Murmurations of Dirichlet Characters

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(Work by Lee, Oliver, Pozdnyakov)

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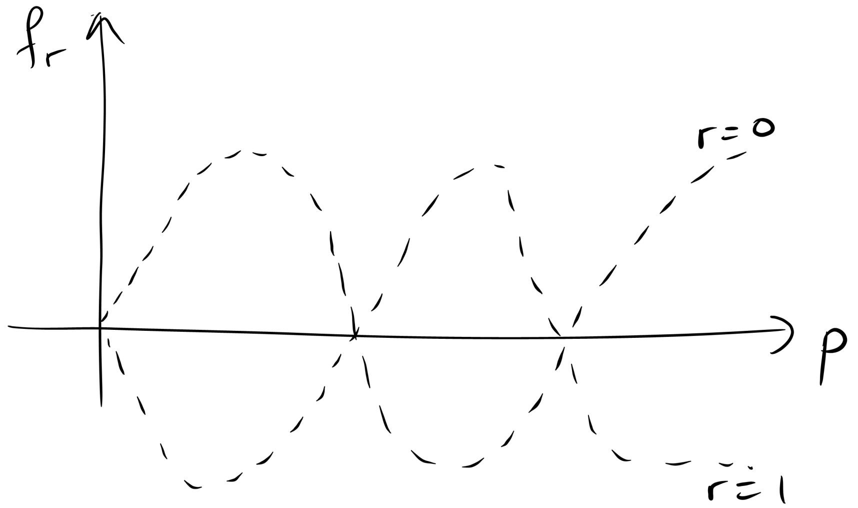
Recap: E elliptic curve over $\mathbb{Q}$

$$a_p(E) = p + 1 - \#E(\mathbb{F}_p)$$

$$\mathcal{E}_r(N_1, N_2) = \{\text{elliptic curves } E/\mathbb{Q},\ \text{rank } r,\ \text{conductor} \in [N_1, N_2]\}.$$

$$f_r(p) = \frac{1}{\#\mathcal{E}_r(N_1,N_2)} \sum_{E \in \mathcal{E}_r[N_1,N_2]} a_p(E) \qquad \left(|a_p(E)| \le 2\sqrt{p}\right)$$



Dirichlet Characters mod N ← prime

$\chi : \mathbb{Z} \longrightarrow \mathbb{C}$

- $\chi(a) = 0$ when $(a, N) > 1$. $\qquad (|\chi(a)| = 1 \text{ or } 0)$
- χ is completely multiplicative.

• χ is periodic with period N.

Gauss sum: $\tau(\chi) = \sum_{a \bmod N} \chi(a) e^{2\pi i a/N}$

$\left(|\tau(\chi)| = \sqrt{N} \text{ for } \chi \neq \chi_0, \text{ where } \chi_0(n) = 1_{(n,N)=1} \right)$

trivial character

Correspondence/Dictionary between elliptic curves and Dirichlet Characters:

$a_p(E) \rightsquigarrow \chi(p)$ (suitably weighted)

rank of $E \rightsquigarrow \chi$ odd $(\chi(-1) = -1)$
or χ even $(\chi(-1) = +1)$

cond of $E \rightsquigarrow$ cond of χ
(= modulus of χ as N prime)

$D_+(N) = \{$ (primitive) characters χ mod N, even $\}$

$D_-(N) = \{$ " " " , odd $\}$

$\# D_\pm(N) \asymp N$.

We aim to study the following double sum:

(want $\left|\frac{x}{\tau(\chi)}\right| \sim |\sqrt{x}|$ to match elliptic case)

Fix $\frac{1}{2} < \delta < 1$ (assume RH)

$$\frac{1}{X^{\delta}/\log X} \cancel{\frac{1}{X}} \sum_{\substack{N \in [X, X+X^{\delta}] \\ (\text{prime})}} \sum_{\chi \in D_+(N)} \frac{\chi(p)}{\tau(\chi)} \cancel{X}$$

(will assume N prime to simplify matters)

$$\text{Let } P_{\pm}(y, X, \delta) := \frac{\log X}{X^{\delta}} \sum_{\substack{N \in [X, X+X^{\delta}] \\ N \text{ prime}}} \sum_{\chi \in D_{\pm}(N)} \frac{\chi(\lceil yX \rceil^{p})}{\tau(\chi)}$$

where $\lceil x \rceil^{p}$ = smallest prime $\geq x$.

<u>Thm</u>. Assume RH. Then:

$$P_{\pm}(y, X, \delta) \xrightarrow[\text{as } X \to \infty]{} \begin{cases} \cos(2\pi y) & \text{if } + \\ -i \sin(2\pi y) & \text{if } - \end{cases}$$

(without RH, could take $\delta \in (\frac{7}{12}, 1)$)

<u>Proof</u>: We have $\frac{1}{\tau(\chi)} = \frac{\chi(-1)}{N} \tau(\overline{\chi})$

("baby version of trace formula")

$$\Rightarrow \sum_{D_+(N)} \frac{\chi(p)}{\tau(\chi)} = \frac{1}{N} \sum_{D_+(N)} \lambda(p)\tau(\overline{\chi}) \qquad (p \neq N)$$

Consider: $$\sum_{\chi \bmod N} \chi(p)\tau(\overline{\chi}) = \sum_{\chi \bmod N} \chi(p) \sum_{a \bmod N} \overline{\chi}(a) e^{\frac{2\pi i a}{N}}$$

$$= \sum_{a \bmod N} e^{\frac{2\pi i a}{N}} \underbrace{\sum_{\chi} \chi(p)\overline{\chi}(a)}_{\varphi(N)\cdot \mathbb{1}_{p \equiv a\ (N)} \text{ by orthogonality.}}$$

$$= \underline{\varphi(N) e^{\frac{2\pi i p}{N}}}$$

$$\cos\left(\frac{2\pi p}{N}\right) + i\sin\left(\frac{2\pi p}{N}\right) = \frac{1}{\varphi(N)} \sum_{\chi \bmod N} \chi(p)\tau(\overline{\chi})$$

Even function (in p) Odd function (in p)

$$\Rightarrow \cos\left(\frac{2\pi p}{N}\right) = \frac{1}{\varphi(N)} \sum_{\chi \in D_+(N)} \chi(p)\tau(\overline{\chi}) + \frac{\tau(\chi_0)}{\varphi(N)}$$

$\varphi(N) = N-1 \sim N.$

(trivial character not included in $D_+(N)$, as $N > 1$)

Remark: Can think of trivial character as $1 \notin$ {primes}, {composites} :)

So: $$\sum_{\chi \in D_+} \frac{\chi(p)}{\tau(\chi)} = \cos\left(\frac{2\pi p}{N}\right) + O(1)$$

Then: $$\frac{1}{X^\delta/\log X} \sum_{N\in[X,X+X^\delta]} \sum_{\chi\in D_+(N)} \frac{\chi(\lceil yX\rceil^p)}{\tau(\chi)} = \cos\left(\frac{2\pi \lceil yX\rceil^p}{X}\right) + \text{error}$$

$\approx$

As $\lceil yX\rceil^p \simeq yX$, so $\cos(2\pi y) + \text{error}$

$\downarrow$ as $X\to\infty$

0

$$\left(\text{RH gives } \sum_{N\in[X,X+X^\delta]} 1 \sim \frac{X^\delta}{\log X}\right)$$

□

Get same behaviour with odd characters! :)

We can also take N over larger intervals, e.g. $[X, 2X]$
(get Riemann sum which converges to integral)

Looking ahead:

- Results extend to imprimitive characters, and composite conductors N.
- Next week, χ quadratic character (Legendre symbol)
  much thinner set, so harder!
- Use of "trace formula".

$$\left(\frac{1}{\tau(\chi)} = \frac{\chi(-1)}{N}\tau(\overline{\chi}) \quad \leftrightsquigarrow \quad \sum\left\{\begin{matrix}\text{geometry}\\ \text{physical}\end{matrix}\right\} = \sum\left\{\begin{matrix}\text{spectral}/\\ \text{frequency}\end{matrix}\right\}\right)$$