Murmurations of newforms III

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(work of Nina Zubrilina)

Notation:

$$A = \prod_p \left(1 + \frac{p}{(p+1)^2(p-1)}\right) \qquad B = \prod_p \frac{p^4 - 2p^2 - p + 1}{(p^2-1)^2}$$

$$c(r) = \prod_{p \mid r} \left(1 + \frac{p^2}{p^4 - 2p^2 - p + 1}\right) \qquad D_k = \frac{12}{(k-1)\pi \prod_p \left(1 - \frac{1}{p^2+p}\right)}$$

Recall: $Y = (1+o(1))X^{1-\delta_2}$, $\quad P \ll X^{1+\delta}$, $\quad y = \frac{P}{X}$

$$\frac{\sum^{\flat}_{N \in [X, X+Y]} \sum_{f \in H^{new}(N,k)} \lambda_f(P)\sqrt{P}\,\varepsilon(P)}{\sum^{\flat}_{N \in [X, X+Y]} \sum_{f \in H^{new}(N,k)} 1} =$$

$$= D_k A \sqrt{y} + (-1)^{\frac{k}{2}-1} D_k B \sum_{r \leq 2\sqrt{y}} c(r)\sqrt{4y - r^2}\, U_{k-2}\left(\frac{r}{2\sqrt{y}}\right) - D_k \delta_{k=2} \pi y$$

$$+ O(\text{error})$$

(everything here is small y)

For $k=2$, can simplify this, as there is a lot of cancellation. True order of magnitude smaller than what individual terms suggest.

Theorem (Zubrilina)

$$M_2(y) = y^{\frac{1}{4}} \frac{2BD_2}{\pi} \sum_{\substack{1 \le d \le 2\sqrt{y} \\ d\ \square\text{-free}}} Q(d)\, d^{\frac{1}{2}}\, \zeta\left(-\tfrac{1}{2}, \left\{\tfrac{2\sqrt{y}}{d}\right\}\right) + O(1)$$

where $Q(d) = \prod_{p \mid d} \frac{p^2}{p^4 - 2p^2 - p - 1}$ and $\zeta(s,a)$ is the Hurwitz zeta function

(analytic continuation of $\sum_{n=1}^{\infty} \frac{1}{(n+a)^s}$)

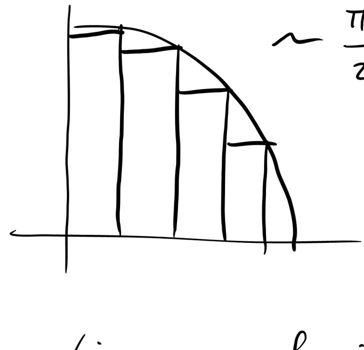
Proof (sketch):

Step 1: Essentially doing Moebius inversion (at least morally)

Note: $c(r) = \sum_{\substack{d \mid r \\ d\ \square\text{-free}}} Q(d)$ (consider $r = p_1 \cdots p_k$)

$$\Rightarrow \sum_{1 \le r \le 2\sqrt{y}} c(r) \sqrt{y - \frac{r^2}{4}} = \sum_{\substack{1 \le d \le 2\sqrt{y} \\ d\ \square\text{-free}}} Q(d) \underbrace{\sum_{1 \le r \le \frac{2\sqrt{y}}{d}} \sqrt{y - \frac{d^2 r^2}{4}}}_{\sim \frac{\pi y}{2d}}$$

(Consider Riemann sum for quarter-circle)



Define $F(y,d) = \frac{\pi y}{2d} - \sum \sqrt{y - \frac{d^2 \pi^2}{4}}$ (i.e. area of circle minus Riemann sum)

Now define $m(y)$ as $M_2(y)$, without D_2:

$$m(y) = A\sqrt{y} - \pi y + y B \pi \underbrace{\sum_{\substack{1 \le d \le 2\sqrt{y} \\ d\ \square\text{-free}}} \frac{Q(d)}{d}} - 2B \sum_{\substack{1 \le d \le 2\sqrt{y} \\ d\ \square\text{-free}}} F(d,y)\, Q(d)$$

Note: $$\sum_{\substack{d \geq 1 \\ \square\text{-free}}} \frac{Q(d)}{d} = \prod_p \left(1 + \frac{p}{p^4 - 2p^2 - p + 1}\right) = B^{-1} \quad :\!\smile$$

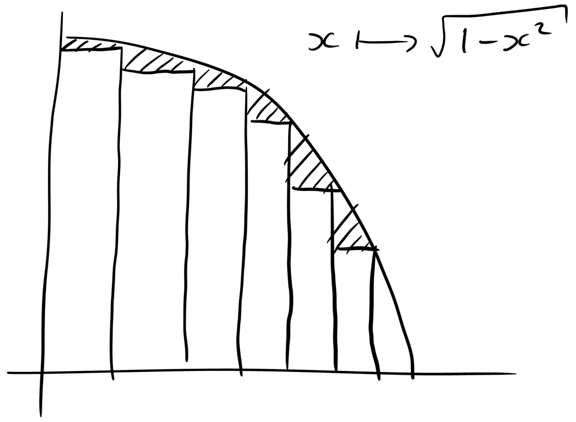
(so ignoring the tail, this cancels with $-\pi y$)

Tail: $$\sum_{d > 2\sqrt{y}} \frac{Q(d)}{d} = O(y^{-1}) \qquad \left(Q(d) \approx \frac{1}{d^2}\right)$$

So this shows that:

$$m(y) = A\sqrt{y} - 2B \sum_{\substack{1 \le d \le 2\sqrt{y} \\ d\ \square\text{-free}}} F(d,y)\, Q(d) \qquad (*)$$

Want to control the error, $\longrightarrow$ have to asymptotically expand the error in approximation.



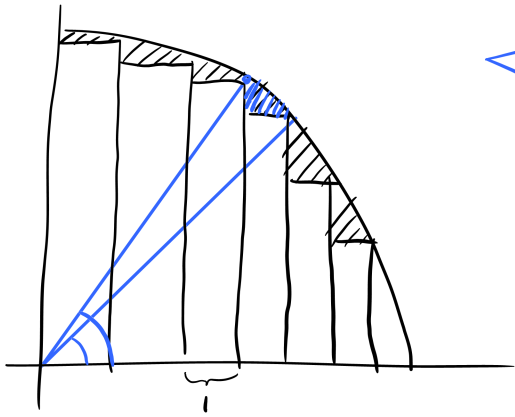
Natural tool: Use <u>Euler-Maclaurin</u> summation

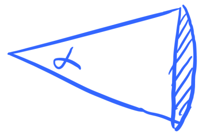
But, $\sqrt{1-x^2}$ not differentiable at $x = 1$, trouble?

(might be possible to work around this?)

Instead, do geometric argument:

Use cosines of angles, and apply trig!
(with many computations)





Let $A(t) = \frac{\pi}{4} - \sum_{1 \le s \le \frac{1}{t}} t\sqrt{1 - s^2 t^2}$

<u>Lemma</u>: $A(t) = \frac{t}{2} + t^{\frac{3}{2}} P\left(\frac{1}{t}\right) + O\left(t^{\frac{5}{2}}\right)$

where $P(t) = \sqrt{2}\, \zeta\left(-\frac{1}{2}, \{t\}\right)$

Insert into above expression for $m(y)$: $\circledast$

$$m(y) = A\sqrt{y} - 2B \sum_{\substack{1 \le d \le 2\sqrt{y} \\ d\ \square\text{-free}}} Q(d) \frac{2y}{d}\left(\frac{d}{4\sqrt{y}} + \frac{d^{\frac{3}{2}}}{2^{\frac{3}{2}} y^{\frac{3}{4}}} P\left(\frac{d}{2\sqrt{y}}\right) + O\left(\frac{d^{\frac{5}{2}}}{y^{5/4}}\right)\right) + O(1)$$

Main term of sum cancels with $A\sqrt{y}$.

$$\sum_{d\ \square\text{-free}} Q(d) = \prod_p \left(1 + \frac{p^2}{p^4 - 2y^2 - p + 1}\right) = \frac{A}{B} \quad :)$$

(same trick with extending sum to ∞ and bounding tail)

$A\sqrt{y} - B \sum Q(d)\sqrt{y}$ cancels!

Obtain: $m(y) = y^{1/4}\sqrt{2}\, B \sum Q(d)\sqrt{d}\, P\left(\frac{d}{2\sqrt{y}}\right)$ $\boxtimes$

Can also averege over larger intervds:

Geometric Averaging:

$Z = cX, \quad P \ll X^{6/5}, \quad y = \frac{P}{X}.$

$$\frac{\sum\limits_{N \in [X,Z]}^{\square} \sum\limits_{f \in H^{new}(N,k)} \lambda_f(P)\sqrt{P}\,\varepsilon(f)}{\sum\limits_{N \in [X,Z]}^{\square} \sum\limits_{f \in H^{new}(N,k)} 1} = \frac{2}{c^2-1}\int_1^c u M_k\left(\frac{y}{u}\right) du + o_y(1)$$

Proof strategy is straightforward. Just write $Z = X + gY$ with $Y \sim X^{1-\delta_1}$ and averge the previous result over $[X, X+Y], [X+Y, X+2Y], \ldots$:)

(replace sums with integrals, and check error terms)

Theorem:

$\Phi : (0,\infty) \to \mathbb{C}$ compactly supported smooth weight.

Define: $$M_\Phi(y) = \frac{\int_0^\infty M_2\left(\frac{y}{u}\right)\Phi(u)\,u^2\,du}{\int_0^\infty \Phi(u)\,u^2\,du}$$

Then: $M_\Phi(y) = 1 + o(1)$ as $y \to \infty$.

($o_y(1)$ is typo!)

<u>Idea</u>: Rewrites above sum using Mellin inversion, as a contour integral.

Can do standard trick of regularising L-function by comparing to Riemann Zeta function.

By moving contour to the left, and bounding the residues along the way, gives the result!