Modular Forms and L-functions

Lectures

1. Characters

- **Character:** Let G be abelian topological group. A (unitary) **Character** of G is a continuous homomorphism: $\chi : G \to U(1) = \{z \in \mathbb{C} : |z| = 1\}$

  Notation: The set of all characters of G, denoted $\hat{G}$ forms a group under multiplication.

  Examples:

  - $G = \mathbb{R}$. Every character is of the form $x \mapsto e^{2\pi ixy}$ for some $y \in \mathbb{R}$. $\hat{\mathbb{R}} \cong \mathbb{R}$.
  - $G = \mathbb{Z}$. Character depends only on $\chi(1)$. $\hat{\mathbb{Z}} \cong U(1)$.
  - $G = \mathbb{Z}/N\mathbb{Z}$. $\chi(1)$ is n-th root of unity. $\hat{G} \cong \mu_N = \{\zeta \in \mathbb{C}^\times : \zeta^N = 1\}$.
  - $G = \mathbb{R}^\times$. Then $\hat{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$.
  - $G = (\mathbb{Z}/N\mathbb{Z})^\times$. Then $\hat{G}$ is finite abelian with order $\phi(N)$.
    If $N = 2^r \cdot p_1^{r_1} \dots p_k^{r_k}$, with p_i odd primes, then:

    $$\begin{aligned}(\mathbb{Z}/N\mathbb{Z})^\times &\cong (\mathbb{Z}/2^r\mathbb{Z})^\times \times (\mathbb{Z}/p_1^{r_1}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_k^{r_k}\mathbb{Z})^\times \\ &\cong C_2 \times C_{2^{r-2}} \times C_{p_1^{r_1} - p_1^{r_1-1}} \times \cdots \times C_{p_k^{r_k} - p_k^{r_k-1}}\end{aligned}$$

    where C_n is the cyclic group of order n.

- **Locally compact group:** A topological group G for which the underlying topology is Hausdorff and locally compact (i.e. for all $g \in G$, there exists an open set U and compact set K s.t. $g \in U \subset K$)

  Examples: Any finite group, any discrete group. Any compact space. Euclidean space $\mathbb{R}^n$. Unit interval $[0, 1]$. The circle group $U(1)$.

- **Pontryagin duality:** If G is locally compact, then $G \to \hat{G}$ is an isomorphism (non-canonically).

- **Orthogonality relations:** Let G be finite abelian group. with $\chi, \chi \in \hat{G}$ characters of G. Then

  $$\sum_{g \in G} \overline{\chi(g)}\chi'(g) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ |G| & \text{if } \chi = \chi' \end{cases}$$

  Also, let $g, g' \in G$ be elements of G. Then

  $$\sum_{\chi \in \hat{G}} \overline{\chi(g)}\chi(g') = \begin{cases} 0 & \text{if } g \neq g' \\ |G| & \text{if } g = g' \end{cases}$$

- Let G be finite abelian group. Then $\#\hat{G} = \#G$ and G and $\hat{G}$ are (non-canonically) isomorphic.

- L^1 **- function:** A function $f : \mathbb{R} \to \mathbb{C}$, such that $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$.
- **Fourier transform:** Let f be an L^1-function. Then the Fourier transform is:

$$\hat{f}(y) := \int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)dx = \int_{\mathbb{R}} \chi_y(-x)f(x)dx$$

- **Schwartz space:** The set of functions over $\mathbb{R}$ that are "sufficiently nice".

$$S(\mathbb{R}) = \{f \in C^{\infty}(\mathbb{R}) : \forall k, n \geq 0, |x^n f^{(k)}(x)| \to 0 \text{ as } |x| \to \infty\}$$

- **Fourier inversion formula:** If $f \in S(\mathbb{R})$, then $\hat{f} \in S(\mathbb{R})$ and $\hat{\hat{f}} = f(-x)$
- **Fourier series:** IF $G = \mathbb{R}/\mathbb{Z}$ and $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, then get Fourier series

$$c_n(f) = \int_0^1 e^{-2\pi inx} f(x)dx = \int_{\mathbb{R}/\mathbb{Z}} \chi_n(-x)f(x)dx$$

- **Finite Fourier transform:** If $G = \mathbb{Z}/N\mathbb{Z}$ and $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, then get

$$\hat{f}(\zeta) = \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \zeta^{-a} f(a)$$

- (Finite Fourier inversion):

$$f(x) = \frac{1}{N} \sum_{\zeta \in \mu_N} \zeta^x \hat{f}(\zeta)$$

  which follows from the lemma:

$$\sum_{\zeta \in \mu_N} \zeta^k = \begin{cases} N & \text{if } k \equiv 0 \text{ mod } N \\ 0 & \text{otherwise} \end{cases}$$

- **Poisson summation formula:** (for $\mathbb{R}^n$) Let $f \in S(\mathbb{R}^n)$. Then

$$\sum_{a \in \mathbb{Z}^n} f(a) = \sum_{b \in \mathbb{Z}^n} \hat{f}(b)$$

- (Functional equation of Θ-function) Let

$$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

  Then $\Theta(1/t) = t^{1/2}\Theta(t)$
  Corollary: Let $\vartheta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$. Then $\vartheta(-1/z) = (z/i)^{1/2}\vartheta(z)$

- *Examples of Fourier transforms:*
  - Linearity
  - Scaling
  - Time shifting
  - Let $g(x) = e^{-\pi x^2}$. Then $\hat{g}(y) = g(y)$

2. Γ-function and Mellin transform

- **Gamma function:** An analytic function of s:

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y} \quad s \in \mathbb{C}, \mathrm{Re}(s) > 0$$

Properties:

  - $\Gamma(s+1) = s\Gamma(s)$ (by integration by parts)
  - $\Gamma(1) = 1 \implies \Gamma(n) = (n-1)!$
  - $$\Gamma(s) = \frac{1}{s(s+1)\dots(s+N-1)}\Gamma(s+N) \quad \forall N \geq 1$$

- Weierstrauss product formula:

$$\Gamma(s)^{-1} = e^{\gamma s} s \prod_{n\geq 1}\left(1+\frac{s}{n}\right)e^{-s/n}$$

where γ is the Euler-Mascheroni constant:

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log(n)\right)$$

- Duplication formula:

$$\pi^{1/2}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)$$

- Reflection formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

- Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function. We say f is **rapidly decreasing** at ∞ if, for all $n \in \mathbb{Z}$, $y^n f(y) \to 0$ as $y \to \infty$.

  We say f has **moderate growth at 0**, if, for some $m \geq 0$, $|y^m f(y)|$ is bounded as $y \to 0$

- **Mellin transform:** Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous functon rapidly decresaing at ∞, and with moderate growth at 0 for some $m \geq 0$. The Mellin transform is

$$M(f,s) := \int_0^\infty f(y) y^s \frac{dy}{y}$$

which converges absolutely, defining an anlytic funciton of s for $\mathrm{Re}(s) > m$.

Examples:

  - If $f(y) = e^{-y}$, then $M(f,s) = \Gamma(s)$.
  - If $f(y) = \dfrac{1}{e^y - 1}$, then $M(f,s) = \Gamma(s)\zeta(s)$

- **Riemann-zeta function:** Let $\zeta(s)$ be the function

$$\zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s} \quad \text{for } \mathrm{Re}(s) > 1$$

Then $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$, only singularity is simple pole at $s = 1$, with residue 1.

- **Bernoulli numbers:** Numbers given as coefficients of generating function of $t/(e^t - 1)$:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} = \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots\right)^{-1} = 1 - \frac{t}{2} + \frac{1}{6}\frac{t^2}{2!} - \dots$$

Examples: $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, and $B_{12} = -691/(2 \cdot 3 \cdot 5 \cdot 7 \cdot 13)$

Fact: $B_n = 0$ if $n \geq 3$ odd.

Corollary: $\zeta(0) = -1/2 = B_1$ and for all $n \geq 2$, $\zeta(1-n) = -B_n/n$

- **Functional equation:** Let

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Then $Z(s)$ has simple poles at $s = 1, 0$ with residues $+1, -1$ and $Z(s) = Z(1-s)$

- **Euler product formula:** If $\mathrm{Re}(s) > 1$, then

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}}\right)$$

3. Dirichlet L-functions

- **Dirichlet series:** Let a_n be a sequence with at most polynomial growth. Then the Dirichlet series is:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- **Dirichlet character:** Let $N \geq 1$. A Dirichlet character (mod N) is a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

- **Dirichlet L-function:** Let

$$L(\chi, s) = \sum_{\substack{n\geq 1 \\ (n,N)=1}} \chi(n) n^{-s}$$

This converges absolutely for $\mathrm{Re}(s) > 1$.

- **Euler product:** Since χ is multiplicative, we have the Euler product

$$L(\chi, s) = \prod_{\substack{p \text{ prime} \\ (p,N)=1}} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}} + \dots\right) = \prod_{(p,N)=1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

- **Analytic continuation:** Let $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any periodic function. Define

$$L(\psi, s) := \sum_{n\geq 1} \psi(n) n^{-s}$$

which converges for $\mathrm{Re}(s) > 1$.

The Mellin transform is:

$$\Gamma(s) L(\psi, s) = \sum_{n\geq 1} \psi(n) M(e^{-ny}, s) = M(f, s) \quad \text{with} \quad f(y) = \sum_{n=1}^{N} \psi(n) \frac{e^{(N-n)y}}{e^{Ny} - 1}$$

which extends $L(\psi, s)$ meromorphically to $\mathrm{Re}(s) > 0$.

- For any $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, $L(\psi, s)$ has a meromorphic continuation to $\{\mathrm{Re}(s) > 0\}$, holomorphic for $s \neq 1$. At $s = 1$, at worst simple pole, residue $= \frac{1}{N} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \psi(n)$.

- Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Then
  - If $\chi \neq \chi_0$, then $L(\chi, s)$ is analytic at $s = 1$.
  - For χ_0, the residue at $s = 1$ is

$$\mathrm{Res}_{s=1} L(\chi_0, s) = \frac{\phi(N)}{N} = \prod_{p|N} \left(1 - \frac{1}{p}\right)$$

- **Dirichlet's Theorem on Primes in Arithmetic Progression:** Let $a, n \in \mathbb{Z}$ such that $1 \leq a < N$ with $(a, N) = 1$. Then there exists infinitely many primes p such that $p \equiv a \pmod{N}$.

  Lemmas:

- $$\sum_p p^{-x} \sim -\log(x-1) \text{ as } x \to 1_+$$

- Let $\chi \neq \chi_0$. Then

$$\sum_{(p,N)=1} \chi(p)p^{-x}$$

is bounded as $x \to 1_+$.

- **Lattice:** Let V be finite dim $\mathbb{R}$-vector space, and let (e_i) be a $\mathbb{R}$-basis of V. A **lattice** in V is a subgroup $\Lambda \subset V$ of the form

$$\sum_{1 \leq i \leq d} \mathbb{Z}e_i$$

Properties:

(a) Λ is a discrete subgroup. I.e. for any norm on V, $\{x \in \Lambda : ||x|| < R\}$ is finite, for all R.

(b) Λ is cocompact. I.e. V/Λ is compact.

Furthermore, Λ is lattice if and only if (a) and (b) hold.

- **Norm:** Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm is a function $|| \cdot || : V \to \mathbb{R}$ such that

  - $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in V$.
  - $||\alpha x|| = |\alpha|||x||$ for all $\alpha \in \mathbb{R}$ or $\mathbb{C}$, and for all $x \in V$.
  - If $||x|| = 0$, then $x = 0$ is the zero vector.

- **Equivalent norms:** Let $||\cdot||$ and $||\cdot||'$ be two norms on V. These norms are **equivalent** if, there exists constants $A, B > 0$ such that

$$A||x|| \leq ||x||' \leq B||x||$$

for all $x \in V$.

- Let V be a finite dimensional vector space over $\mathbb{R}$. Then any two norms on V are equivalent.

- Let $\Lambda \subset V$ be a lattice, with $|| \cdot ||$ any norm on V. Then

$$\sum_{0 \neq x \in \Lambda} \frac{1}{||x||^\alpha} \text{ converges } \iff \alpha > \dim(v)$$

4. Modular group

- **Upper half plane:** Upper half plane is

$$\mathfrak{h} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

- **GL**$_2(\mathbb{R})$: General linear group over $\mathbb{R}$; the set of 2×2 invertible matrices over $\mathbb{R}$.

$$\mathrm{GL}_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : ad - bc \neq 0 \right\}$$

- **Group action:** $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{C}\backslash\mathbb{R}$ on the left by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d} = \gamma(z)$$

- **GL**$_2(\mathbb{R})^+$: The set of 2×2 matrices over $\mathbb{R}$ with positive determinant.

$$\mathrm{GL}_2(\mathbb{R})^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : ad - bc > 0 \right\}$$

- $\mathrm{GL}_2(\mathbb{R})^+$ maps $\mathfrak{h}$ to itself.
- **PGL**$_2(\mathbb{R})^+$:

$$\mathrm{PGL}_2(\mathbb{R})^+ := \frac{\mathrm{GL}_2(\mathbb{R})^+}{\mathbb{R}^\times \cdot I} \cong \frac{\mathrm{SL}_2(\mathbb{R})}{\{\pm I\}}$$

- $\mathrm{PGL}_2(\mathbb{R})^+$ is the full group of holomorphic automorphisms of $\mathfrak{h}$.
- **SL**$_2(\mathbb{Z})$: Special linear grup over $\mathbb{Z}$; the set of 2×2 matrices over $\mathbb{Z}$ with determinant 1.

$$\mathrm{SL}_2(\mathbb{R})^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}$$

- If $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, then $\bar{\Gamma}$ is image in $\mathrm{PSL}_2(\mathbb{Z})$
- **Modular group:** If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $\bar{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ is called the **modular group**.
- Let

$$S = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  Then $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T \rangle$.

- **Fundamental domain:** Let

$$\mathcal{D} = \left\{ z \in \mathfrak{h} : -\frac{1}{2} < \mathrm{Re}(z) \leq \frac{1}{2},\ |z| \geq 1,\ \text{and if } |z| = 1,\ \text{then } \mathrm{Re}(z) \geq 0 \right\}$$

  Then $\mathcal{D}$ is a fundamental set for $\mathrm{PSL}_2(\mathbb{Z})$. I.e. every orbit of $\mathrm{PSL}_2(\mathbb{Z})$ on h intersects $\mathcal{D}$ in exactly one point.

- **Stabilizers:** For $z \in \mathfrak{h}$, let $\bar{\Gamma}_z = \{\gamma \in \bar{\Gamma} : \gamma(z) = z\}$ be the stabiliser subgroup of z. If $z \in \mathcal{D}$, then:
  - $\bar{\Gamma}_z = \{1\}$ if $z \notin \{i, \rho\}$.

- $\bar{\Gamma}_i = \langle S \rangle \cong \mathbb{Z}/2\mathbb{Z}$
- $\bar{\Gamma}_\rho = \langle TS \rangle \cong \mathbb{Z}/3\mathbb{Z}$

Remark: $\mathrm{PSL}_2(\mathbb{Z})$ is freely generated by $\langle S \rangle$ and $\langle TS \rangle$.

- **Hyperbolic measure:** The (hyperbolic) measure $\frac{dxdy}{y^2}$ is invariant under $\mathrm{SL}_2(\mathbb{R})$.
- The quotient $\bar{\Gamma}\backslash\mathfrak{h}$ has finite measure for any finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$
- **Principal congruence subgroup:** Let $N \geq 1$. The principal congruence subgroup mod N is

$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I \bmod N\} = \ker\left(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right)$$

$$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod N\}$$
$$\Gamma_1(N) = \{\gamma \in \Gamma_0(N) : d \equiv 1 \mod N\}$$

Note: $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$

- **Congruence subgroup:** Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup. Γ is a **congruence subgroup** if $\Gamma(N) \subset \Gamma$ for some N.

  Remark about non-congruence subgroup:

  - $$\frac{\#\{\text{congruence subgroups of index } \leq M\}}{\#\{\text{all subgroups of index } \leq M\}} \longrightarrow 0 \quad \text{as } M \to \infty$$
  - For $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, every subgroup of finite index is a congruence subgroup!

- **j-operator:** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, and let $z \in \mathfrak{h}$. Define

$$j(\gamma, z) := cz + d$$

Note that, by action of γ,

$$\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = (cz+d) \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma(z) \\ 1 \end{pmatrix}$$

Properties:

- $j(\gamma\delta, z) = j(\gamma, \delta(z)) \cdot j(\delta, z)$
- $j(\gamma^{-1}, z) = j(\gamma, \gamma^{-1}(z))^{-1}$
- $\frac{d}{dz}\gamma(z) = \frac{\det(\gamma)}{j(\gamma,z)^2}$
- $\mathrm{Im}\gamma(z) = \frac{\det\gamma \mathrm{Im}(z)}{|j(\gamma,z)|^2}$

5. Modular forms (of level 1)

- **Modular:** A complex valued function on $\mathfrak{h}$ is **modular of weight** k if, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have
$$f(\gamma(z)) = j(\gamma, z)^k f(z) \quad = (cz+d)^k f(z)$$
Note: If above property holds for $\gamma, \delta \in \mathrm{SL}_2(\mathbb{Z})$, then it holds for $\gamma\delta$.
- If k is odd, then f identically zero.
- Let k be even. Then f is modular of weight k if and only if
$$f(z+1) = f(z) \qquad \text{and} \qquad f(-1/z) = z^k f(z)$$
- **Eisenstein series:** Let $k \geq 4$ be even. Define (for $z \in \mathfrak{h}$)
$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$
Properties of $G_k(z)$:
  - (i) Series for G_k converges to a holomorphic function of $\mathfrak{h}$ which is modular of weight k.
  - (ii) We have the q-expansion
$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad \text{where} \quad \sigma_r(n) = \sum_{1\leq d|n} d^r$$
which arises from the lemma that, for all $k \geq 2$,
$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{m\geq 1} m^{k-1} q^m \quad \text{where } q = e^{2\pi i z}$$
- **Weierstrauss $\wp$-function:** Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice, where ω_1 and ω_2 are linearly independent over $\mathbb{R}$. Then the Weierstrauss $\wp$-function is
$$\begin{aligned}\wp(u) &= \frac{1}{u^2} + \sum_{0\neq w\in\Lambda}\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right) \\ &= \frac{1}{u^2} + \sum_{c,d\in\mathbb{Z}^2\setminus(0,0)}\left(\frac{1}{(u-cw_1-dw_2)^2} - \frac{1}{(cw_1+dw_2)^2}\right)\end{aligned}$$
- Laurent expansion of $\wp$ at origin:
$$\wp(u) = \frac{1}{u^2} + \sum_{\substack{k\geq 4 \\ k \text{ even}}} (k-1) G_k(\Lambda) u^{k-2} \quad \text{where} \quad G_k(\Lambda) = \sum_{0\neq w\in\Lambda} \frac{1}{w^k} = \frac{1}{w_2^k} G_k\left(\frac{w_1}{w_2}\right)$$
- **Normalised Eisenstein series:** Define
$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Examples:

$$E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \dots$$

$$E_6 = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - \dots$$

$$E_8 = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n = 1 + 480q + 61920q^2 + 1050240q^3 + 7926240q^4 + \dots$$

$$E_{10} = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n = 1 - 264q - 135432q^2 - 5196576q^3 - 69341448q^4 - \dots$$

$$E_{12} = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n = 1 + \frac{65520}{691}q + \frac{134250480}{691}q^2 + \frac{11606736960}{691}q^3 + \dots$$

$$E_{14} = 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n = 1 - 24q - 196632q^2 - 38263776q^3 - 1610809368q^4 - \dots$$

- **Meromorphic/holomorphic at infinity:** Let f be complex function on $\mathfrak{h}$ such that $f(z+N) = f(z)$ for some $N \geq 1$. Let $\tilde{f}(q_N) = f(z)$ where $q_N = e^{2\pi iz/N}$. If f holomorphic for $\text{Im}(z) > Y$, then $\tilde{f}$ holomorphic on $\{0 < |q_N| < e^{-2\pi Y/N}\}$.

  We say f is **meromorphic/holomorphic** at infinity if $\tilde{f}$ **meromorphic/holomorphic** at $q_n = 0$.

  I.e. Let f have convergent series representation

  $$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inz/N} \quad \text{for } \text{Im}(z) > Y$$

  Then:

  - If $c_n = 0$ for sufficiently small n, then **meromorphic** at infinity.
  - If $c_n = 0$ for all $n < 0$, then **holomorphic** at infinity.

- **Modular form:** A modular form of weight $k \in 2\mathbb{Z}$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ which is

  (i) modular of weight k.

  (ii) holomorphic at infinity.

- **Weak modular form:** Same as above, but can replace (ii) with memormorphic at infinity.

- **Valence formula:** Let f be a (weak) modular form of weight k, not indeitcally zero. Then

  $$\sum_{\substack{z\in\mathcal{D}\\ z\neq i,\rho}} \text{ord}_{z=z_0} f + \frac{1}{2}\text{ord}_{z=i} f + \frac{1}{3}\text{ord}_{z=\rho} f + \text{ord}_{\infty} f = \frac{k}{12}$$

  Applications:

  - $M_k = 0$ if $k < 0$.

- $M_0 = \mathbb{C}$.
- $M_2 = \{0\}$

- For all $k \geq 0$,

$$\dim_{\mathbb{C}} M_k \leq 1 + \frac{k}{12}$$

Corollary: $M_k = \mathbb{C}E_k$ for $k \in \{4, 6, 8, 10\}$. $E_4^2 = E_8$ and $E_4 E_6 = E_{10}$.

- **Delta function:** Define

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots = \sum_{n=1} \tau(n) q^n$$

$\tau(n)$ is **Ramanujan τ-function**.

Corollary: $M_{12} = \mathbb{C}E_4^3 \oplus \mathbb{C}\Delta$

- **Cusp form:** Let $S_k = \{f \in M_k : \text{ord}_\infty f \geq 1\}$. These are the space of cusp forms of weight 1. Note $\Delta \in S_{12}$.
  - $M_k = S_k \oplus \mathbb{C} \cdot E_k$
  - $f \mapsto f\Delta$ is an isomorphism between m_k and S_{k+12}

- Let $k \geq 0$ be even. Then

$$\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \bmod 12 \\ 1 + \lfloor \frac{k}{12} \rfloor & \text{otherwise} \end{cases} \tag{1}$$

- **Basis:** A basis for M_k is

$$\{E_4^a E_6^b \Delta^c : b \in \{0, 1\}, a, c \geq 0, 4a + 6b + 12c = k\}$$

Any element of M_k can be expressed uniquely as a polynomial in E_4 and E_6 (i.e E_4 and E_6 are algebraically independent)

- **Basis (weak modular forms):** A basis for the set of weak modular forms of weight k, $M_k^!$ is

$$\{E_4^a E_6^b \Delta^c : b \in \{0, 1\}, a \geq 0, c \in \mathbb{Z}, 4a + 6b + 12c = k\}$$

Properties of $\tau(n)$:

  - (i) $\tau(n) \in \mathbb{Z}$ for all $n \geq 1$.
  - (ii) $\tau(n) \equiv \sigma_{11}(n) \bmod 691$.

- **Modular function j:** Noting $\Delta(z) \neq 0$ for all $z \in \mathfrak{h}$, we define the j function:

$$j(z) = \frac{E_4(z)^3}{\Delta(z)}$$

This is holomorphic on H, modular of weight 0, and memorphic at ∞ with $\text{ord}_\infty j = -1$.

- **Modular function:** A modular function (of level 1) is a meromorphic function f on H, modular of weight 0 and meromorphic at ∞.

- j defines a bijection $j : \mathrm{SL}_2(\mathbb{Z})\backslash H \to \mathbb{C}$.
- The field of modular functions equals $\mathbb{C}(j)$ (i.e. every modular function is rational function of j)

- j has a triple zero at $z = \rho$.
- $j - 1728 = E_6^2/\Delta$ has a double zero at $z = i$.
- The q-expansion is given as:

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots$$

- **Picard's Little Theorem:** If $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant entire function, then f omits at most one value

- **Weight $k = 2$:** There are no non-zero weight 2 modular forms (of level 1). However, we can define $G_2(z)$ as

$$G_2(z) := \sum_{m=\infty}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0 \\ \text{if } m=0}}^{\infty} \frac{1}{(mz+n)^2} = \frac{\pi^2}{3}\left(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n\right)$$

Note this is **conditionally convergent**, and so swapping the sums does not yield the same function $G_2(z)$. So $G_2(-1/z) \neq z^2 G_2(z)$, but we do have

$$G_2(-1/z) = z^2 G_2(z) - 2\pi i z$$

6. Hecke operators

- **Slash operator:** Let $k \in \mathbb{Z}$, and let f a (complex) function on $\mathfrak{h}$, and $\gamma \in \mathrm{GL}_2(\mathbb{R})^+$. The slash operator is

$$f|_k\gamma := \frac{(\det(\gamma))^{k/2}}{j(\gamma, z)^k} f(\gamma(z)) = \frac{(ad-bc)^{k/2}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right) \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  This defines (right) action of $\mathrm{GL}_2(\mathbb{R})^+$

  Thus $f|_k(\gamma\delta) = (f|_k\gamma)|_k\delta$.

- **Modular:** A funcitno $f : \mathfrak{h} \to \mathbb{C}$ is modular of weight k if and only if

$$f|_k\gamma = f \quad \text{for all } \gamma \in \Gamma$$

- If $f : \mathfrak{h} \to \mathbb{C}$ is modular of weight k and $\gamma \in \mathrm{GL}_2(\mathbb{R})^+$, then $g = f|_k\gamma$ is **not** necessarily modular of weight k.

- If $\gamma_1, \dots, \gamma_r \in \mathrm{GL}_2(\mathbb{R})^+$ such that the family of cosets $\Gamma\gamma_1, \dots, \Gamma\gamma_r$ is invariant under right action of Γ, then

$$f \text{ modular of weight } k \implies \sum_{i=1}^{r} f|_k\gamma_i \text{ modular of weight } k$$

- Let $m \geq 1$. Let $\Delta_m = \{\gamma \in \mathrm{Mat}_2(\mathbb{Z}) : \det(\gamma) = m\}$ (i.e. 2×2 matrices over $\mathbb{Z}$ with determinant m). Then

$$\Delta_m = \coprod_{\delta \in \Pi_m} \Gamma\delta \quad \text{(disjoint union)}$$

  where

$$\Pi_m = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d > 0, ad = m, b \in R_d \right\}$$

  where R_d is any complete set of integers mod d. (e.g. $R_d = \{0, \dots, d-1\}$). So Π_m is a set of representatives of Γ in Δ_m.

  Examples:

  - If $m = 1$, then $\Pi_1 = \{I\}$ and $\Delta_1 = \Gamma$.
  - If $m = 2$, then

$$\Pi_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

  - If $m = 3$, then

$$\Pi_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

  - If $m = 4$, then

$$\Pi_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

  - If $m = p$ prime, then we have $|\Pi_p| = p + 1$, with

$$\Pi_3 = \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & p \end{pmatrix}, \dots, \begin{pmatrix} 1 & p-1 \\ 0 & p \end{pmatrix} \right\}$$

- In general, $|\Pi_m| = \sum_{d|m} d = \sigma(m)$.

- **Hecke operators:** Let $f : \mathfrak{h} \to \mathbb{C}$, let k be integer, and let $m \geq 1$. We define the Hecke operator T_m as
$$f|_k T_m = m^{k/2-1} \sum_{\delta \in \Pi_m} f|_k \delta$$
Note: f modular weight k implies $f|_k T_m$ modular weight k

  Notation: $T_m f = f|_k T_m$ (is fine since T_m commute)

- - If $(m, m') = 1$, then
$$f|_k T_m|_k T_{m'} = f|_k T_{mm'}$$
  - For p prime, and $r \geq 1$, we have
$$f|_k T_{p^r}|_k T_p = f|_k T_{p^{r+1}} + p^{k-1} f|_k T_{p^{r-1}}$$
  - For all $m, m' \geq 1$, T_m and $T_{m'}$ commute.

- - $f \in M_k \implies T_m f \in M_k$
  - $f \in S_k \implies T_m f \in S_k$
  - If $f = \sum_{n \in \mathbb{Z}} a_n q^n$ weak modular form, weight k, then
$$T_m f = \sum_{n \in \mathbb{Z}} b_n q^n, \quad b_n = \sum_{1 \leq d | (m,n)} d^{k-1} a_{mn/d^2}$$

  Examples:

  - If $m = 1$, then T_1 is the identity operator.
  - If $m = 2$, then $T_2 f = \sum b_n q^n$ where
$$b_n = \begin{cases} a_{2n} & \text{if } n \text{ odd} \\ a_{2n} + 2^{k-1} a_{n/2} & \text{if } n \text{ even} \end{cases}$$
  - If $m = 3$, then $T_3 f = \sum b_n q^n$ where
$$b_n = \begin{cases} a_{3n} & \text{if 3 not divides } n \\ a_{3n} + 3^{k-1} a_{n/3} & \text{if 3 divides } n \end{cases}$$
  - If $m = 4$, then $T_4 f = \sum b_n q^n$ where
$$b_n = \begin{cases} a_{4n} & \text{if } n \text{ odd} \\ a_{4n} + 2^{k-1} a_n & \text{if } n \equiv 2 \bmod 4 \\ a_{4n} + 2^{k-1} a_n + 4^{k-1} a_{n/4} & \text{if } n \equiv 0 \bmod 4 \end{cases}$$
  - If $m = p$ prime, then $T_p f = \sum b_n q^n$ where
$$b_n = \begin{cases} a_{pn} & \text{if } p \text{ not divides } n \\ a_{pn} + p^{k-1} a_{n/p} & \text{if } p \text{ divides } n \end{cases}$$

- Suppose $m \geq 1$, $\lambda \in \mathbb{C}$, $f \in \sum_{n \geq 0} a_n q^n \in M_k$ and $f \neq 0$ with $T_m f = \lambda f$. Then

- (i) $a_m = \lambda a_1$, more generally if $(m,n)=1$, $a_{mn} = \lambda a_n$
- (ii) If $a_0 \neq 0$ (i.e. $f \notin S_k$), then $\lambda = \sigma_{k-1}(m)$

- **Hecke eigenform:** f is a Hecke eigenform if $T_m f = \lambda_m f$ (for some $\lambda_m \in \mathbb{C}$) for **all** $m \geq 1$ (i.e. f is eigenvector for all Hecke operators)

- Let $f = \sum_{n\geq 0} a_n q^n \in M_k$ for $k > 0$. Suppose $a_0 \neq 0$. Then f is a Hecke eigenform if and only if $f = a_0 E_k$. If so, then $\lambda_m = \sigma_{k-1}(m)$

- If f, g are modular of weight k, we define

$$\Omega(f,g) = f(z)\bar{g}(z)\, y^k \,\frac{dxdy}{y^2}$$

  This is Γ-invariant.

- **Peterson inner product** Let $f, g \in M_k$, at least one a cusp form. Then

$$\langle f, g\rangle := \int_{\mathcal{D}} \Omega(f,g) = \int_{\Gamma\backslash\mathfrak{h}} \Omega(f,g)$$

  (can integrate over any (measurable) fundamental set)

- For all $m \geq 1$, $\langle T_m f, g\rangle = \langle f, T_m g\rangle$ (i.e. inner product is **self-adjoint**)

- There exists a basis for S_k composed of Hecke eigneforms (simultaneuous eigenvectors for $\{T_m\}$) Moreover, the eignenvalues of T_m are all **real**.

- **Multiplicity one:** Let $f, g \in M_k\backslash\{0\}$ such that $\forall p, \exists \lambda_p$ with $T_p f = \lambda_p f$, $T_p g = \lambda_p g$. Then $f = cg$ for some $c \in \mathbb{C}$

- **(Strong mulriplicity one):** Let $f, g \in M_k\backslash\{0\}$ such that $T_p f = \lambda_p f$ for all but *finitely* many primes p. Then $f = cg$.

- **Normalised Hecke eigenform:** A cusp form which is a Hecke eigenform with $a_1 = 1$ is said to be **normalised**. In this case $T_n f = a_n f$ for all $n \geq 1$.

  S_k has a unique basis of normalised Hecke eigenforms.

  Examples:

  - For $k = 12$, normalised eigenform is Δ.
  - For $k \in \{16, 18, 20, 22, 26\}$, normalised eigenform is $E_{k-12}\Delta$.
  - For $k = 24$, , the space of cusp forms S_{24} has basis $\{E_4^3\Delta, \Delta^2\}$, where

$$\begin{aligned} E_4^3\Delta &= q + 696q^2 + 162252q^3 + 12831808q^4 + \dots \\ \Delta^2 &= q^2 - 48q^3 + 1080q^4 + \dots \end{aligned}$$

  Computing how T_2 acts on the basis we get

$$\begin{aligned} T_2(E_4^3\Delta) &= 696q + 21220461q^2 + \dots = 696E_4^3\Delta + 20736000\Delta^2 \\ T_2(\Delta^2) &= q + 1080q^2 + \dots = 1 \cdot E_4^3\Delta + 384\Delta^2 \end{aligned}$$

  The matrix of T_2 with respect to this basis is therefore

$$T_2 = \begin{pmatrix} 696 & 1 \\ 20736000 & 384 \end{pmatrix}$$

Thus, there are two normalised eigenforms h_1, h_2 with coefficients in $\mathbb{Q}(\sqrt{144169})$.

$$h_1 = E_4^3\Delta + (-156 + 12\sqrt{144169})\Delta^2$$
$$h_2 = E_4^3\Delta + (-156 - 12\sqrt{144169})\Delta^2$$

- For $k = 28$, the space of cusp forms S_{28} has basis $\{E_4^4\Delta, E_4\delta^2\}$, where

$$E_4^4\Delta = q + 936q^2 + 331452q^3 + 53282368q^4 \dots$$
$$E_4\Delta^2 = q^2 + 192q^3 - 8280q^4 \dots$$

Computing how T_2 acts on the basis we get

$$T_2(E_4^4\Delta) = 936q + 187500096q^2 + \dots = 936E_4^4\Delta + 186624000E_4\Delta^2$$
$$T_2(E_4\Delta^2) = q - 8280q^2 + \dots = 1 \cdot E_4^4\Delta - 9216E_4\Delta^2$$

The matrix of T_2 with respect to this basis is therefore

$$T_2 = \begin{pmatrix} 936 & 1 \\ 186624000 & -9216 \end{pmatrix}$$

Thus, there are two normalised eigenforms h_1, h_2 with coefficients in $\mathbb{Q}(\sqrt{18209})$.

$$h_1 = E_4^4\Delta + (-5076 + 108\sqrt{18209})E_4\Delta^2$$
$$h_2 = E_4^4\Delta + (-5076 - 108\sqrt{18209})E_4\Delta^2$$

- Note $\Delta = \sum_{n\geq 1}\tau(n)q^n \in S_{12}$. $\dim(S_{12}) = 1$ and $\tau(1) = 1$ implies Δ is normalised eigenform. Thus $T_m\Delta = \tau(n)\Delta$ for all $m \geq 1$.

  Therefore

  - $\tau(mn) = \tau(m)\tau(n)$ if $(m,n) = 1$.
  - $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$

- **Maeda's conjecture:** For every $k > 26$, the characteristic polynomial of T_2 acting on S_k is conjectured to be **irreducible** over $\mathbb{Q}$ (and has Galois group the full symmetric group).

  This has been verified computationally for all weights $k \leq 12\,000$

- Let $f = \sum a_n q^n \in S_k$. Then there exists C such that $|a_n| \leq Cn^{k/2}$.

- (Deligne, 1972) Let f be a normalised eigenform. Then, for p prime, $|a_p| \leq 2p^{(k-1)/2}$ (*very difficult*). This implies that, for any $f \in S_k$, we have $a_n = \mathcal{O}(n^{\frac{k-1}{2}}\log n)$.

  Also implies that $|\tau(p)| < 2p^{11/2}$

7. L-functions of Modular forms

- Let f be non-zero cusp form, $f = \sum_{n=1}^{\infty} a_n q^n \in S_k$. Define

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

  As $a_n = \mathcal{O}(n^{k/2})$, this converges absolutely for $\text{Re}(s) > \frac{k}{2} + 1$.

- $L(f,s)$ is an entire function of s, and satisfies the functional equation

$$\Lambda(f,s) := 2(2\pi)^{-s}\Gamma(s)L(f,s) = (-1)^{k/2}\Lambda(f,k-s)$$

- **Euler product:** Suppose $f = \sum_{n\geq 1} a_n q^n \in S_k$ is normalised eigenform. Then

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}} \quad \text{for } \text{Re}(s) > \frac{k}{2} + 1$$

- **Eisenstein series:** For the Eisenstein series E_k, the Dirichlet series naturally attached to E_k is

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) n^{-s} = \zeta(s)\zeta(s+1-k)$$

8. Epstein zeta function

(associated to a quadratic form)

- *Setup:* Let V be real inner product space of $\dim N < \infty$ with norm $||v|| = (v,v)^{1/2}$. Let $\Lambda \subset V$ be a lattice.
- **Dual lattice:** Let V be real inner product space, with Λ a lattice. Then the dual lattice Λ' is:

$$\Lambda' = \{v \in V : \forall x \in \Lambda, (x,v) \in \mathbb{Z}\}$$

  Examples:

  - Dual lattice of $\mathbb{Z}^n$ is $\mathbb{Z}^n$.
  - In $\mathbb{R}$, dual lattice of $\mathbb{Z}a$ is $\frac{1}{a}\mathbb{Z}$.
  - In $\mathbb{R}^2$, dual lattice of $\mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is

$$\mathbb{Z} \cdot \frac{\mathbf{Rb}}{\mathbf{a} \cdot \mathbf{Rb}} + \mathbb{Z} \cdot \frac{\mathbf{Ra}}{\mathbf{b} \cdot \mathbf{Ra}}$$

    where $\mathbf{R}$ is the 90° rotation matrix.

  If $\Lambda = \oplus_{i=1}^N \mathbb{Z}e_i$ then $\Lambda' = \oplus_{i=1}^N \mathbb{Z}e_i'$, where (e_i') is dual basis given by $(e_i, e_j') = \delta_{ij}$

- **Measure:** If $(v_1, \ldots, v_N)$ are coordinates with respect to an orthonormal basis, then $d\mu_V = dv_1 dv_2 \ldots dv_N$.
- **Covolume:** Let $m(\Lambda)$ be the covolume of Λ

$$m(\Lambda) = \int_{V/\Lambda} d\mu_V = \int_{\mathcal{F}_V} d\mu_V \quad \text{where } \mathcal{F}_\Lambda = \left\{ \sum_{i=1}^N x_i e_i : 0 \le x_i < 1 \right\} \text{ and } (e_i) \text{ basis for } \Lambda$$

  $\mathcal{F}_\Lambda$ is the **fundamental parallelopiped** for Λ

- **Epstein Zeta function**: The Epstein zeta function of $(\Lambda, || \cdot ||)$ is

$$G(\Lambda, s) = \sum_{0 \neq x \in \Lambda} \frac{1}{||x||^{2s}}$$

  This is absolutely convergent if $\mathrm{Re}(s) > N/2$.

  E.g.

  - If $V = \mathbb{R}$ with Euclidean norm, $\Lambda = \mathbb{Z}$. Then

$$G(\Lambda, s) = \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{|n|^{2s}} = 2\zeta(2s)$$

  - If $V = \mathbb{C}$ with Euclidean norm, $\Lambda = \mathbb{Z}[i]$. Then

$$G(\Lambda, s) = \sum_{x+iy \in \in \mathbb{Z}[i] \setminus 0} \frac{1}{(x^2+y^2)^s} = \frac{1}{4}\zeta_{\mathbb{Q}(i)}(s)$$

- **Functional equation:** $Z(\Lambda, s) := \pi^{-s}\Gamma(s)G(\Lambda, s)$ has a memomorphic continuation to $\mathbb{C}$, holomorphic except for simple poles at $s = N/2, 0$ with residues $m(\Lambda)^{-1}, -1$ respectively. It satisfies functional equation:

$$Z(\Lambda, s) = m(\Lambda)^{-1}Z(\Lambda', N/2 - s)$$

  In particular, $G(\Lambda, 0) = -1$ (e.g. $\zeta(0) = -\frac{1}{2}$)

- **Poisson summation for Λ:** Let $f \in \mathcal{L}(V)$ with Fourier transform

$$\hat{f}(v) = \int_V e^{-2\pi i(u,v)} f(u) d\mu_V(u)$$

  Then

$$\sum_{x\in\Lambda} f(x) = m(\Lambda)^{-1} \sum_{x\in\Lambda'} \hat{f}(x)$$

- Let $\Theta_\Lambda(t) = \sum_{x\in\Lambda} e^{-\pi||x||^2 t}$. Then it satisfies the transformation law

$$\Theta_\Lambda(t) = t^{-N/2} m(\Lambda)^{-1}\Theta_{\Lambda'}(1/t)$$

- **Real analytic Eisenstein series**

$$G(z, s) = \sum_{m,n\in\mathbb{Z}} \frac{y^s}{|mz + n|^{2s}} \quad y = \mathrm{Im}(z)$$

  - $G(\gamma(z), s) = G(z, s)$ for all $\gamma \in SL_2(\mathbb{Z})$
  - Define $\mathcal{E}(z, s) := \pi^{-s}\Gamma(s)G(z, s)$. Then $\mathcal{E}$ has meromorphic continuation to $\mathbb{C}$, with simple poles at $s = 1, 0$, with residues $+1, -1$ respectively, and $\mathcal{E}(z, s) = \mathcal{E}(z, 1 - s)$.

- **Kronecker limit formula:** Let G' be the s-derivative of G. Then we have

$$G'(z, 0) = 4\zeta'(0) - \log(y|\eta^4|) = -\log(4\pi^2 y|\eta^4|)$$

  where $\eta = \eta(z) = q^{1/24}\prod_{i=1}^{\infty}(1 - q^n)$ and where $q^{1/24} = e^{2\pi iz/24}$.

  Corollary:

  - $\eta(-1/z) = (z/i)^{1/2}\eta(z)$
  - $\Delta(z) = \eta(z)^{24} = q\prod_{i=1}^{\infty}(1 - q^n)^{24}$

- To prove KLF, use the following lemma. Define

$$H(z, s) = \pi^{-s}\Gamma(s)\sum_{n\in\mathbb{Z}} \frac{1}{|z + n|^{2s}}$$

  - If $z \notin \mathbb{R}$, then

$$H(z, s) = \pi^{\frac{1}{2}-s}|y|^{1-2s}\Gamma\left(s - \frac{1}{2}\right) + H_1(z, s)$$

    where $H_1(z, s)$ is entire function of s.

  - **Case $y \neq 0$:**

$$H_1(z, 0) = \begin{cases} -2\log|1 - e^{2\pi iz}| & \text{if } y > 0 \\ -2\log|1 - e^{-2\pi iz}| & \text{if } y < 0 \end{cases}$$

- **Case $y = 0$:** If $x \in \mathbb{R}\backslash\mathbb{Z}$, then $H(x,s)$ has analytic continuation to $\mathbb{C}\backslash\{1/2\}$ and

$$H(x,0) = -2\log|1-e^{2\pi i x}|$$

- Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character mod N, with $\chi \neq \chi_0$.
  - Then
$$L(\chi,0) = -\frac{1}{N}\sum_{\substack{0<n<N\\(n,N)=1}} \chi(n)n$$
  - If $\chi(-1) = 1$ (i.e. χ is **even**), then $L(\chi,0) = 0$ and
$$L'(\chi,0) = -\sum_{\substack{0<n<N/2\\(n,N)=1}} \chi(n)\log|1-e^{2\pi i n/N}|$$

Applications to number theory

- **Quadratic field:** $K = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}\backslash\{0,1\}$ squarefree.
- **Ring of integers:** $\mathcal{O}_K = \mathbb{Z}[\theta] = \mathbb{Z} + \mathbb{Z}\theta$ where
$$\theta = \begin{cases} \sqrt{d} & \text{if } d \not\equiv 1 \pmod 4 \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4 \end{cases}$$
- **Norm:** Let $x = a + b\sqrt{d}$, and let $x' = a - b\sqrt{d}$. Then the **norm** is $N_{K/\mathbb{Q}}(x) = xx' = a^2 - b^2 d$.
- **Discriminant:**
$$d_k = \begin{vmatrix} 1 & \theta \\ 1 & \theta' \end{vmatrix}^2 = \begin{cases} 4d & \text{if } d \not\equiv 1 \pmod 4 \\ d & \text{if } d \equiv 1 \pmod 4 \end{cases}$$
- **Units:** Let $\mathcal{O}_K^\times$ denote the units (i.e. elements x such that $N_{K/\mathbb{Q}}(x) = \pm 1$). Then, if $D < 0$, then
$$\mathcal{O}_K^\times = \begin{cases} \{\pm i, \pm 1\} & \text{if } D = -1 \\ \{\pm 1, \pm e^{\pm 2\pi i/3}\} & \text{if } D = -3 \\ \{\pm 1\} & \text{otherwise} \end{cases}$$
  Else, if $D < 0$, then $\mathcal{O}_K^\times = \{\pm 1\} \times \langle \epsilon \rangle$ where ϵ is the **fundamental unit**.
- Let $K = \mathbb{Q}(\sqrt{d})$ with discriminant d_k. Then
  - $(p) = p^2$ is ramified if and only if $p|d_k$.
  - There exists a unique Dirichlet character $\chi_K : (\mathbb{Z}/|d_k|\mathbb{Z}) \to \{\pm 1\}$ such that if $p \not| d_k$ then
$$\chi_K(p) = \begin{cases} +1 & \text{if } p \text{ splits} \\ -1 & \text{if } p \text{ inert} \end{cases}$$
    Moreover $\chi(-1) = \operatorname{sgn}(d)$

- **Dedekind ζ-function:** Let K be number field, with ring of integers $\mathcal{O}_K$. The Dedekind ζ-function of K is:

$$\zeta_K(s) = \sum_{0 \neq I \subset \mathcal{O}_K} \frac{1}{N(I)^s}$$

  Examples:

  - If $K = \mathbb{Q}$, then $\mathcal{O}_K = \mathbb{Z}$, $I = (n)$ and $\zeta_K(s) = \zeta(s)$ (Riemann zeta function)
  - If $K = \mathbb{Q}(i)$, then $\mathcal{O}_K = \mathbb{Z}[i]$, $I = (a + bi)$. Can choose canonical generator for each I s.t a is positive odd. Thus

$$\zeta_K(s) = \sum_{k=0}^{\infty} \sum_{a>0, a \text{ odd}} \frac{1}{(2^k(a + bi))^s}$$

  - Generally, if K principal ideal domain, then just have $I = (x)$ gives $N(I) = |N_{K/\mathbb{Q}}(x)|$

- **Euler product:** $\zeta_k(S)$ converges for $\mathrm{Re}(s) = \sigma > 1$ and has an Euler product:

$$\zeta_K(s) = \prod_{primeidealsP} \frac{1}{1 - N(p)^{-s}}$$

- $\zeta_K(s) = \zeta(s)L(\chi_K, s)$

- **Analytic class number formula:** Let $K = \mathbb{Q}(\sqrt{d}), d < 0$ be imaginary quadratic field. Let h_K be class number of K. Let w_k be the number of roots of unity in K. I.e.

$$w_K = \begin{cases} 4 & \text{if } K = \mathbb{Q}(i) \\ 6 & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ 2 & \text{otherwise} \end{cases}$$

  Then $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$, with simple pole at $s = 1$ with

$$\zeta_k(0) = -\frac{h_k}{w_k} \qquad \text{and} \qquad \mathrm{Res}_{s=1}\zeta_K(s) = \frac{2\pi}{|d_K|^{1/2}} \cdot \frac{h_K}{w_K}$$

- Let K be imaginary quadratic. Then

$$h_K = -\frac{w_K}{2 \cdot |d_K|} \sum_{\substack{0<n<|d_K| \\ (n,d_K)=1}} n\chi_K(n)$$

- Let $K = \mathbb{Q}(\sqrt{-d})$, $d > 3$ and $d \equiv 3 \pmod 4$. Then

$$h_K(2 - \chi_K(2)) = \sum_{\substack{0<n<d/2 \\ (n,d)=1}} \chi(x)$$

- Let $K = \mathbb{Q}(\sqrt{-q})$, q prime > 3 and $q \equiv 3 \pmod 4$. Then
$$h_k = \begin{cases} R - N & q \equiv 7 \pmod 8 \\ \frac{1}{3}(R - N) & q \equiv 3 \pmod 8 \end{cases}$$
where
$$R = \# \text{ of quadratic residues in interval } (0, q/2)$$
$$N = \# \text{ of quadratic non-residues in interval } (0, q/2)$$

- **Real quadratic:** Let K be real quadratic field. Then $\zeta_K(s)$ has meromorphic continuation to $\mathbb{C}$, with simple poles at $s = 1$ and simple zero at $s = 0$.
$$\zeta_K'(0) = -\frac{h_K}{w_K}\log(\epsilon) = -\frac{h_K}{2}\log(\epsilon)$$
and
$$\text{Res}_{s=1}\zeta_K(s) = \frac{4}{d_k^{1/2}}\frac{h_k \log \epsilon}{w_k} = \frac{2}{d_k^{1/2}} h_k \log \epsilon$$
where $\epsilon > 1$ is the **fundamental unit** of K

- **Fundamental unit:** Let $K = \mathbb{Q}(\sqrt{d})$, with ring of integers $\mathcal{O}_K$. Then $\mathcal{O}_K^\times \cong \{\pm 1\} \times \mathbb{Z}$ and $\epsilon > 1$ is a generator of $\mathcal{O}_K^\times$ mod ± 1.

  Examples :

  - For $K = \mathbb{Q}(\sqrt{2})$, then $\epsilon = 1 + \sqrt{2}$.
  - If $K = \mathbb{Q}(\sqrt{3})$, then $\epsilon = 2 + \sqrt{3}$
  - If $K = \mathbb{Q}(\sqrt{5})$, then $\epsilon = \frac{1}{2}(1 + \sqrt{5})$
  - If $K = \mathbb{Q}(\sqrt{7})$, then $\epsilon = 8 + 3\sqrt{7}$

- Let $x_1, x_2 \in \mathbb{R}^\times$. Then for $\text{Re}(s) > 0$:
$$\frac{1}{|x_1 x_2|^s} = \frac{2\Gamma(s)}{\Gamma(s/2)^2}\int_0^\infty \frac{1}{(ux_1^2 + u^{-1}x_2^2)^s}\frac{du}{u}$$

- For any Dirichlet character $\chi \neq \chi_0$, $L(\chi, 1) \neq 0$.

- **Class formula for real quadratic:** Let K be real quadratic field, with fundamental unit $\epsilon > 1$. Then

  -
$$h_k = \frac{1}{\log \epsilon} \sum_{\substack{0 < n < d_k/2 \\ (n, d_k) = 1}} \chi_k(n) \cdot \log\left(\sin \frac{n\pi}{d_K}\right)$$

  - Let η be
$$\eta = \prod_{\substack{0 < n < d_k/2 \\ (n, d_k) = 1}} \left(\sin \frac{n\pi}{d_K}\right)^{-\chi(n)}$$

    Then $\eta \in \mathcal{O}_k^\times$ and $\eta = \epsilon^{h_K}$

Quadratic field examples