Murmurations in the weight aspect

Simon L Rydin Myerson, Murmurations study group, number theory seminar, University of Warwick

after Bober, Booker, Lee & Lowry-Duda (B2L2)

Recall the big picture:

We have $\mathcal{F}$ a family of L-functions

$$L_\pi(s) = \sum f_\pi(n) n^{-s},$$

each having its own separate functional equation $\Lambda(s) = w_\pi \overline{\Lambda}(1-s)$ where

$$\Lambda(s) := N_{arith}^{s/2} L_{\pi,\infty}(s) \cdot L_\pi(s)$$

and normalized so that the Ramanujan conjecture is $f_\pi(n) \ll n^\epsilon$. Then $L_\pi(s)$ has root number w_π (± 1 if $f_\pi(n)$ is real), the (arithmetic) conductor is N_{arith} and the analytic conductor is

$$N_\pi := \exp\left(2\mathrm{Re}\left(\frac{L'_{\pi,\infty}(1/2)}{L_{\pi,\infty}(1/2)}\right)\right) N_{arith}.$$

Previously we always had N_{arith} instead of N_π. B2L2 take a family with $N_{arith} = 1$, namely $\mathcal{F} = \bigcup_k H_k(1)$ where $H_k(1)$ is a basis of weight k newforms for $\Gamma_0(1) = \Gamma_1(0) = \Gamma_1 = \mathrm{PSL}_2(Z)$.

The Katz-Sarnak philosophy

For fixed smooth Φ supported in $[0,1]$, we expect

$$\frac{\sum_{p\in[N^a,2N^a]}\sum_{\substack{\pi\in\mathcal{F},w_\pi=w\\ N_\pi\in[N,2N]}}\Phi((N_\pi-N)/N)f_\pi(p)\sqrt{p}}{\sum_{p\in[N^a,2N^a]}\sum_{\substack{\pi\in\mathcal{F},w_\pi=w\\ N_\pi\in[N,2N]}}\Phi((N_\pi-N)/N)}\to\begin{cases}0 & (a<1)\\ const & (a>1).\end{cases}$$

Murmurations: $a=1$. For piecewise smooth Φ we guess that

$$\frac{\sum_{p\in[yN,yN+X]}\sum_{\substack{\pi\in\mathcal{F},w_\pi=w\\ N_\pi\in[N,N+Y]}}\Phi((N_\pi-N)/Y)f_\pi(p)}{\sum_{p\in[yN,yN+X]}\sum_{\substack{\pi\in\mathcal{F}\\ N_\pi\in[N,N+Y]}}\Phi((N_\pi-N)/Y)}\sim wN^{-1/2}M_\Phi(y)$$

provided we take the limit over a sequence of (N,X,Y) with $X,Y<N$ and with enough terms in the sum, namely

$$\sum_{p\in[yN,yN+X]}\sum_{\substack{\pi\in\mathcal{F},w_\pi=w\\ N_\pi\in[N,N+Y]}}1>N^{1+\epsilon}.$$

When this is false, we see noise and a picture like murmurations of starlings. May need more than $N^{1+\epsilon}$ to see lower-order terms in the asymptotic? (B2L2 see extra noise when these should be present.)

Theorem (B2L2)

Assume GRH for the L-functions of Dirichlet characters and modular forms. Fix $\varepsilon \in (0, \frac{1}{12})$, $\delta \in \{0, 1\}$, and a compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. Let $K, H \in \mathbb{R}_{>0}$ with $K^{\frac{5}{6}+\varepsilon} < H < K^{1-\varepsilon}$, and set $N = (K/4\pi)^2$. Then as $K \to \infty$, we have

$$\frac{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \le H}} \sum_{f \in H_k(1)} \lambda_f(p)}{\sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \le H}} \sum_{f \in H_k(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}} \left(\frac{\nu(E)}{|E|} + o_{E,\varepsilon}(1) \right),$$

where

$$\nu(E) = \frac{1}{\zeta(2)} \sum\nolimits^{*}_{\substack{a,q \in \mathbb{Z}_{>0} \\ \gcd(a,q)=1 \\ (a/q)^{-2} \in E}} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{q}{a}\right)^3 = \frac{1}{2} \sum_{t=-\infty}^{\infty} \prod_{p \nmid t} \frac{p^2 - p - 1}{p^2 - p} \cdot \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) dy,$$

and $*$ indicates terms at the endpoints of E are halved. *Notice: M_Φ is a distribution, the guess from the last slide is therefore false as a short average over p may diverge even with plenty of terms.*

Our object of study

Tragically the letter p is now replaced by n for the rest of the discussion. Please remember than n is a prime. The reason is that p, q and ℓ are used for other things.

Much of the technical difficulty comes from the sharp cutoffs $(a/q)^{-2} \in E, |k - K| \leq H$. If these were both smoothed there would by a power saving and no GRH for modular form L-functions.

At first I will stick to the proof sketch from B2L2 since we are really just doing routine manipulations. This deals with a semi-smoothed version, from which the result above follows. We study

$$\Sigma_0 = \sum_{\substack{n \,\mathrm{prime} \\ n/N \in E}} \log n \sum_{k \in 2\delta + 4\mathbb{Z}} W\left(\frac{k - k_0}{4h}\right) \sum_{f \in H_k(1)} \lambda_f(n) + O_{E,\varepsilon}\left(hK^{2+\varepsilon}\right).$$

First steps

Definition

$$\psi_D(m) = \left(\frac{d}{m/\gcd(m,\ell)}\right) \quad (D = d\ell^2)$$

$$\phi_{t,n} = \arcsin\left(\frac{t}{2\sqrt{n}}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad (t \in \mathbb{Z}, t^2 < 4n).$$

The Eichler-Selberg trace formula for $\sum_{f\in H_k(1)} \lambda_f(n)$ gives

$$\Sigma_0 = \frac{(-1)^\delta}{\pi} \sum_{\substack{n \text{ prime} \\ n/N \in E}} \log n \sum_{k \in 2\delta + 4\mathbb{Z}} W\left(\frac{k-k_0}{4h}\right) \sum_{\substack{t\in\mathbb{Z} \\ t^2 < 4n}} L(1, \psi_{t^2-4n}) \cos((k-1)\phi_{t,n}).$$

Second steps

We perform Poisson summation in k to get

$$\frac{(-1)^\delta}{\pi} \sum_{\substack{n \,\mathrm{prime} \\ n/N \in E}} \log n \sum_{\substack{t \in \mathbb{Z} \\ t^2 < 4n}} L(1, \psi_{t^2-4n}) \sum_{k \in 2\delta + 4\mathbb{Z}} W\left(\frac{k-k_0}{4h}\right) \cos((k-1)\phi_{t,n})$$

Definition

$$T = K^{1+\epsilon_0}/h.$$

By elementary arguments, the terms with $\ell \neq 0$ turn out to be $O(h^{-1}K^3 \log K)$; because the terms with $|t| > T$ are tiny, we get only a tiny error on replacing $2\phi_{t,n}$ by $tn^{-1/2}$. Hence

$$\Sigma_0 = \frac{(-1)^\delta h}{\pi} \sum_{\substack{n \,\mathrm{prime} \\ n/N \in E}} \log n \sum_{\substack{t \in \mathbb{Z} \\ |t| \leq T}} L(1, \psi_{t^2-4n}) \cos\left(\frac{(k_0-1)t}{2\sqrt{n}}\right)$$
$$\widehat{W}\left(\frac{ht}{\pi\sqrt{n}}\right) + O_{E,\varepsilon_0}\left(\frac{K^3 \log K}{h}\right).$$

Lemma 4.5

Now we reach the real number theory. We want to apply the PNT in APs, in the form (Lemma 4.5):
Assume GRH for Dirichlet L-functions. Let $t \in \mathbb{Z}$ and $A, B \in \mathbb{R}$ with $\frac{t^2}{4} < A < B$, and let $\Phi \in C^1([A, B])$. Set $M = \max_{u \in [A,B]} |\Phi(u)|$ and $V = \int_A^B |\Phi'(u)|\, du$. Then

$$\sum_{\substack{n \in [A,B] \\ n \text{ prime}}} L(1, \psi_{t^2-4n}) \Phi(n) \log n = L(1, \overline{\psi}_t) \int_A^B \Phi(u)\, du$$

$$+ O_\varepsilon\left(M^{\frac{4}{5}} (M + V)^{\frac{1}{5}} B^{\frac{9}{10}+\varepsilon}\right) \quad \forall \varepsilon \in (0, \tfrac{1}{10}],$$

where

$$\overline{\psi}_t(m) = \frac{1}{\varphi(m^2)} \sum_{\substack{n \bmod m^2 \\ (n,m)=1}} \psi_{t^2-4n}(m).$$

Lemma 4.3

Actually to prove this we need (Lemma 4.3):
$L(s,\overline{\psi}_t)$ continues analytically to $\Re(s) > \frac{1}{2}$ and satisfies

$$L(1,\overline{\psi}_t) = Cf(t),$$

where

$$C = L(1,\overline{\psi}_1) = \prod_p \left(1 - \frac{1}{(p-1)^2(p+1)}\right) = 0.6151326573181718\ldots$$

and

$$f(t) = P(1,t) = \prod_{p|t} \left(1 + \frac{1}{p^2 - p - 1}\right).$$

Finally getting somewhere

Definition

$$\lambda_{k_0} = \frac{k_0 - 1}{4\pi\sqrt{N}}$$

$$x_{k_0}(\alpha) = \frac{k_0 - 1}{4\alpha h}$$

$$E = [\alpha_2^{-2}, \alpha_1^{-2}]$$

By the lemmas above we get

$$\Sigma_0 = \frac{2h(-1)^\delta}{\pi}\left(\frac{k_0 - 1}{4\pi}\right)^2 \int_{\lambda_{k_0}\alpha_1}^{\lambda_{k_0}\alpha_2} \sum_{\substack{t \in \mathbb{Z} \\ |t| \leq T}} L(1, \overline{\psi}_t)\cos(2\pi\alpha t)\widehat{W}\left(\frac{t}{x_{k_0}(\alpha)}\right)\frac{d\alpha}{\alpha^3} + O_{E,\varepsilon,\varepsilon_0}\left(\frac{K^{3+\varepsilon}}{h^{\frac{1}{5}}}\right).$$

Proposition 5.1

Now the (to me) most satisfying result in the paper is needed, which is Proposition 5.1:
Assume GRH for Dirichlet L-functions. Let $\alpha, \theta, x \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ with $x, q \geq 1$, $\gcd(a, q) = 1$, $\alpha = \frac{a}{q} + \theta$, and $|\theta| \leq \frac{1}{q^2}$.
Then,

$$\sum_{t \in \mathbb{Z}} L(1, \overline{\psi}_t) \cos(2\pi\alpha t) \widehat{W}\left(\frac{t}{x}\right) = \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} x W(x\theta) + O\big(qx^{-1} \max(1, x|\theta|)\big) + O_\varepsilon\big(q^3 x^{-\frac{7}{4}+\varepsilon} \max(1, x|\theta|)^{\frac{7}{2}}\big).$$

I wish I had time to get into the proof!
Morally, we can now 'just' use the circle method to estimate

$$\int_{\lambda_{k_0}\alpha_1}^{\lambda_{k_0}\alpha_2} \sum_{\substack{t \in \mathbb{Z} \\ |t| \leq T}} L(1, \overline{\psi}_t) \cos(2\pi\alpha t) \widehat{W}\left(\frac{t}{x_{k_0}(\alpha)}\right) \frac{d\alpha}{\alpha^3}.$$

This is not entirely straightforward especially at the endpoints.