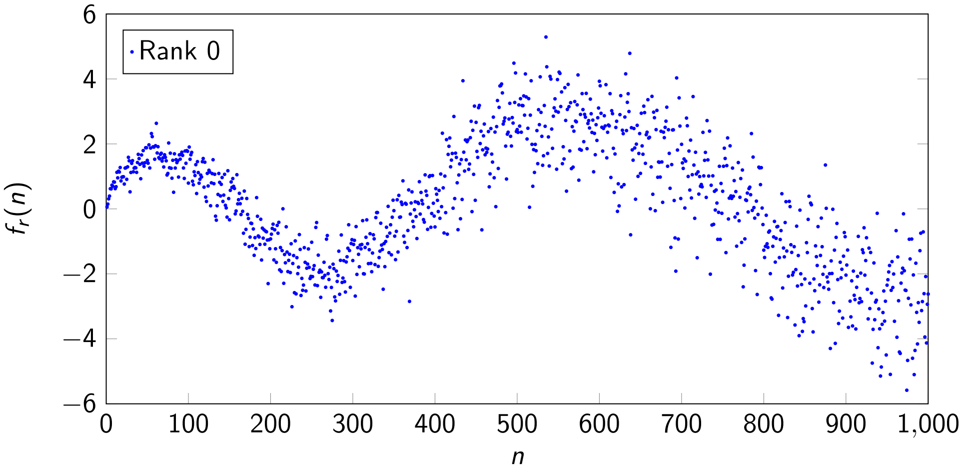

Figure: Scatter plot of $(n, f_r(n))$ for ranks $r = 0$ (blue) and $r = 1$ (red) with conductor N between $N_1 = 7\,500$ and $N_2 = 10\,000$ (He–Lee–Oliver–Pozdnyakov 2022).

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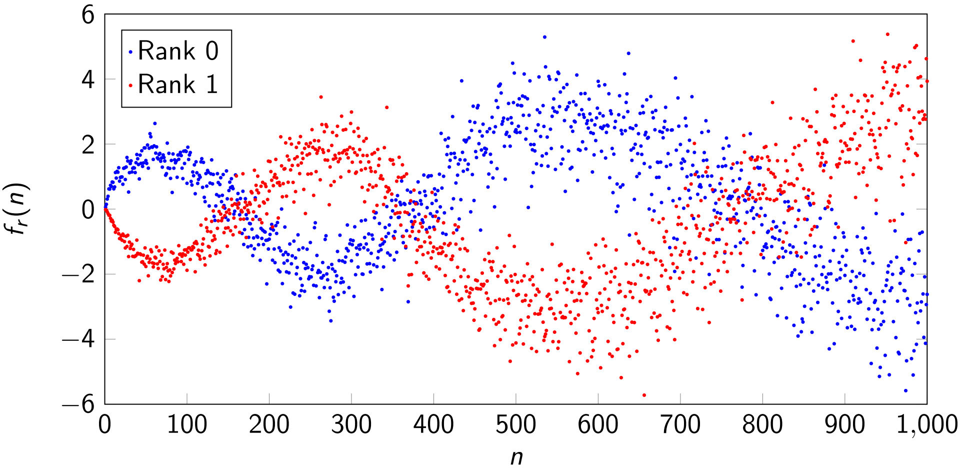

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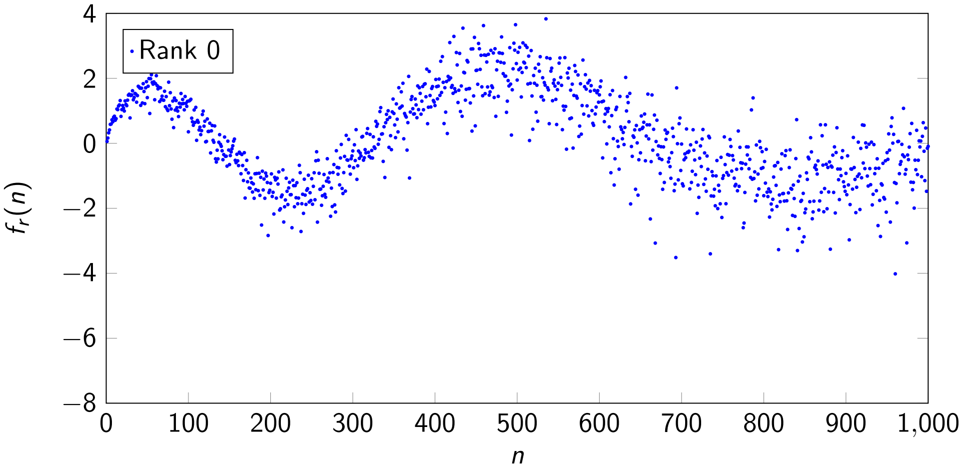

Figure: Scatter plot of $(n, f_r(n))$ for ranks $r = 0$ (blue) and $r = 2$ (green) with conductor N between $N_1 = 5\,000$ and $N_2 = 10\,000$ (He–Lee–Oliver–Pozdnyakov 2022).

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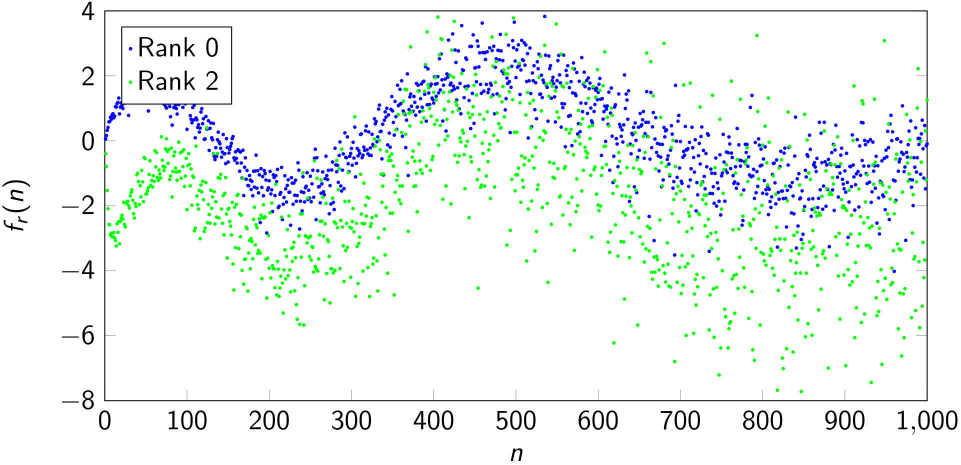

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Figure: *Murmurations* - Alain Delorme

More murmurations

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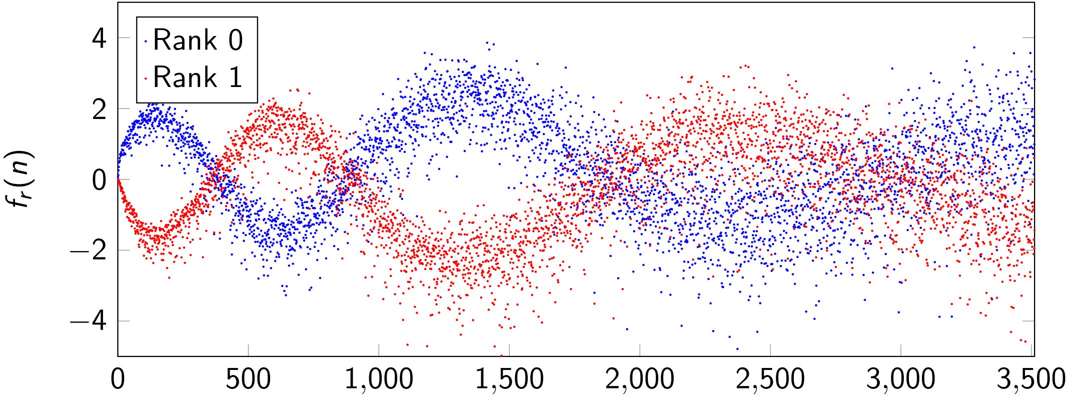

Figure: Scatter plot of $(n, f_r(n))$ for ranks $r = 0$ (blue) and $r = 1$ (red) with conductor N between $N_1 = 2^{14}$ and $N_2 = 2^{15}$ for all $p_n \leq 2^{15}$.

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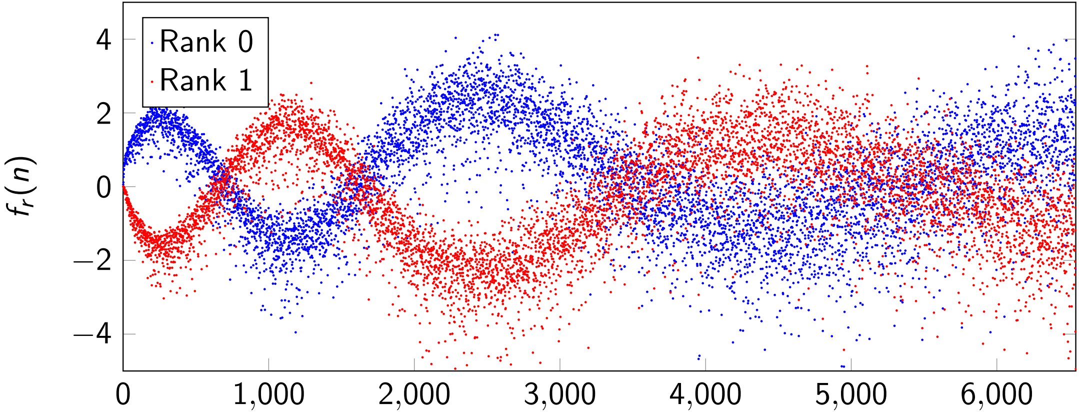

Figure: Scatter plot of $(n, f_r(n))$ for ranks $r = 0$ (blue) and $r = 1$ (red) with conductor N between $N_1 = 2^{15}$ and $N_2 = 2^{16}$ for all $p_n \leq 2^{16}$.

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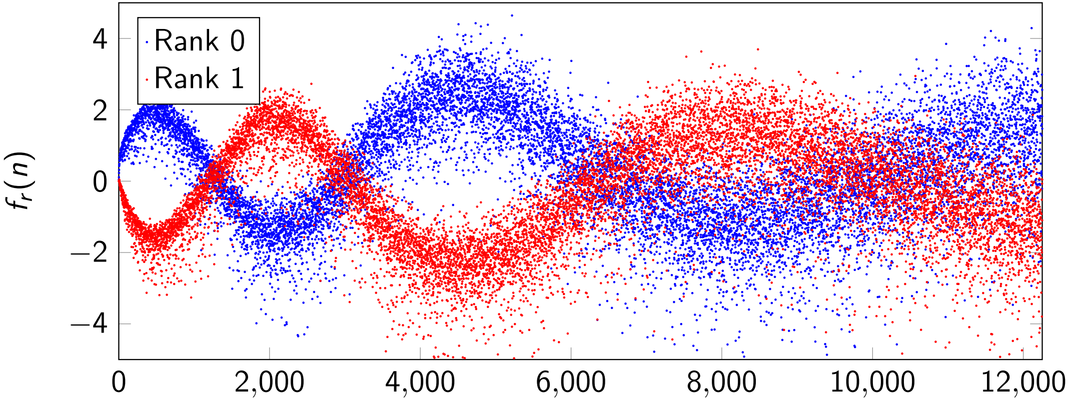

Figure: Scatter plot of $(n, f_r(n))$ for ranks $r = 0$ (blue) and $r = 1$ (red) with conductor N between $N_1 = 2^{16}$ and $N_2 = 2^{17}$ for $p_n \leq 2^{17}$.

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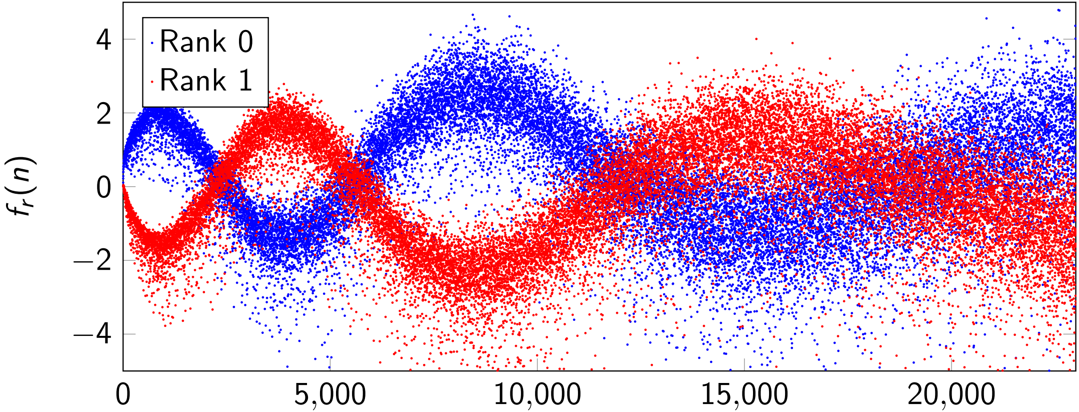

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- This phenomenon is not specific to elliptic curves, and can be seen for many families of arithmetic L-functions. E.g. Dirichlet characters, higher dimension abelian varieties, newforms for $\Gamma_0(N)$, higher genus curves, etc.

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- This phenomenon is not specific to elliptic curves, and can be seen for many families of arithmetic L-functions. E.g. Dirichlet characters, higher dimension abelian varieties, newforms for $\Gamma_0(N)$, higher genus curves, etc.
- This phenomenon appears to only occur in primitive arithmetic L-function, e.g. no oscillations are visible when plotting L-functions of products of elliptic curves.

Murmurations for Dirichlet characters

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Theorem (Lee–Oliver–Pozdnyakov 2023)

Assume RH. Let $\mathcal{D}_+(N)$ (resp. $\mathcal{D}_-(N)$) denote the set of primitive even (resp. odd) Dirichlet characters mod N. Fix some $\delta \in (\frac{1}{2}, 1)$, and let $y := P/X$. Then

$$\lim_{X\to\infty} \frac{\log X}{X^\delta} \sum_{\substack{N\in[X,X+X^\delta] \\ N \text{ prime}}} \sum_{\chi\in\mathcal{D}_\pm(N)} \frac{\chi(P)}{G(\chi)} = \begin{cases} \cos(2\pi y), & \text{if } +, \\ -i\sin(2\pi y), & \text{if } -, \end{cases}$$

where $G(\chi) := \sum_{a=1}^{m} \chi(a) e^{2\pi i a/m}$ is the Gauss sum of χ.

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- Proof uses the Fourier expansion of additive characters in terms of Dirichlet characters, the prime number theorem, and elementary analysis on $\mathbb{R}$ (Pozdnyakov 2023).

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To obtain averages over some geometric interval $[X, cX]$, integrate over the interval $[1, c]$:

Theorem (Lee–Oliver–Pozdnyakov 2023)

Fix some $c > 1$. Let $y := P/X$. Then

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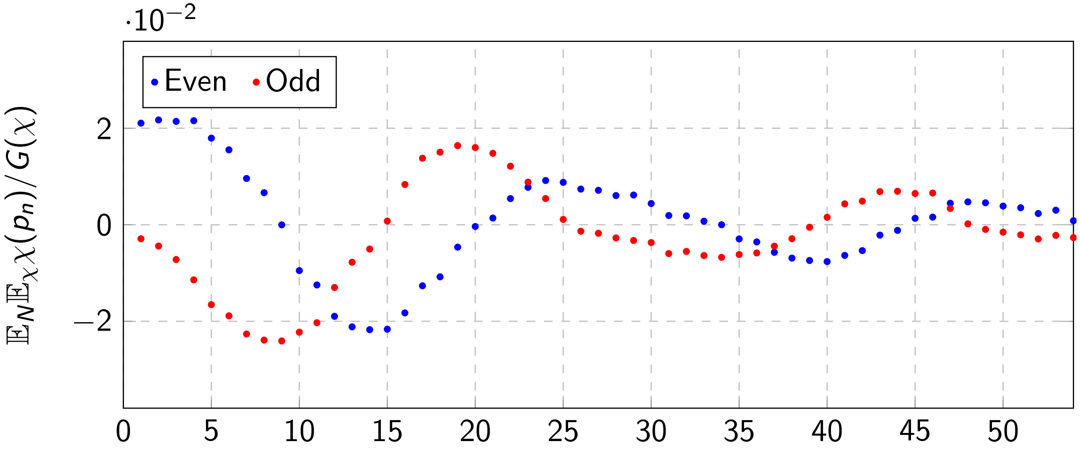

Figure: Scatter plot of $(n, \mathbb{E}_N \mathbb{E}_\chi \chi(p_n)/G(\chi))$ for even (blue) and odd (red) primitive Dirichlet characters χ with level N between $N_1 = 2^6$ and $N_2 = 2^7$ for all $p_n \leq 2^8$.

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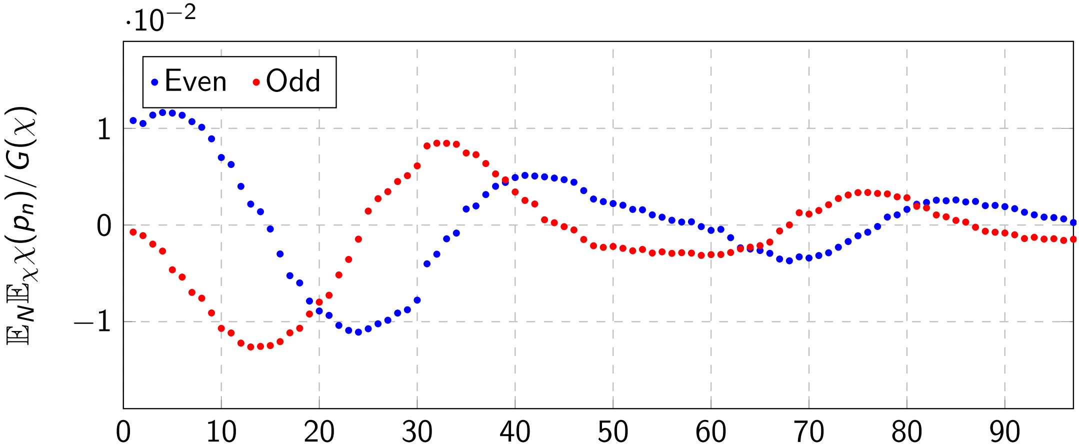

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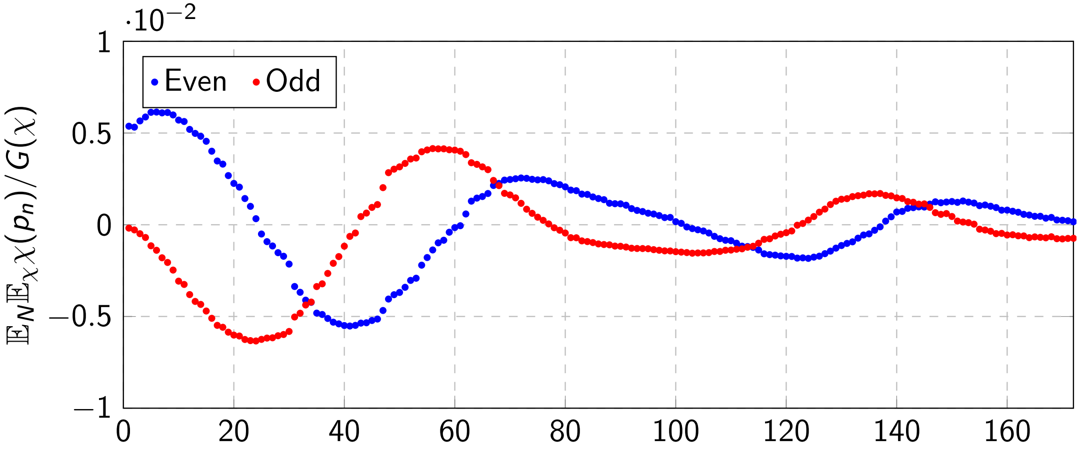

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Let $\mathcal{D}_\pm(N)$ be as before and let $\mathcal{I}_+(N)$ (resp. $\mathcal{I}_-(N)$) denote the set of imprimitive even (resp. odd) nontrivial Dirichlet characters mod N. Fix $\delta \in (0,1)$ and $y := P/X$. Then

$$\lim_{X\to\infty} \frac{1}{X^\delta} \sum_{\substack{N\in[X,X+X^\delta] \\ N\not\equiv 2 \bmod 4}} \left(\sum_{\chi\in\mathcal{D}_\pm(N)} \frac{\chi(P)}{G(\chi)} \pm \frac{1}{N} \sum_{\chi\in\mathcal{I}_\pm(N)} G(\overline{\chi})\chi(P) \right) = \begin{cases} \frac{5}{\pi^2}\cos(2\pi y), & \text{if } +, \\ -i\frac{5}{\pi^2}\sin(2\pi y), & \text{if } -, \end{cases}$$

where $G(\chi)$ is the Gauss sum of χ. Similarly, for some fixed $c > 1$,

$$\lim_{X\to\infty} \frac{1}{X} \sum_{\substack{N\in[X,cX] \\ N\not\equiv 2 \bmod 4}} \left(\sum_{\chi\in\mathcal{D}_\pm(N)} \frac{\chi(P)}{G(\chi)} \pm \frac{1}{N} \sum_{\chi\in\mathcal{I}_\pm(N)} G(\overline{\chi})\chi(P) \right) = \begin{cases} \frac{5}{\pi^2}\int_1^c \cos\left(\frac{2\pi y}{u}\right)du, & \text{if } +, \\ -i\frac{5}{\pi^2}\int_1^c \sin\left(\frac{2\pi y}{u}\right)du, & \text{if } -. \end{cases}$$

Murmurations of newforms for $\Gamma_0(N)$

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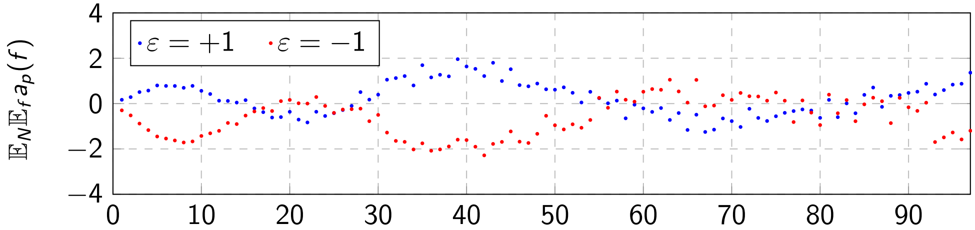

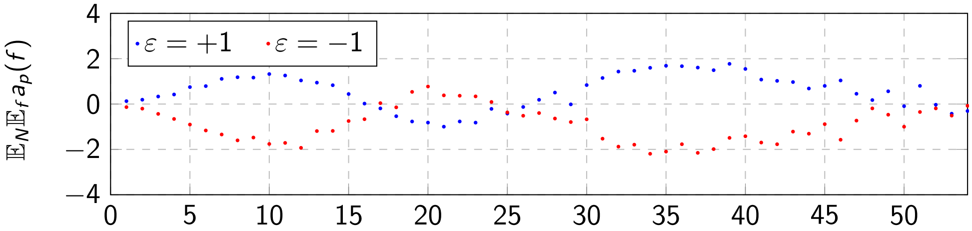

Figure: Scatter plot of $(n, \mathbb{E}_N \mathbb{E}_f a_{p_n}(f))$ over all newforms $f \in H_k^{\text{new}}(N)$ with root number ε and level $N \in [2^8, 2^9]$ for all $p_n \leq 2^9$. Top plot is weight $k = 2$ and bottom plot is weight $k = 4$.

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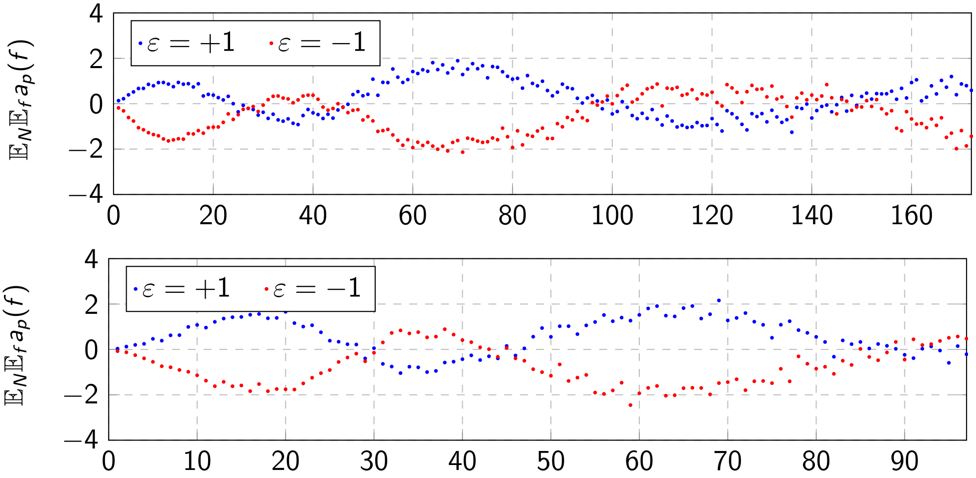

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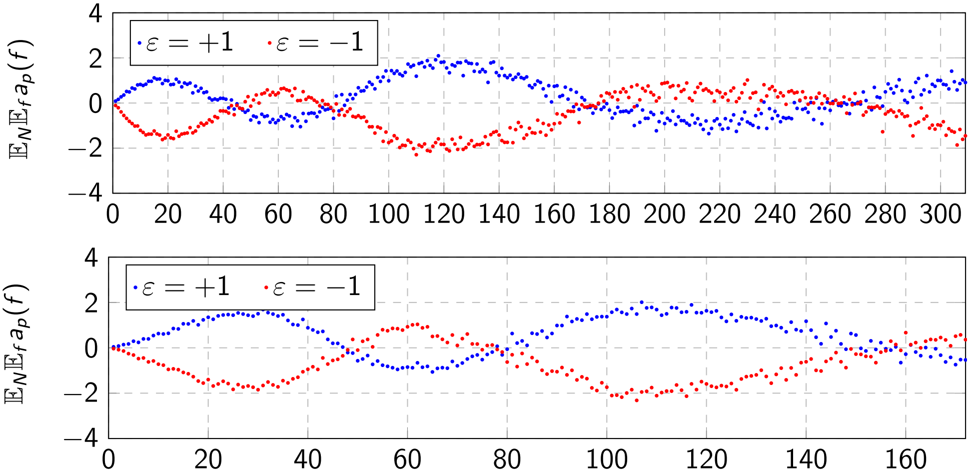

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Zubrilina's breakthrough

Theorem (Zubrilina 2023)

Let $H_k^{new}(N)$ be a basis of trivial character weight k newforms for $\Gamma_0(N)$. Let $X, Y, P \to \infty$ with P prime, and assume that $Y = (1 + o(1))X^{1-\delta_2}$ and $P \ll X^{1+\delta_1}$ for some $\delta_1, \delta_2 > 0$ with $2\delta_1 < \delta_2 < 1$. Let $y := P/X$. Then

$$\frac{\sum_{N\in[X,X+Y]}^{\square-free} \sum_{f\in H_k^{new}(N)} \varepsilon(f) a_f(P) P^{1-k/2}}{\sum_{N\in[X,X+Y]}^{\square-free} \sum_{f\in H_k^{new}(N)} 1} = M_k(y) + O_\varepsilon\left(X^{-\delta'+\epsilon} + \frac{1}{P}\right)$$

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$$\frac{\sum_{N \in [X, X+Y]}^{\square-free} \sum_{f \in H_k^{new}(N)} \varepsilon(f) a_f(P) P^{1-k/2}}{\sum_{N \in [X, X+Y]}^{\square-free} \sum_{f \in H_k^{new}(N)} 1} = M_k(y) + O_\varepsilon\left(X^{-\delta'+\epsilon} + \frac{1}{P}\right)$$

where $M_k(y)$ is the weight k murmuration density function:

$$M_k(y) := D_k\left(A\sqrt{y} + (-1)^{k/2-1} B \sum_{1 \leq r \leq 2\sqrt{y}} c(r)\sqrt{4y - r^2}\, U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) - \delta_{k=2}\pi y\right)$$

$A = \prod_p (1 + \frac{p}{(p+1)^2(p-1)})$, $B = \prod_p \frac{p^4 - 2p^2 - p + 1}{(p^2-1)^2}$, $c(r) = \prod_{p|r}(1 + \frac{p^2}{p^4 - 2p^2 - p + 1})$, $D_k = \frac{12}{(k-1)\pi \prod_p (1 - \frac{1}{p^2+p})}$

Murmuration density function

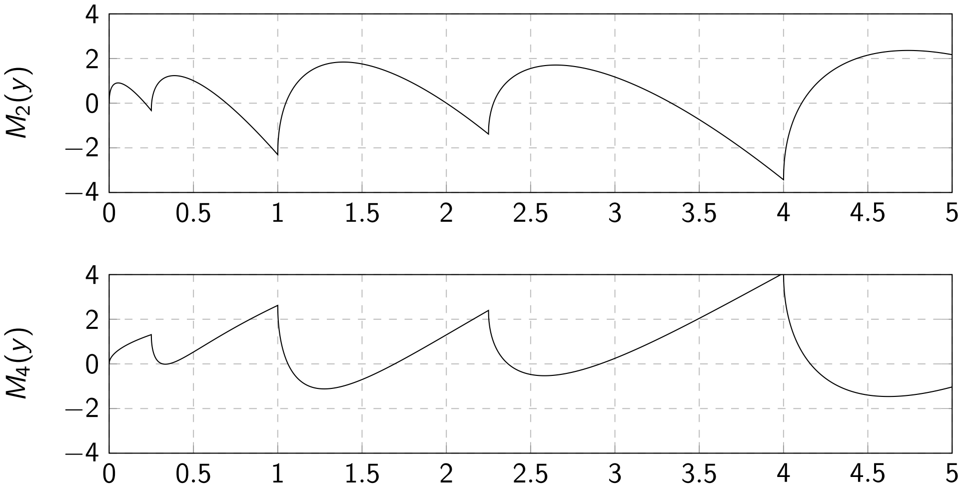

Figure: Murmuration density function $M_k(y)$ for weights $k = 2$ and $k = 4$.

Murmuration density function

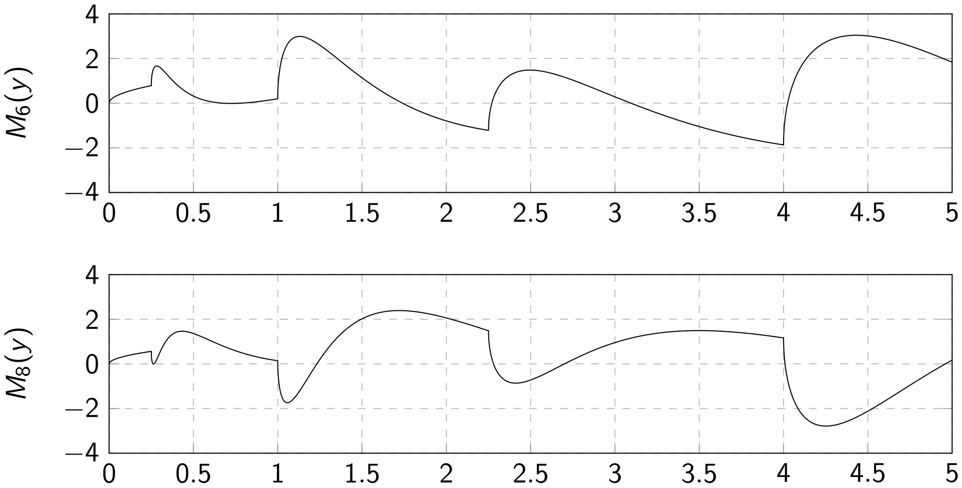

Figure: Murmuration density function $M_k(y)$ for weights $k = 6$ and $k = 8$.

Zubrilina's breakthrough

To obtain averages over some geometric interval $[X, cX]$, integrate $uM_k(y/u)$ over the interval $[1, c]$:

Theorem (Zubrilina 2023)

Let $P \ll X^{6/5}$, let $c > 1$ be constant and $y := P/X$ Then as $X \to \infty$:

$$\frac{\sum_{N\in[X,cX]}^{\square-free} \sum_{f\in H_k^{new}(N)} \varepsilon(f)a_f(p)p^{1-k/2}}{\sum_{N\in[X,cX]}^{\square-free} \sum_{f\in H_k^{new}(N)} 1} = \frac{2}{(c^2-1)} \int_1^c uM_k(y/u)du + o_y(1)$$

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Theorem (Zubrilina 2023)

Let $P \ll X^{6/5}$ and $y := P/X$. Then as $X \to \infty$, the dyadic average

$$\frac{\sum_{N\in[X,2X]}^{\square-free} \sum_{f\in H_2^{new}(N)} \varepsilon(f) a_f(P)}{\sum_{N\in[X,2X]}^{\square-free} \sum_{f\in H_2^{new}(N)} 1}$$

converges to the function

$$\begin{cases} \alpha\sqrt{y} - \beta y & \text{if } y \in [0, 1/4], \\ \alpha\sqrt{y} - \beta y + \gamma\pi y^2 - \gamma(1-2y)\sqrt{y-1/4} - 2\gamma y^2 \arcsin(1/2y - 1) & \text{if } y \in [1/4, 1/2], \\ \alpha\sqrt{y} - \beta y + 2\gamma y^2(\arcsin(1/y - 1) - \arcsin(1/2y - 1)) & \\ \qquad -\gamma(1-2y)\sqrt{y-1/4} + 2\gamma(1-y)\sqrt{2y-1} & \text{if } y \in [1/2, 1], \end{cases}$$

where $\alpha \approx 6.38936$, $\beta \approx 11.3536$, and $\gamma \approx 2.6436$.

Zubrilina's breakthrough

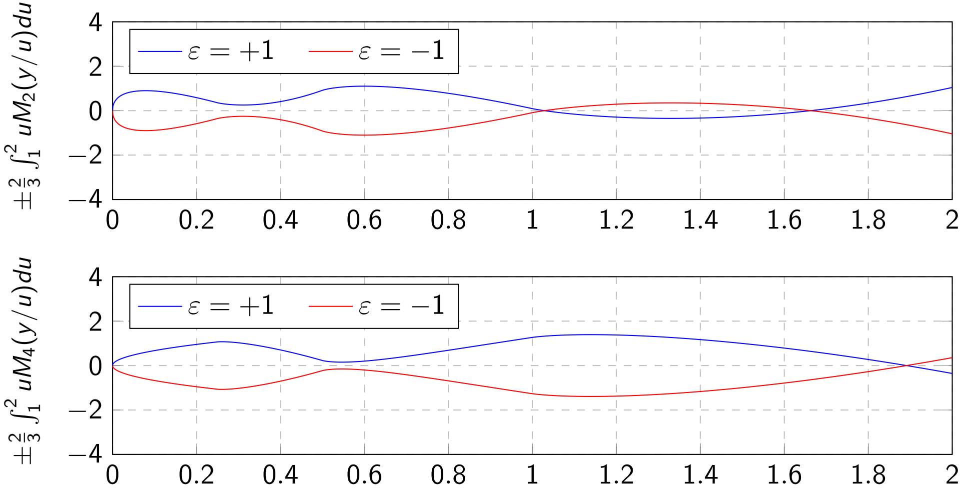
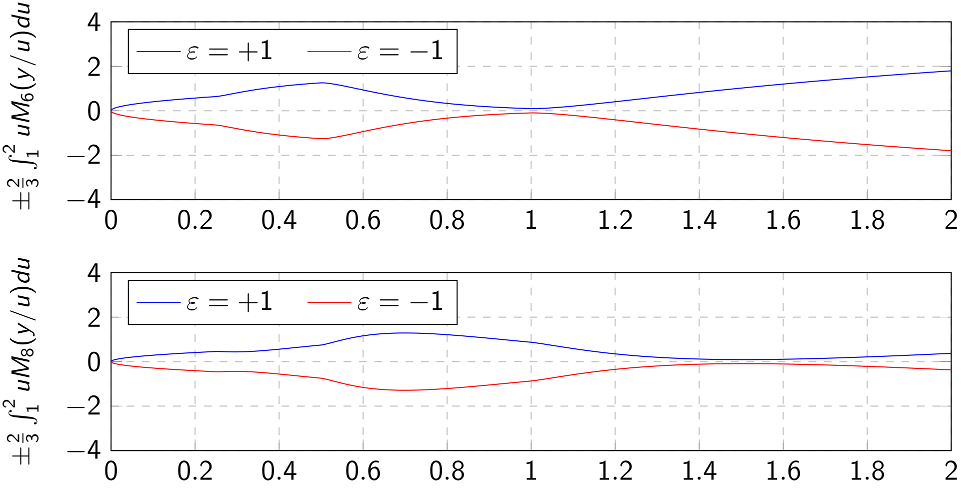


Figure: Plots of $\pm\frac{2}{3}\int_1^2 uM_k(y/u)du$ for weights $k = 2$ and $k = 4$.

Zubrilina's breakthrough



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Theorem (Yamauchi 1973, Skoruppa-Zagier 1988)

For weight $k = 2$, N squarefree, and a prime $P \nmid N$,

$$Tr(T_p \circ W_N) = \frac{H_1(-4PN)}{2} + \sum_{0<r\leq 2\sqrt{P/N}} H_1(r^2N^2 - 4PN) - P - 1$$

where $H_1(-d)$ is the Hurwitz class number.

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- Can express $H_1(-d) = \sum_{f \in \mathbb{N} : f^2 | d} h(-d/f^2) + O(1)$.
- Apply the class number formula!

Peter's Letter

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In general, given some suitable family $\mathcal{F}$ of L-functions with a natural ordering (usually by conductor), and a constant $\theta > 0$, we can study the double averages:

$$\frac{\sum_{P\sim N^\theta}\sum_{\pi\in\mathcal{F}}\Phi(N_\pi/N)a_\pi(P)}{\sum_{P\sim N^\theta}\sum_{\pi\in\mathcal{F}}\Phi(N_\pi/N)},$$

where $\Phi : (0,\infty) \to \mathbb{R}$ is a smooth nonnegative weight function.

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$$\frac{\sum_{P \sim N^\theta} \sum_{\pi \in \mathcal{F}} \Phi(N_\pi / N) a_\pi(P)}{\sum_{P \sim N^\theta} \sum_{\pi \in \mathcal{F}} \Phi(N_\pi / N)},$$

where $\Phi : (0, \infty) \to \mathbb{R}$ is a smooth nonnegative weight function.

Sarnak remarked that these double averages are related to the 1-level densities of the zeros of $L(s, \pi)$. Using random matrix theory, Katz and Sarnak predicted that these averages for $\theta < 1$ behave differently to $\theta > 1$. The murmurations phenomenon arises at the sharp phase transition when $\theta = 1$!

Murmurations in the weight aspect

Another breakthrough by Bober–Booker–Lee–Lowry-Duda:

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Theorem (Bober–Booker–Lee–Lowry-Duda 2023)

Assume GRH. Fix $\epsilon > 0$ small and $\delta \in \{0,1\}$. Fix a compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. Let $K, H \in \mathbb{R}_{>0}$ with $K^{\frac{5}{6}+\epsilon} < H < K^{1-\epsilon}$. As $K \to \infty$:

$$\frac{\sum_{p/N \in E} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \leq H}} \sum_{f \in H_k^{new}(1)} \lambda_f(p)}{\sum_{p/N \in E} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \leq H}} \sum_{f \in H_k^{new}(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}} \left(\frac{\nu(E)}{|E|} + o_{E,\epsilon}(1) \right),$$

where $H_k^{new}(1)$ is a basis of level 1 weight k newforms and where

$$\nu(E) = \frac{1}{\zeta(2)} \sum_{\substack{a,q \in \mathbb{Z}_{>0} \\ gcd(a,q)=1 \\ (a/q)^{-2} \in E}} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{q}{a}\right)^3 = \frac{1}{2} \sum_{t=-\infty}^{\infty} \prod_{p \nmid t} \frac{p^2 - p - 1}{p^2 - p} \cdot \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) dy.$$

Murmurations in the weight aspect

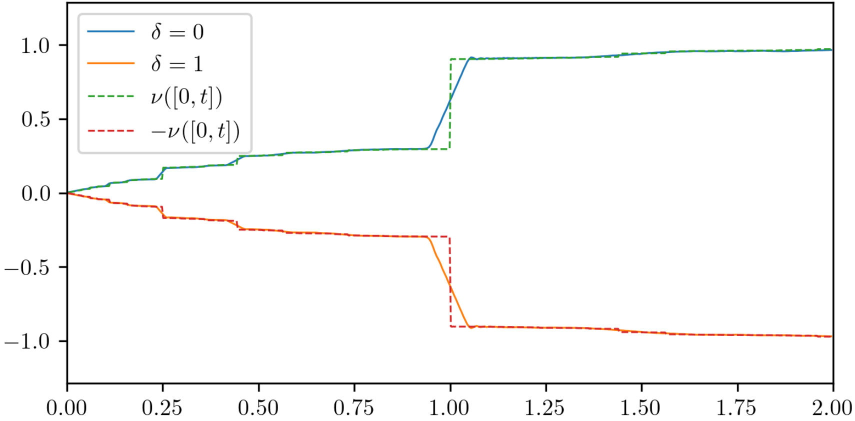

Figure: A comparison of $(-1)^{\delta}\nu([0,t])$ and the left-hand side of the main theorem, scaled by $t\sqrt{N}$, for $K = 3830$, $H = 100$, and $t \in [0,2]$. (Bober–Booker–Lee–Lowry-Duda 2023)

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Preprint, Available at: arXiv:2310.07746.

He, Y.-H., Lee, K.-H., Oliver, T., Pozdnyakov, A. (2022)
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Murmurations of Dirichlet characters
Preprint, Available at: arXiv:2307.00256

References

Sarnak, P. (2023)
Letter to Drew Sutherland and Nina Zubrilina

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Murmurations of arithmetic L-functions
Talk at *Arithmetic statistics* conference, CIRM.

Sutherland, A. (2022)
Letter to Michael Rubinstein and Peter Sarnak

Zubrilina, N. (2023)
Murmurations
Preprint, Available at: arXiv:2310.07681.

Suggested talk schedule

- **Week 3** (26 Jan): Work through He–Lee–Oliver–Pozdnyakov machine learning paper. Predicting ranks of elliptic curves using logistic regression. Background on other machine learning strategies.
- **Week 4** (02 Feb): Work through Drew Sutherland's and Peter Sarnak's letters. Give some background on existing conjectures and theorems on horizontal/vertical trace distributions of $a_p(f)$ (Sato-Tate conjecture, Katz-Sarnak philosophy, Birch, Serre, etc.)
- **Week 5 - 6** (09, 16 Feb): Murmurations of Dirichlet characters (Lee–Oliver–Pozdnyakov)
- **Week 7 - 9** (23 Feb; 01, 08 Mar): Murmurations of weight k newforms (Nina Zubrilina)
- **Week 10** (15 Mar): Murmurations of modular forms in the weight aspect (Bober–Booker–Lee–Lowry-Duda)