Algebra
Examples Class 1.

29/10/2019

Q8: · Point of this was for Q9.

RTP: S is saturated $\iff$ $R \setminus S$ is a union of prime ideals.

($\Leftarrow$) ✓. Easy by definition.

· ($\Rightarrow$) Assume S saturated and consider complement. Take $x \in R \setminus S$.
Saturated property $\Rightarrow$ $x/1$ is not a unit in $S^{-1}R$. So $\exists$ maximal ideal of $S^{-1}R$ containing $x/1$. By our knowledge of localisation, this maximal ideal corresponds to a prime ideal of R containing x and $P \cap S = \phi$.

So $x \in P \subset R \setminus S$ for any $x \in R \setminus S$.
Hence: $R \setminus S$ is a union of prime ideals.

(OR: Note $(x) \subseteq R \setminus S$, apply Zorn to take maximal member in $R \setminus S$ containing x, and show its prime.)

Or: Apply Zorn to set of ideals I containing x and $I \cap S = \phi$. Maximal such is a prime ideal.

If $S = 1 + I$:
$R \setminus S' =$ union of prime ideals P with $P \cap S = \phi$.

Note: $P \cap S = \phi \iff I + P \neq R$ (otherwise $1 \in I + P$ contradiction)

Such a P lies in a maximal ideal Q
containing $I + P$

So union of primes P with $P \cap S = \emptyset$ $=$ Union of max ideals Q with $Q \supseteq I$.

$\therefore \quad S' = \{ x \mid x \notin \text{max ideal} \supseteq I \}$

$\quad = \{ \underset{x+I}{x \ (\mathrm{mod}\ I)} \text{ is a unit of } R/I \}$.

Thus: $R \setminus \bigcup (\quad I \subseteq Q, \ Q \ \underline{\text{maximal}})$

Q9: (Kaplansky). R is an integral domain.
R is a UFD $\iff$ Each non-zero prime ideal contains a ~~prime~~ principal prime ideal (f)
($\equiv p$ is a prime element).

$\Rightarrow$ Straightforward (just use factorisation $p_1 p_2 \cdots p_k$).

$(\Leftarrow)$ For this, we use Q8.
Want a saturated mult. closed set S.

Must include!

Set: $S = \{ r \in R \mid r$ is either unit, or finite product of prime elements $\}$.

We hope: $R \setminus S = \{0\}$

NB! We need the units in order to have $1 \in S$ and for S to be saturated.

Q8: $\Rightarrow$ $R \setminus S = \bigcup$ prime ideals P with $P \cap S = \emptyset$.

We're supposing that each non-zero P contains a principal prime ideal (f) f prime element

If $\bigcup_{\text{prime ideals}} \supset 0$ then we have $P \neq 0$ with $P \cap S = \emptyset$
But $f \in P$ where $\underset{\text{non-zero}}{(f)} \subseteq P$

But $f \in S$ $\times\!\!\times$ $P \cap S = \emptyset$ [Exercise in using the set S].

Q18: Let $S = 1 + I$

We show $S^{-1}I \subseteq \mathrm{Jac}(S^{-1}R)$

Recall that $\mathrm{Jac}(S^{-1}R) = \bigcap$ max ideals of $S^{-1}R$.

Max ideals of $S^{-1}R \longleftrightarrow$ maximal ideals containing I.

$S^{-1}Q \longleftrightarrow Q$.

$I \subseteq Q$. So $S^{-1}I \subseteq S^{-1}Q$.

But $\mathrm{Jac}(S^{-1}R) = \bigcap_{\substack{Q \supseteq I \\ \max}} S^{-1}Q$

So $S^{-1}I \subseteq \mathrm{Jac}(S^{-1}R)$.

Q17: $I \subseteq \mathrm{Jac}\, R$, We have $\theta : M \to N$ R-module map. This induces map $\bar{\theta} : M/IM \to N/IN$

RTP $\bar{\theta}$ surjective $\Rightarrow$ θ surjective.

Note: $\bar{\theta}$ surjective $\Leftrightarrow$ $\mathrm{Im}\,\theta + IN = N$.

Consider: $I(N/\mathrm{Im}\,\theta) = IN + \mathrm{Im}\,\theta / \mathrm{Im}\,\theta = N/\mathrm{Im}\,\theta$ if $\bar{\theta}$ surjective.

Thus, multiplying by I, nothing much happens.

Nakayama: $M = 0 \Leftrightarrow (\mathrm{Jac}(R))M = M$.

Same proof: $M = 0 \Leftrightarrow IM = M$ for $I \subseteq \mathrm{Jac}\, R$

(if $M \neq 0$ then there is a proper quotient $M/N \cong R/P$ for some ideal P. So $(\mathrm{Jac}\, R)M \subsetneq M$ (and $IM \subsetneq M$ for any $I \subseteq \mathrm{Jac}\, R$).

So in this case: $N/\mathrm{Im}\,\theta = 0$.
Thus $\mathrm{Im}\,\theta = N$, θ surjective. ☺

Q6: Suppose R_p has no non-zero nilpotent elements for all primes P. Take $x \in R$. Suppose it's nilpotent. $\mathrm{Ann}(x) \subsetneq R$ since $x \neq 0$. $(x \neq 0)$
So $\exists$ max ideal $Q \supseteq \mathrm{Ann}(x)$.

Consider R_Q. $x/1$ is nilpotent because x is.
"$S^{-1}R$", $S = R \setminus Q$.
So by supposition, $x/1$ must be zero.
So $\exists y \in S$ s.t. $yx = 0$, $y \in \mathrm{Ann}(x) \subseteq Q$
and so $y \notin S$ ※.

So there does not exist a non-zero nilpotent x.
(all nilpotents are 0).

What about int. dom? No!
E.g. $R = \mathbb{Z}/6\mathbb{Z}$, 2 prime ideals: $2\mathbb{Z}/6\mathbb{Z}$, $3\mathbb{Z}/6\mathbb{Z}$.
$R_p \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ which are IDs.
But R is not an ID.

Q7: $\phi : M \longrightarrow N$

TFAE:
ϕ surjective
ϕ_P sur $\forall$ primes P
ϕ_Q sur $\forall$ max Q.

① $\Rightarrow$ ② $\Rightarrow$ ③ obviously ✓.

③ $\Rightarrow$ ① : We have SES:

Want to show this is zero.

$$0 \longrightarrow M \xrightarrow{\phi} N \longrightarrow N/\mathrm{Im}\,\phi \longrightarrow 0$$

Localising:

$$0 \longrightarrow M_Q \xrightarrow{\phi_Q} N_Q \longrightarrow (N/\mathrm{Im}\,\phi)_Q \longrightarrow 0. \quad (*)$$

This is still exact sequence. As $(*)$ is exact, we know $(N/\mathrm{Im}\,\phi)_Q = N_Q/\mathrm{Im}\,\phi_Q = 0$ $\forall Q$ maximal.

As ϕ_Q surj.

Thus: $(N/\mathrm{Im}\,\phi)_Q = 0 \quad \forall Q$

Hence: $N/\mathrm{Im}\,\phi = 0$. Thus ϕ surjective.

[Injective works in the same way, same proof technique]

Q15: Given R local ring, M, N f.g. R-modules
Prove that $M \otimes_R N = 0 \implies M = 0$ or $N = 0$.

Equivalently: $\left.\begin{matrix} M \neq 0 \\ N \neq 0 \end{matrix}\right\} \implies M \otimes_R N \neq 0$.

Proof: M f.g. $\implies \exists$ quotient $M/M_1 \cong R/p$ where p is max ideal.
But the max. ideal is unique in a local ring.

N f.g. $\implies N/N_1 \simeq R/p$

So $M \otimes_R N$ has image $\underbrace{R/p \otimes_R R/p}_{M/M_1 \otimes N/N_1} \cong R/p$ non-zero.

So $M \otimes_R N \neq 0$.

Crucial thing, if finite-gen, then we have this quotient R/p that we can use.

Note: $M \xrightarrow{\theta} M/M_1$, $N \xrightarrow{\phi} N/N_1$
$M \otimes N \xrightarrow{\theta \otimes \phi} M/M_1 \otimes N/N_1$

Q12: Element of $M \otimes N$ not of form $m \otimes n$.

Example: M: 2-dimensional k-vector space.
N: $= M$.

$M \otimes_k M$ is 4-dimensional vector space
e_1, e_2 f_1, f_2 basis.

Then: $e_1 \otimes f_1 + e_2 \otimes f_2$ is not of form $m \otimes n$.

Basis is: $e_1 \otimes f_1$, $e_2 \otimes f_1$, $e_2 \otimes f_2$, $e_1 \otimes f_2$.
Basis of $M \otimes N$ ↗

All pairs of form: $(\lambda_1 e_1 + \lambda_2 e_2) \otimes (\mu_1 f_1 + \mu_2 f_2)$
$\neq e_1 \otimes f_1 + e_2 \otimes f_2$

for any λ_i, μ_j