Murmurations of newforms

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Introduction: (He – Lee – Oliver – Pozdnyakov)

$E/\mathbb{Q}$ elliptic curve

Aim: Predict rank(E)

$$\{a_p \mid p \leq T\}$$
$$\Big|$$
$$\text{rank } E$$

$$\mathcal{E} = \mathcal{E}_r[N_1, N_2] = \{ E/\mathbb{Q} \text{ rank } r,\ N \in [N_1, N_2] \}.$$

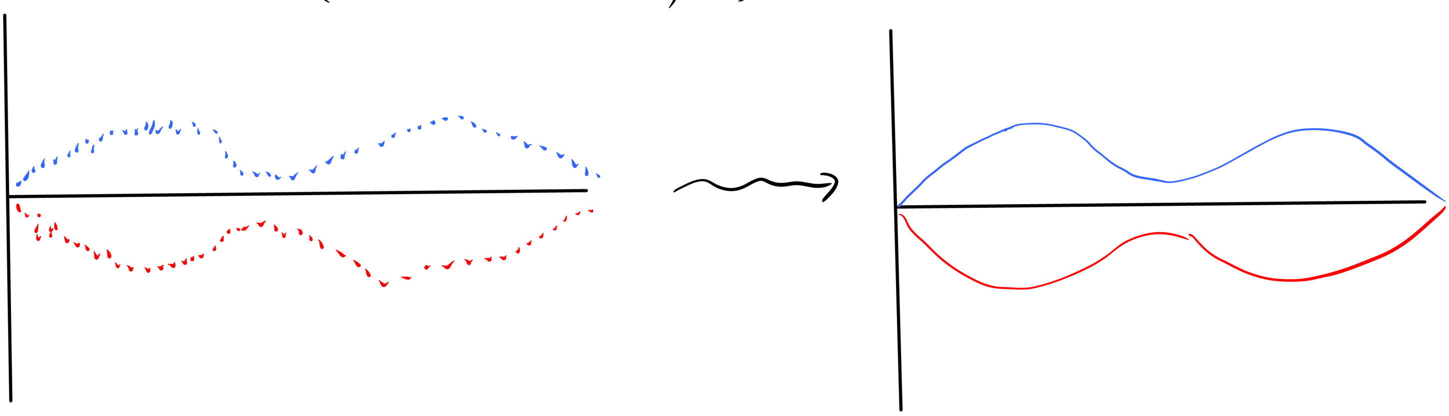
Computed $\sum_{E \in \mathcal{E}} a_E(p) / |\mathcal{E}|$

Sutherland:

For weight 2 newforms on $\Gamma_0(N)$, $N \in [2^{13}, 2^{14})$

Galois orbits at most 2^d, $d \in [0, 5]$.

(As d increases, fuzziness decreases :)



Blue: $\bar{a}_p$ avg over even.

Red: $\bar{a}_p$ avg over odd

Thm (1) (Zubrilina)

Let $H^{new}(N)$ be a Hecke basis for trivial character weight k cusp newforms for $\Gamma_0(N)$ with $f \in H^{new}(N)$ renormalised to have lead coeff 1. Let $\varepsilon(f)$ be the root number of f, let $a_p(f)$ be p^{th}-Fourier coeff of f. $\lambda_f(p) = a_p / p^{\frac{k-1}{2}}$.

Let X, Y, and P be parameters going to infinity with P prime. Assume further that $Y = (1+o(1)) X^{1-\delta_1}$ and $P \ll X^{1+\delta_2}$ for some $\delta_1, \delta_2 > 0$. $2\delta_1 < \delta_2 < 1$.

Let $y = \frac{P}{X}$. then:

$$\frac{\sum^{\square}_{N \in [X, X+Y]} \sum_{f \in H^{new}(N)} \lambda_f(p) \sqrt{p}\, \varepsilon(f)}{\sum^{\square}_{N \in [X, X+Y]} \sum_{f \in H^{new}(N)} 1} =$$

$$= D_k A \sqrt{y} + (-1)^{\frac{k-1}{2}} D_k B \sum C(r) \sqrt{4y - r^2}\, U_{k-2}\left(\frac{r}{2\sqrt{y}}\right)$$
$$- D_k \delta_{k=2} \pi y + O_\varepsilon\left(X^{-\delta' + \varepsilon} + \frac{1}{P}\right)$$

where $U_n(\cos\theta) = \dfrac{\sin(\theta + n\theta)}{\sin\theta}$ and $\delta' = \min\left\{\frac{\delta_2}{2} - \delta_1, \frac{1}{9} + \frac{\delta_2}{9} - \delta_1\right\}$

and $A = \ldots$ $B = \ldots$ $C(r) = \ldots$ $D_k = \ldots$

Define weight k murmuration density:

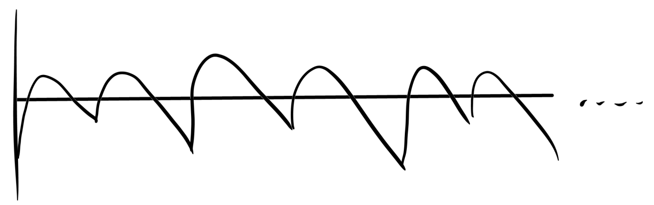
$$M_k(y) := D_k A\sqrt{y} + (-1)^{\frac{k}{2}-1} D_k B \sum_{1\le r\le 2\sqrt{y}} c(r)\sqrt{4y-r^2}\, U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) - D_k \delta_{k=2}\pi y$$

Thm 2

Let $k=2$. $M_2(y)$ as above. Then as $y\to\infty$:

$$M_2(y) = y^{1/4}\frac{2BD_2}{\pi}\sum_{\substack{1\le d\le 2\sqrt{q}\\ d\ \square\text{-free}}} Q(d)\, d^{1/2}\, \zeta\left(-\tfrac{1}{2}, \left\{\tfrac{2\sqrt{y}}{d}\right\}\right) + O(1)$$

where $\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$, and $Q(d) = \prod_{p\mid d}\frac{p^2}{p^4-2p^2-p+1}$

Sketch of $M_2(y)$: 

(is a convergent sum of periodic functions)

Thm 3

Let $p \ll X^{6/5}$, let $c>1$ be a constant, $Z = cX$, $y = \frac{p}{X}$. Then as $X\to\infty$:

$$\frac{\sum\sum \lambda_f(p)\sqrt{p}\,\varepsilon(f)}{\sum\sum 1} = \frac{2}{c^2-1}\int_1^c u\, M_k\left(\frac{y}{u}\right)du + o_y(1)$$

Q: Is this just $o(1)$?? (not uniform in y?)

where for $k=c=2$:

$$\begin{cases} \alpha\sqrt{y}-\beta y & \text{on } [0,\frac{1}{4}] \\ \alpha\sqrt{y}-\beta y+\gamma\pi y^2-\gamma(1-2y)\sqrt{y-\frac{1}{4}} \\ \quad -2\gamma y^2\arcsin\left(\frac{1}{2y}-1\right) & \text{on } [\frac{1}{4},\frac{1}{2}] \\ \alpha\sqrt{y}-\beta y+2\gamma y^2\left(\arcsin\left(\frac{1}{y}-1\right)-\arcsin\left(\frac{1}{2y}-1\right)\right) \\ \quad -\gamma(1-2y)\sqrt{y-\frac{1}{4}}+2\gamma(1-y)\sqrt{2y-1} & \text{on } [\frac{1}{2},1] \end{cases}$$

(where α, β, γ constants)

Thm 4

Let $\Phi : (0,\infty) \longrightarrow \mathbb{C}$ be a compactly supported smooth weight function and let

$$M_\Phi(y) = \left(\int_0^\infty M_2\left(\frac{y}{u}\right)\Phi(u)u^2\,\frac{du}{u}\right) \Big/ \int_0^\infty \Phi(u)u^2\,\frac{du}{u}.$$

Then $M_\Phi(y)$ is cts on $(0,\infty)$, $M_\Phi(0)=0$ and as $y\to\infty$, $M_\Phi(y) = 1 + o_y(1)$

Is this also $o(1)$??

Setup of the proofs:

N $\square$-free positive integer, p prime, $p\nmid N$.

$a_f(p) = \lambda_f(p)\,p^{\frac{k-1}{2}}$, $\varepsilon(f)$ root number

$(-1)^{k/2}\varepsilon(f)$ eigenvalues of f under the Atkin-Lehner involution W_N.

<u>Goal</u>: average $a_f(p)\,\varepsilon(f)$, by computing:

$$\mathrm{Tr}\left((-1)^{k/2} T_p \circ W_N \,,\ S^{new}(N) \right)$$

<u>Thm</u> (Trace Formula, Yamuchi, Skorappa-Zagier)

For N $\square$-free and $p \nmid N$:

$$\sum_{f \in H_k^{new}(N)} \sqrt{p}\,\lambda_f(p)\,\varepsilon(f) = \frac{H_1(-4pN)}{2} + (-1)^{\frac{k}{2}-1} U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{p}}\right) \cdot \sum_{0<r\leq 2\sqrt{\frac{p}{N}}} H_1(r^2N^2-4pN) - \delta_{k=2}(p+1)$$

where $H_1(-d)$ = Hurwitz class number $\left(\begin{array}{c}\text{disappears if}\\ d \neq 3\square \text{ or } 4\square\end{array}\right)$

$$H_1(-d) = \sum_{\substack{f\in\mathbb{N}\\ f^2\mid d}} h\left(\frac{-d}{f^2}\right) + O(1)$$ where h is Gauss class number.

$$\mathrm{Tr}\left(T_\ell \circ W_{n_1},\ S_{2k-2}(n_1 n_2)\right) =$$

$$= -\frac{1}{2} \sum_{\substack{n'\mid n_1\\ n_1/n' = \square}} \mu\left(\sqrt{\frac{n_1}{n'}}\right) \sum_{\substack{s^2\leq 4\ell n'\\ \sqrt{n_1 n'}\mid s}} P_{2k-2}\left(\frac{s}{\sqrt{n'}}, \ell\right) \sum_{\substack{t\mid n_2\\ n_2/t\ \square\text{-free}}} H_t(s^2-4\ell n')$$

where $P_k(s,\ell)$ coeff x^{k-2} in the power series of $(1-sx+\ell x^2)^{-1}$

Assure $p \neq 2$:

- Square factors of $4PN$ are $1, 4$.
- If prime $r \geq 1$, $q^2 \mid N(r^2N - 4P)$

other: $q^2 \mid r^2N - 4P$

$q \mid N$, $q \mid 4P$, $q = 2$, N even, $N = 2N'$ N' odd.

Then for any d with $4d^2 \mid r^2N^2 - 4PN$

$$\frac{r^2N^2 - 4PN}{4d^2} = \frac{r^2N^2 - 2PN}{d^2} \quad \text{always } 2 \text{ or } 3 \bmod 4.$$

The trace formula becomes:

$$\sum_{f \in H^{new}(N)} \sqrt{P}\, \lambda_f(p)\, \varepsilon(P) = \frac{h(-4PN)}{2} + \frac{h(-PN)}{2} - \delta_{k=2}(P) + O(1)$$

$$+ (-1)^{k/2}\, U_{k-2}\left(\frac{r\sqrt{N}}{2\sqrt{P}}\right) \sum_{1 \leq r \leq 2\sqrt{\frac{P}{N}}} \; \sum_{d^2 \mid r^2N - 4P} h\left(N(r^2N - 4P)/d^2\right)$$