Analytic Number Theory

Lectures

1. Arithmetic functions

- **Arithmetic function:** A function $f : \mathbb{N} \to \mathbb{R}$. $(\mathbb{N} = \{1, 2, 3, \ldots, \})$

  f is **multiplicative** if $f(mn) = f(m)f(n)$ for all $(m, n) = 1$. f is **completely multiplicative** if $f(mn) = f(m)f(n)$ for all m, n.

- **Convolution:** Let f, g be arithmetic functions. The convolution is:

$$f * g(n) = \sum_{ab=n} f(a)g(b)$$

  Fact: Convolution is commutative and associative. If f, g multiplicative, then $f * g$ multiplicative. If $1 * f = g$, then $\mu * g = f$. The identity is $\delta = 1 * \mu$

- **Mobius inversion:** Let f, g be arithmetic functions such that $1 * f = g$:

$$g(n) = \sum_{d|n} f(d) \quad \text{for all } n \geq 1$$

  Then we have that $\mu * g = f$:

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) \quad \text{for all } n \geq 1$$

- **Von Mangoldt function:** Defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

- $$1 * \Lambda = \log \quad \Longrightarrow \quad \sum_{a|n} \Lambda(a) = \log n$$

- **Partial summation:** If $a_n \in \mathbb{C}$ and $f : \mathbb{R}_+ \to \mathbb{R}$ s.t. f' is continuous then

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt$$

  where $A(x) = \sum_{n \leq x} a_n$.

- Lemma 2:

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + \mathcal{O}\left(\frac{1}{x}\right)$$

  where $\gamma = 0.577\ldots$ is Euler's constant

- Lemma 3
$$\sum_{n\leq x} \log n = x\log x - x + \mathcal{O}(\log x)$$

- Let $\tau(n)$ denote the divisors function, $\tau(n) = 1 * 1(n) = \sum_{d|n} 1$. Then
$$\sum_{n\leq x} \tau(n) = x\log x + \mathcal{O}(x)$$
$$\sum_{n\leq x} \tau(n) = x\log x + (2\gamma - 1)x + \mathcal{O}(x^{1/2})$$

- **Prime number theorem:** Let $\pi(x) = \sum_{p\leq x} 1$ be the number of primes $\leq x$. Then
$$\pi(x) \sim \frac{x}{\log x}$$
This is equivalent to $\psi(x) \sim x$ by Lemma 5.

- **Chebyshev, 1850** We have $\psi(x) \asymp x$. I.e. there exists constants c_1, c_2 such that, for all large x:
$$c_1 x \leq \psi(x) \leq c_2 x$$
We can explicitly show $c_1 = \log 2 - \epsilon$ and $c_2 = 2\log 2 + \epsilon$ works.

- **Lemma 5:** Relation between $\pi(x)$ and $\psi(x)$:
$$\pi(x) = \frac{\psi(x)}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right)$$
In particular, $\pi(x) \asymp \frac{x}{\log x}$ and $\pi(x) \sim \frac{x}{\log x}$ iff $\psi(x) \sim x$.

- **Lemma 6:** (Merten's first theorem)
$$\sum_{p\leq x} \frac{\log p}{p} = \log x + \mathcal{O}(1)$$

- **Lemma 7:** (Merten's second theorem)
$$\sum_{p\leq x} \frac{1}{p} = \log\log x + b + \mathcal{O}\left(\frac{1}{\log x}\right)$$
where b is some constant.

- **Lemma 8:** (Merten's third theorem)
$$\prod_{p\leq x} \left(1 - \frac{1}{p}\right)^{-1} = c\log x + \mathcal{O}(1)$$
where $c > 1$ is some constant.

  Fact: The constant c is $c = e^{\gamma} \approx 1.78...$

- **Chebyshev, 1850s:** If limit for $\pi(x)$ exists, it must be 1.
$$\text{If } \pi(x) \sim c\frac{x}{\log x}, \text{ then } c = 1$$

2. Dirichlet series and the Rimenann zeta function

- **Complex exponentiation:** Let $n \in \mathbb{N}$ and $s \in \mathbb{C}$, then we define

$$n^s = e^{s \log n}$$

- **Dirichlet series:** Let $a_n : \mathbb{N} \to \mathbb{C}$ be a complex sequence. The Dirichlet series is:

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{where } a_n \in \mathbb{C}$$

- **Riemann Zeta function:**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

  which converges for $\sigma > 1$.

- There is an abscissa of convergence σ_c s.t. $F(s)$ converges for all $\sigma > \sigma_c$ and diverges for all $\sigma < \sigma_c$, and if $\sigma > \sigma_c$ there is a neighbourhood of s in which $F(s)$ converges uniformly In particular, f is holomorphic at s.

- If $\sum \frac{a_n}{n^s} = \sum \frac{b_n}{n^s}$ for all s in some half-place $\sigma > \sigma_0$ (where both converge), then $a_n = b_n$ for all n.

- If F_f and F_g are both absolutely convergent at s, then so is $F_f \cdot F_g$ and $F_f(s)F_g(s) = F_{f*g}(s)$

- For $\sigma > 1$,

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt$$

- **Product Convergence:** If $a_n \in \mathbb{C}$, then we say that $\prod_{n=1}^{\infty} a_n$ **converges** if the partial products $\prod_{n=1}^{\infty}$ converge to a non-zero value $s \in \mathbb{C} \backslash \{0\}$.

- $\prod_{n=1}^{\infty} a_n$ converges if and only if for any $\epsilon > 0$, there is N such that

$$\left| \prod_{k=n}^{m} a_k - 1 \right| < \epsilon \quad \text{for all } m > n \geq N$$

  In particular, $\lim_{n \to \infty} a_n = 1$.

- We say that $\prod(1 + a_n)$ **converges absolutely** if $\prod(1 + |a_n|)$ converges.

- If $\prod(1 + a_n)$ converges abosllutely, then it converges.

- If $a_n > 0$ for all $n \geq 1$, then

$$\prod(1 + a_n) \text{ converges} \iff \sum a_n \text{ converges}$$

- **Euler product:** If f is multiplicative and $\sum \frac{|f(n)|}{n^\sigma}$ converges, then

$$F_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right)$$

If f is completely multiplicative, then $f(p^k) = f(p)^k$, so

$$F_f(s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

Examples of Euler products:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad = \prod_p \left(1 - \frac{1}{p^s}\right)$$

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \quad = \prod_p \left(1 + \frac{1}{p^s}\right)$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta(s)^2$$

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k)$$

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}$$

$$\sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta'(s)$$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} = \log \zeta(s)$$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \quad \text{NB! Gives relation between } \zeta \text{ and } \Lambda \text{, and thus } \psi$$

- **Gamma function:** Define

$$\gamma = \lim_{N \to \infty} \left(\sum_{n \le N} \frac{1}{n} - \log N \right) = 0.577$$

and define the gamma function as:

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

This gives entire function (analytic for all $s \in \mathbb{C}$)

- $\Gamma(s)$ has no zeros, and has poles at $s = 0, -1, -2, \ldots$. The residue at $s = -n$ is $\dfrac{(-1)^n}{n!}$.

- **Euler definition**:
$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1}$$
and rewriting, we get
$$\Gamma(s) = \lim_{N\to\infty} N^s \frac{(N-1)!}{s(s+1)\dots(s+N-1)}$$
- We have $\Gamma(s+1) = s\Gamma(s)$. In particular, for positive integers n, we have $\Gamma(n) = (n-1)!$
- **Reflection formula**:
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
In particular, $\Gamma(1/2) = \sqrt{\pi}$.
- **Duplication formula**:
$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s)$$
- **Integral formula**: If $\sigma > 0$, then
$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt$$
- Particular values of Γ function:
$$\begin{aligned}
&\Gamma(1) = 1, \quad \Gamma(2) = 1, \quad \Gamma(3) = 2, \quad \Gamma(4) = 6, \quad \Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{N} \\
&\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}, \\
&\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \quad \text{for } n \in \{0,1,2,\dots\} \\
&\Gamma'(1) = -\gamma
\end{aligned}$$
- **Functional equation**: $\zeta(s)$ can be extended to a meromorphic function on $\mathbb{C}$, and for all s
$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$
or equivalently
$$\zeta(s) = \chi(s)\zeta(1-s) \quad \text{where} \quad \chi(s) = \pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(s/2)}$$
- **Bernoulli numbers:** We define the Bernoulli numbers by the generating function
$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$$
We have the recursive formula $B_0 = 1$, and for $k \geq 2$
$$\sum_{0\leq n\leq k-1}\binom{k}{n}B_n = 0$$
Examples:
$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}\dots$$

- Particular values of ζ:

$$\zeta(0) = B_1 = -\frac{1}{2} \qquad \zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \text{ for } n = 0, 1, 2, \dots$$

$$\text{E.g. } \zeta(-1) = -\frac{1}{12}, \quad \zeta(-2) = 0, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-4) = 0, \quad \zeta(-5) = -\frac{1}{252}, \dots$$

$$\zeta(2n) = (-1)^{n+1} 2^{2n-1} \pi^{2n} \frac{B_{2n}}{(2n)!}$$

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945},$$

$$\zeta'(0) = -\frac{1}{2} \log(2\pi), \quad \frac{\zeta'(0)}{\zeta(0)} = \log(2\pi)$$

- Integral representations of ζ:

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} du \qquad \text{for } \sigma > 1 \text{ (but also valid for } \sigma > 0)$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s \int_1^\infty \frac{f(u)}{u^{s+1}} du \qquad \text{for } \sigma > -1 \quad (\text{where } f(x) = \frac{1}{2} - \{x\})$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s(s-1) \int_1^\infty \frac{F(u)}{u^{s+2}} du \qquad \text{for } \sigma > -1 \quad (\text{where } F(x) = \int_0^x f(u) du)$$

$$\zeta(s) = s \int_0^\infty \frac{f(u)}{u^{s+1}} du \qquad \text{for } -1 < \sigma < 0$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \qquad \text{for } \sigma > 1$$

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1-s) \int_{\mathcal{C}} \frac{z^{s-1}}{e^z - 1} dz \qquad \text{for } \sigma > 1 \quad \text{where } \mathcal{C} \text{ is Hankel contour}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx \qquad \text{for } 0 < \sigma < 1$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1} dx \qquad \text{for } -1 < \sigma < 0$$

- Expansion of $\zeta(s)$ around $s = 0$:

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2} \log(2\pi) s + O(s^2)$$

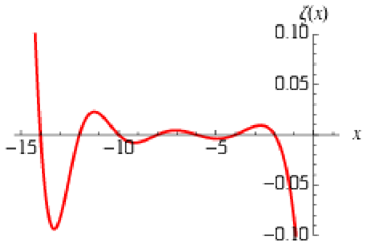
- Expansion of $\zeta(s)$ around $s = 1$:

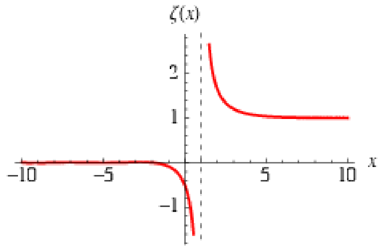
$$\zeta(s) = \frac{1}{s-1} + \gamma + O((s-1))$$

- Values of ζ along the real line between $\sigma = -15$ and $\sigma = 10$:

- **Jensen's inequality:** Suppose $f(z)$ is analytic in a disc with radius R, centre a, and that $|f(z)| \le M$ in this disc, and $f(a) \neq 0$. Then

$$|\#\text{ zeros of } f \text{ in disc centre } a, \text{ radius } r < R| \le \frac{\log \frac{M}{|f(a)|}}{\log \frac{R}{r}}$$





- When $0 < \delta \le \sigma \le 2$, and $|t| \ge 1$,

$$|\zeta(s)| \ll (1 + |t|^{1-\sigma}) \cdot \min\left(\frac{1}{|\sigma - 1|}, \log\left(|t| + 4\right)\right)$$

- Let $N(T)$ be the number of zeros of $\zeta(s)$ in $0 \le \sigma \le 1$, $0 \le t \le T$. THen for any $T \ge 4$

$$N(T+1) - N(T) \ll \log T$$

Corollary: $N(T) \ll T \log T$

3. Explicit formula

- **Riemann–von Mangoldt explicit formula:**

$$\psi(x) = \sum_{n \le x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)$$

  where ρ is over all zeros of $\zeta(s)$ in the critical strip $0 \le \sigma \le 1$

- **Borel–Carathéodory:** Let f be a holomorphic on $|z| \le R$ s.t. $f(0) = 0$ and $\mathrm{Re} f(z) \le M$ for all $|z| \le R$. Then for any $r < R$

$$\sup_{|z| \le r} |f(z)| \ll_{r,R} M$$

  We also get $\sup_{|z|\le r} |f'(z)| \ll_{r,R} M$

- If $\sigma_0 > 0$, then

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s} ds = \begin{cases} 1 & \text{if } y > 1 \\ 0 & \text{if } y < 1 \end{cases} + \mathcal{O}\left(\frac{y^{\sigma_0}}{T \log y}\right)$$

- **Perron's formula:** If $F(s) = \sum \frac{a_n}{n^s}$ is absolutely convergent for $\sigma > \sigma_a$ and $\sigma_0 > \max(0, \sigma_a)$, then for any $T \ge 1$

$$\sum_{n \le x} a_n = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma + iT} F(s)\frac{x^s}{s} ds + \mathcal{O}\left(2^{\sigma_0}\frac{x}{T}\sum_{x/2 < n < 2x} \frac{|a_n|}{|n - x|} + \frac{x^{\sigma_0}}{T}\sum_n \frac{|a_n|}{n^{\sigma_0}}\right)$$

  This gives a type of converse to Dirichlet series (given $F(s)$, determine $\sum_{n \le x} a_n$)

- Suppose $f(z)$ is analytic in a domain containing $|z| \le 1$, where $|f(z)| \le M$ and $f(0) \ne 0$. Let $0 < r < R < 1$. Then for $|z| \le r$

$$\frac{f'}{f}(z) = \sum_k \frac{1}{z - z_k} + \mathcal{O}_{r,R}\left(\log \frac{M}{|f(0)|}\right)$$

  where the sum is over z_k, zeros of f where $|z_k| \le R$

- For $|t| \ge 2$

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho \\ |t - \gamma| \le 1}} \frac{1}{s - \rho} + \mathcal{O}(\log |t|)$$

  uniformly for $-1 \le \sigma \le 2$. (i.e. the sum is over all zeros of ζ in the critical strip with imaginary part between $\mathrm{Im}(s) - 1$ and $\mathrm{Im}(s) + 1$)

- For any $T \ge 4$, there is some $T \le T_1 \le T + 1$ such that

$$\left|\frac{\zeta'}{\zeta}(\sigma + iT_1)\right| \ll (\log T)^2$$

  uniformly for $-1 \le \sigma \le 2$.

  ...

- **Stirling's formula:** For $|s| \geq \delta$ and $|\arg(s)| < \pi - \delta$

$$\frac{\Gamma'}{\Gamma}(s) = \log s + \mathcal{O}(1) \quad \text{and} \quad \log\Gamma(s) = s\log s + \mathcal{O}(s).$$

Note: For real values: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, and thus $\log n! = n\log n - n + O(\log n)$.

- If $\sigma \leq -1$ and $|s + 2k| \geq \frac{1}{4}$, then get

$$\left|\frac{\zeta'}{\zeta}(s)\right| \ll \log\left(|s| + 1\right)$$

- **Explicit formula:** If x is not an integer, then, fpr any $T \geq 1$

$$\psi(x) = \sum_{n\leq x}\Lambda(n) = x - \sum_{\substack{\rho \\ |\gamma|\leq T}}\frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) + \mathcal{O}\left(\frac{x}{T}\left(\log(xT)^2 + \frac{\log x}{\langle x\rangle}\right)\right)$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power.
Corollary: If x is not an integer, then

$$\psi(x) = \sum_{n\leq x}\Lambda(n) = x - \lim_{T\to\infty}\sum_{\substack{\rho \\ |\gamma|\leq T}}\frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)$$

Corollary 2: Assuming the Riemann Hypothesis, then

$$\psi(x) = x + \mathcal{O}(x^{1/2}(\log x)^2)$$

- (Generalised explicit formula:) If $s \neq 1$, $\zeta(s) \neq 0$, and x is not an integer, then

$$\sum_{n\leq x}\frac{\Lambda(n)}{n^s} = \frac{x^{1-s}}{1-s} - \lim_{T\to\infty}\sum_{\substack{\rho \\ |\gamma|\leq T}}\frac{x^{\rho-s}}{\rho - s} - \frac{\zeta'}{\zeta}(s) + \sum_{k=1}^{\infty}\frac{x^{-2k-s}}{2k+s}$$

- Littlewood has shown error term of $\psi(x)$ at least $x^{1/2}\log\log\log x$. That is, if $\psi(x) = x + E(x)$, then:

$$\limsup_{x\to\infty}\frac{E(x)}{x^{1/2}\log\log\log x} > 0 \quad \text{and} \quad \liminf_{x\to\infty}\frac{E(x)}{x^{1/2}\log\log\log x} < 0$$

4. Zeros of $\zeta(s)$

- If $\sigma > \frac{1}{2}(1+t^2)$, then $\zeta(s) \neq 0$
  In particular, $\zeta(s) \neq 0$ if $8/9 \leq \sigma \leq 1$ and $|t| \leq 7/8$.
- **de la Vallée Poussin 1899:** There is $c > 0$ such that $\zeta(s) \neq 0$ for $\sigma > 1 - \frac{c}{\log(|t|+4)}$
  Conjecture: Does there exist $\epsilon > 0$ such that $\zeta(s) \neq 0$ for $\sigma > 1 - \epsilon$?
- There is $c > 0$ such that
$$\psi(x) = x + \mathcal{O}(xe^{-c\sqrt{\log x}})$$
- If $|t| \geq \frac{7}{8}$ and $\frac{5}{6} \leq \sigma \leq 2$, then
$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + \mathcal{O}(\log|t| + 4)$$
  where ρ is over all zeros in $|\rho - (\frac{3}{2} + it)| \leq \frac{5}{6}$
- **Korobov-Vinogradov-Richert:** There exists $c > 0$ such that
$$\zeta(s) \neq 0 \quad \text{for } \sigma > 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}$$
- Asymptotic formula for $N(T)$:
$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \mathcal{O}(\log T)$$
  In particular, $N(T) \sim \frac{1}{2\pi} T \log T$
- **Normlized zeta function:**
$$\xi(s) = \frac{1}{2}\zeta(s)(s-1)s\pi^{-s/2}\Gamma(s/2)$$
  Functional equation states $\xi(s) = \xi(1-s)$. ξ is entire function and only has zeros at zeros of $\zeta(s)$ in $0 < \sigma < 1$.
- **Landau:** Let $A(x)$ be integrable, bounded in any finite interval, and $A(x) \geq 0$ for large $x \geq X$. Let
$$\sigma_c = \inf\left\{\sigma : \int_X^\infty \frac{A(x)}{x^\sigma} dx < \infty\right\}$$
  Then the function
$$F(s) = \int_1^\infty \frac{A(x)}{x^s} dx$$
  is analytic for $\sigma > \sigma_c$ but not at $s = \sigma_c$.
  Corollary: If $F(s)$ can be defined in some half-plane $\sigma > \sigma_1$ where
$$F(s) = \int_1^\infty \frac{A(x)}{x^s} dx$$
  such that $F(s)$ can be meromorphically continued to the half-plane $\sigma > \sigma_0$ (possibly with poles) and such that there are no poles on the real line $s = \sigma > \sigma_0$, then $F(s)$ has no poles in the entire half-plane $\sigma > \sigma_0$!

- We say that $f = \Omega_\pm(g)$ if

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} \geq c > 0 \quad \text{and} \quad \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq -c < 0$$

  I.e. $\exists c > 0$ s.t. $f(x) \geq cg(x)$ infinitely often, and $f(x) \leq -cg(x)$ infinitely often.

- If σ_0 is the supremum of the real parts of the zeros of $\zeta(s)$, then for any $\sigma < \sigma_0$

$$\psi(x) = x + \Omega_\pm(x^\sigma)$$

  Corollary: The Riemann Hypothesis is equivalent to: for every $\epsilon > 0$, $\psi(x) = x + \mathcal{O}_\epsilon(x^{1/2+\epsilon})$.

- If there is a zero of $\zeta(s)$ at $\rho = \sigma_0 + it$, then

$$\psi(x) = x + \Omega_\pm(x^{\sigma_0})$$

5. Zero density results

- Define $N(\sigma, T)$ as the number of zeros of $\zeta(s)$ with real part $\geq \sigma$ and imaginary part $\leq T$.
- **Ingham:** Let $1/2 < \sigma_0 < 1$. THe number of zeros with $\sigma > \sigma_0$ is $\mathcal{O}(T^{3(1-\sigma_0)}(\log T)^{\mathcal{O}(1)})$. This implies, for all $\epsilon > 0$, there exists a prime number in the interval $[x, x + x^{2/3+\epsilon}]$.
- **Approximate Functional Equation:** When $0 \leq \sigma \leq 1$

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + \mathcal{O}\left(x^{-\sigma} + |t|^{-1/2} x^{1-\sigma}\right)$$

where $x, y \geq \frac{1}{2}$ such that $xy = \frac{|t|}{2\pi}$.
(in particular, can take $x \approx y \approx t^{1/2}$)

- For any $a_n \in \mathbb{C}$,

$$\int_0^T \left| \sum_{n \leq x} a_n n^{it} \right|^2 dt = (T + \mathcal{O}(x)) \sum_{n \leq x} |a_n|^2$$

- **2nd moment of ζ:**

$$\int_{T/2}^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \frac{T}{2} \log T$$

- Corollary:

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim T \log T$$

- For any $a_n \in \mathbb{C}$,

$$\int_0^T \left| \sum_{n \leq x} a_n n^{it} \right|^4 dt \ll (T + x^2) \left(\sum_{n \leq x} |a_n|^2 \tau(n) \right)^2$$

- **4th moment of ζ:**

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt \ll T(\log T)^4$$

(In fact, Ingham showed $\sim \frac{1}{2\pi^2} T(\log T)^4$)
Open conjecture: For all $k \geq 0$,

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt \sim c_k T(\log T)^{k^2}$$

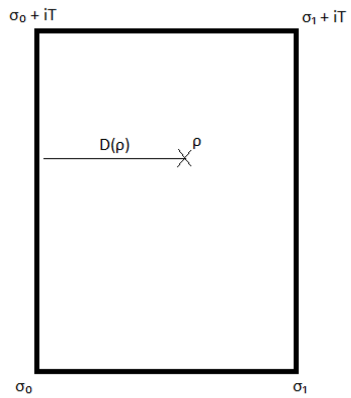
for some $c_k > 0$. Only known for $k = 1, 2$. Ramachandra has showed $\gg T(\log T)^{k^2}$

- If $1/2 < \sigma < 1$

$$\int_0^T |\zeta(\sigma + it)|^2 dt \ll_\sigma T$$

- **Logarithm:** Let $f(s)$ be a function on a rectangular contour $\mathcal{C}$, which is non-zero on $\mathcal{C}$. Then define
$$\log f(s) = \log|f(s)| + i \cdot \arg(f(s))$$
where the argument varies continuously anti-clockwise around $\mathcal{C}$. (just pick a point for arg, then vary continuously)

- **Littlewood:** Let $f(s)$ be analytic on and inside $\mathcal{C}$, and non-zero on $\mathcal{C}$, where $\mathcal{C}$ is a rectangle with vertices at:



Then
$$\sum_\rho D(\rho) = -\frac{1}{2\pi i}\int_{\mathcal{C}} \log f(s)ds$$
where the sum is over all zeros ρ of f inside $\mathcal{C}$, and $D(\rho)$ denotes the horizontal distance from ρ to the left edge of the rectangle

In particular,
$$2\pi\sum_\rho D(\rho) = \int_0^T \log|f(\sigma_0+it)|dt - \int_0^T \log|f(\sigma_1+it)|dt + \int_{\sigma_0}^{\sigma_1} \arg f(\sigma+iT)d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)d\sigma$$

- **Bohr-Landau:** For any $1/2 < \sigma < 1$
$$N(\sigma,T) \ll_\sigma T$$
(almost all zeros of $\zeta(s)$ arbitrarily close to $\sigma = \frac{1}{2}$)

- **Zero density:** A zero density of strength A is the statement:
$$N(\sigma,T) \ll_\sigma T^{A(1-\sigma)}(\log T)^{\mathcal{O}(1)} \quad \text{for all } \frac{1}{2} \leq \sigma \leq 1$$

Density Hypothesis conjecture: We can take $A = 2$.

- Riemann hypothesis $\implies$ Lindelhof hypotehsis $\implies$ Density hypothesis $\implies$ prime between x and $x + x^{1/2+\epsilon}$.
- If we have a zero density result of strength A, then for all $\epsilon > 0$, for all x large enough, there is a prime between x and $x^{\frac{A-1}{A}+\epsilon}$
- For $1/2 \leq \sigma \leq 1$, we have

  $$N(\sigma, T) \ll T^{(1+2\sigma)(1-\sigma)}(\log T)^{O(1)}$$

  In particular,

  $$N(\sigma, T) \ll T^{3(1-\sigma)}(\log T)^{O(1)}$$

  therefore proving zero density of strength $A = 3$.