Murmurations of newforms II

01 Mar

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Last time: Applied the Trace formula

$$\Rightarrow \sum_{f \in H_k^{new}(N)} \sqrt{p}\, \lambda_f(p)\, \varepsilon(f) = \frac{h(-4PN)}{2} + \frac{h(-PN)}{2}$$

$$- P 1_{k=2} + O(1) + \dots \sum_{r,d} h\left(\frac{N(r^2 N - 4P)}{d^2}\right)$$

(Want to average over N in short interval)

For $\mu^2(N) = 1$ and $P \nmid 2N$ prime.

Task: Average over squarefree $N \in [X, X+Y]$ $\quad Y = o(X)$

Useful reference: Iwaniec-Kowalski "Analytic Number Theory"
(often referred to as "the bible"!)

Start with $h(-PN)$. Similar for $h(-4PN)$

Dirichlet class number formula for $K = \mathbb{Q}(\sqrt{d})$

For $d < 0$, $d \equiv 0$ or $1 \bmod 4$:

$$L(1, \chi_d) = \frac{2\pi h}{\sqrt{|d|} \cdot |\mathcal{O}_K^\times|}$$

where $\chi_d(n) = \left(\frac{d}{n}\right)$ is the Kronecker symbol

Is completely multiplicative where $\left(\frac{p}{2}\right) = \left(\frac{2}{p}\right)$ for p prime. This gives:

$$\Rightarrow h(d) = \begin{cases} \frac{\sqrt{|d|}}{\pi} L(1, \chi_d), & d \equiv 0 \text{ or } 1 \bmod 4 \\ 0, & d \equiv 2 \text{ or } 3 \bmod 4 \end{cases}$$

for $d < -4$. Note χ_d mod $4d$.

(can define $h(d)$ as number of primitive binary quadratic forms)

Task I: Average $L(1, \chi_{-PN})$ over N squarefree with $PN \equiv 3 \bmod 4$.

By Pólya-Vinogradov, for $\chi \neq \chi_0 \bmod q$

$$\sum_{T \leq n \leq U} \chi(n) \ll \sqrt{q} \log q$$

(avg is 0 over entire period)

$$\Rightarrow L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n \leq T} \frac{\chi(n)}{n} + O\left(\frac{\sqrt{q} \log q}{T}\right)$$

(for any suitable T, say $T = q$)

($\ll \log q$, by "Siegel's bound")

deeper, but this is all we use.

Henceforth, think $P \approx N \sim X$ and $T = P^{\theta}$, $0 < \theta < 1$.

Will focus on main term:

$$\Rightarrow \sum_{\substack{N \in [X, X+Y] \\ PN \equiv 3 \bmod 4}} \mu^2(N) \sqrt{\frac{N}{X}} L(1, \chi_{-PN})$$

$$\approx \sum_{\substack{N \in [X, X+Y) \\ PN \equiv 3 \bmod 4}} \mu^2(N) \sqrt{\frac{N}{X}} \sum_{n \in 1} \frac{\left(\frac{-PN}{n}\right)}{n}$$

← (will switch order of summation)

$$= S_q + NS_q$$

$n = \square$ main.

$N \sim X$

$$\Rightarrow S_q \approx \sum_{m \in T} \frac{1}{m^2} \sum_{\substack{N \in [X, X+Y] \\ PN \equiv 3 \bmod 4}} \mu^2(N) \left(\frac{-PN}{m^2}\right)$$

$$= \sum_{\substack{m \leq \sqrt{T} \\ (m,p) = 1}} \frac{1}{m^2} \sum_{N \in [X, X+Y]} \mu^2(N) \left(\frac{N}{m^2}\right) \frac{\chi_1(PN) - \chi_2(PN)}{2}$$

detects $PN \equiv 3 \bmod 4$

where χ_1 principal, χ_2 mod 4

* $\left(\frac{N}{m^2}\right) \chi_1(N)$ principal mod $2m$
* $\left(\frac{N}{m^2}\right) \chi_2(N)$ non-principal mod $4m$

Task I': Average $\chi(N)$ over squarefree $N \in [X, X+Y]$.

Use Perron to get $cY + o(Y)$, $c \geq 0$

where $c > 0$ iff $\chi = \chi_0$.

$$\mathcal{L}_\chi(s) = \sum_{n=1}^{\infty} \frac{\mu^2(n)\chi(n)}{n^2} = \prod_p \left(1 + \frac{\chi(p)}{p^s}\right)$$

$$= \frac{\prod_p \left(1 - \frac{\chi^2(p)}{p^{2s}}\right)}{\prod_p \left(1 - \frac{\chi(p)}{p^s}\right)} = \frac{L(s,\chi)}{L(2s,\chi^2)}$$

If $\chi = \chi_0$ is principal, then there's a simple pole at $s = 1$, residue $= \frac{1}{\zeta(2)} \underbrace{\prod_{p \mid q} \frac{1}{1+\frac{1}{p}}}_{\eta(q)}$

Can average $\eta(q)$ using standard techniques

gives: $S_q \sim c_1 Y \quad c_1 > 0$.

Similarly, $NS_q = o(Y)$.

(This was for vanilla average...)

For the hard part, $\sum_r \sum_d$ outside

$$\rightsquigarrow \sum_{\substack{N \in [X, X+Y] \\ \text{congruences}}} \mu^2(N) \frac{L(1, \chi_{r^2N^2 - 4PN})}{\pi d} \sqrt{4PN - r^2N^2}$$

$$\sim \sqrt{4PX - r^2X^2}$$

$$\rightsquigarrow \sum_{n \leq T} \frac{S_{n,d,r}}{nd}, \quad \text{where}$$

$$S_{n,d,r} = \sum_{N \in [X, X+Y]} \mu^2(N) \left(\frac{N}{n}\right) \left(\frac{(r^2N - 4r)/d^2}{n}\right)$$

cf. $\sum_{N \in I} \chi(N) \chi(N+a)$

"defined on ...."

Added difficulty: Can be unbalanced.

<u>Task</u> $S_{n,d,r}$ avg

How to evaluate sum over entire period?

Want to compute: $\sum_{x \leq m} \left(\frac{x}{m}\right) \left(\frac{xr^2 - 4p}{m}\right)$

NMA $m = p^\alpha$, WMA, $\alpha = 1$

$\rightsquigarrow \#\{(x,y) \in \mathbb{F}_p^2 \mid f(x) = y^2\}$, f quadratic

$f(x) = x(x+a)$

$\doteq \#\{xy = u\} = p-1$ (standard ways)

$0 < \tau, \sigma < 1$

* $d \ll y^\tau$, $n \ll y^\sigma$ $\qquad \sum\limits_{N \equiv \text{residue class}} \mu^2(N)$

(Hooley) Equidistribution of squarefrees in congruence classes $\rightsquigarrow$ $\sim c_2 Y$, $c_2 > 0$

* $d \ll y^\tau$, $n \gg y^\sigma$

  Poisson summation $\rightsquigarrow$ $o(Y)$

* $d \gg y^\tau$

  Use Siegel's bound to average over n. $\square$