Murmurations of Dirichlet characters

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(Work by Lee - Oliver - Pozdnyakov)

Speaker: Philip Holdridge.

Aim: Study murmurations of quadratic Dirichlet characters.

Setup: d squarefree, odd, $d \in \mathbb{Z}$. (could be < 0)

Define: $\chi_{8d}(n) = \left(\frac{8d}{n}\right)$ ← real primitive Dirichlet character.

the Kronecker symbol (generalisation of Legendre)

Let $\mathcal{F} = \{d \text{ squarefree, odd, } d \in \mathbb{Z}\}$, $y > 0$ fixed.

$\delta \in (\frac{1}{2}, 1)$ (?) Actually should be $\delta \in (\frac{9}{10}, 1)$!

We study:

$$M_\Phi(y, X, \delta) := \frac{\log X}{X^{1+\delta}} \sum_{p \in [yX, yX + X^\delta]} \sum_{d \in \mathcal{F}} \chi_{8d}(p) \sqrt{p}\, \Phi\left(\frac{d}{X}\right)$$

↑ compactly supported on $[-\beta, \beta]$ say. Φ Schwarz.

Thm: As $X \to \infty$

$$M_\Phi(y, X, \delta) \longrightarrow M_\Phi(y, \delta) = \frac{1}{2} \sum_{\substack{a=1 \\ (a,2)=1}}^{\infty} \frac{\mu(a)}{a^2} \sum_{m=1}^{\infty} (-1)^m \tilde{\Phi}\left(\frac{m^2}{2a^2 y}\right)$$

where $\tilde{\Phi}$ is similar to Fourier transform:

$$\tilde{\Phi}(\xi) := \int_{-\infty}^{\infty} \left(\cos(2\pi \xi x) + \sin(2\pi \xi x)\right) \Phi(x)\, dx$$

("real Fourier transform")

Proof: d squarefree $\iff \mu^2(d) = 1$

$$\left(\text{note: } \mu^2(d) = \sum_{a^2 | d} \mu(a) \right)$$

$$\sum_{\substack{d \in \mathbb{Z} \\ (d,2)=1}} \left(\sum_{a^2|d} \mu(d) \right) \Phi\left(\frac{d}{x}\right) \chi_{8d}(p) \sqrt{p}$$

Remainder term:

Main term

$$M_{\Phi, A}(y, X, \delta) := \sum_{\substack{a^2|d \\ a \leq A}} \mu(a)$$

$$\sum_{\substack{a^2|d \\ a > A}} \mu(a) =: R_{\Phi, A}(y, X, \delta)$$

Want to show remainder term is $o(1)$.

$$R_{\Phi, M}(y, X, \delta) = \frac{\log X}{X^{1+\delta}} \sum_p \sum_{\substack{d \in \mathbb{Z} \\ (d,2)=1}} \sum_{\substack{a^2|d \\ a>A}} \mu(d) \Phi\left(\frac{d}{x}\right) \chi_{8d}(p) \sqrt{p}$$

$$= \frac{\log X}{X^{1/2+\delta}} \sum_p \sum_{\substack{d \in \mathbb{Z} \\ (d,2)=1}} \sum_{\substack{a^2|d \\ a>A}} \mu(d) \Phi\left(\frac{d}{x}\right) \chi_{8d}(p) \sqrt{\frac{p}{X}}$$

(naive bound, taking absolute values, gives $\approx X^{1/2+\varepsilon}$)

Two sources of savings:

- GRH $\Rightarrow \left| \sum_{p \in [yX, yX + X^\delta]} \chi_{8d}(p) \right| \ll X^{1/2+\varepsilon}$

Also, trivially have the bound:

- $$\sum_{\substack{a^2 \mid d \\ a > A}} \mu(a) = \begin{cases} d^{\varepsilon} & \text{if } a^2 \mid d \text{ for some } a > A \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{a > A} \underbrace{\sum_{|b| \leq \frac{\beta X}{a^2}} \Phi\left(\frac{a^2 b}{X}\right)}_{\ll \frac{X}{a^2}} X^{\frac{1}{2}+\varepsilon}$$

This gives $R_{\Phi, A}(y, X, \delta) \ll \dfrac{X^{1+\varepsilon-\delta}}{A}$

so can choose $\underline{A = X^{-1+\delta-2\varepsilon}}$:)

This gets rid of the error term. For the main term:

$$M_{\Phi, A}(y, X, \delta) = \frac{\log X}{X^{1/2+\delta}} \sum_{p} \sum_{\substack{d \in \mathbb{Z} \\ (d,2)=1}} \sum_{\substack{a^2 \mid d \\ a \leq A}} \mu(d) \Phi\left(\frac{d}{X}\right) \chi_{8d}(p) \sqrt{\frac{p}{X}}$$

(<u>can swap order of summation and apply Poisson summation</u>)

$$= \frac{\log X}{X^{\delta}} \sum_{p} \frac{1}{2} \cancel{\left(\frac{16}{p}\right)} \sum_{\substack{(a, \cancel{2p})=1 \\ a \leq A}} \frac{\mu(a)}{a^2} \sum_{k \in \mathbb{Z}} (-1)^k \left(\frac{k}{p}\right) \tilde{\Phi}\left(\frac{kX}{2a^2 p}\right)$$

1 if p is odd

small a will be coprime to p!

$\dfrac{X}{p} \approx y.$

Suppose k is a non-square:

$$(\circledast)\quad \sum_{p\in[yX,\, yX+X^{\delta}]} \left(\frac{k}{p}\right)\tilde{\phi}\left(\frac{kX}{2a^2p}\right) = \left[\tilde{\phi}\left(\frac{kX}{2a^2t}\right)\Psi_k(t)\right]_{yX}^{yX+X^{\delta}}$$

$$- \int_{yX}^{yX+X^{\delta}} \Psi_k(t)\frac{d}{dt}\tilde{\phi}\left(\frac{kX}{2a^2t}\right)dt$$

$$\ll X^{\frac{1}{2}+\varepsilon'}\left(1+\int_0^{\infty}\tilde{\phi}'(u)\,du\right)$$

where $\Psi_k(t) = \sum_{3\le k\le t}\left(\frac{k}{p}\right) \ll X^{\frac{1}{2}+\varepsilon'}$ (by GRH)

We apply this bound when $|k| < X^{\varepsilon'}$ non-square.

Use: $\tilde{\Phi}\left(\frac{kX}{2a^2p}\right) \ll_{\alpha} \left(\frac{kX}{2a^2p}\right)^{-\alpha}$ for any $\alpha > 0$.

$$\Rightarrow \sum_{|k|\geq X^{\varepsilon'}-1} (\circledast \text{ above}) \ll \int_{X^{\varepsilon'}-1}^{\infty}\left(\frac{u}{2a^2y}\right)^{-\alpha}du \ll X^{-\varepsilon'\alpha} \quad \underline{(\text{claim!})}$$

Correct bound should be: $\ll A^{2\alpha}X^{-\varepsilon'(\alpha-1)}$

$$= \left(X^{1-\delta+2\varepsilon}\right)^{2\alpha}X^{-\varepsilon'(\alpha-1)}$$

<u>Required to have δ sufficiently close to 1!</u>

(probably $\delta > \frac{9}{10}$?)