Algebra
Examples Class 2.

12/11/2019.

Q15: Recall: $R \subseteq T$, T fin gen. as R-module by n elements. Show that every max P of R has at most n maximals of T lying over it.

Consider T/PT. This is a R/P-vector space.
Question: $\dim_{R/P}(T/PT) \leq n$

How many max ideals does T/PT have?
(Can quote some results about Artinian rings).

If we have maximal ideals $Q_1, Q_2, \ldots,$ lying over P.
$\overline{Q_i}$ in T/PT. max ideals.

$$Q_1 > Q_1 Q_2 > Q_1 Q_2 Q_3 > \ldots$$
$$\rightarrow \overline{Q_1} > \overline{Q_1}\,\overline{Q_2} > \overline{Q_1}\,\overline{Q_2}\,\overline{Q_3} > \ldots$$

these are strict if Q_i-distinct.

If we had $n+1$ such Q's we'd have a chain of subspaces which is impossible in a vector space of dimension $\leq n$.

So there at most n Q_i's.

Can apply this to Q3:
Let's jump to Q3:

Q3: Describe the set of maximal ideals of $\mathbb{R}[X_1, X_1^{-1}, X_2, X_2^{-1}] = R$

let $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}] = T$.

Note: T is a 2-generated R-module.
So what we've just done shows there are at most 2 maximal ideals lying over each maximal ideal of R.

- Nullstellensatz as applied to R.
Max ideals P are such that R/P is a finite field extension of $\mathbb{R}$.

Choice is $\mathbb{R}$ or $\mathbb{C}$.
Max ideals of T, these are labelled $Q_{a,b}$.
where $a, b \in \mathbb{C}^{\times}$, $Q_{ab} = (X_1 - a, X_2 - b)$.

$$\left\{\begin{matrix} \text{Max ideals} \\ \text{of } T \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{Max ideals} \\ \text{of } R \end{matrix}\right\}$$

$\{(a,b)\ a, b \in \mathbb{C}^{\times}\}$ $\mathbb{C}^{\times 2}$ — Max. ideals of form (a,b) with $a, b \in \mathbb{R}^{\times}$.

Otherwise:

Two max ideals $\xrightarrow[\text{2-1 mapping.}]{}$ P

$Q_{a,b}$, $Q_{\bar{a},\bar{b}}$ $\longrightarrow$ same P

Can think of max ideals in R as covered by the max ideals in T.

$\longrightarrow$.

Q16: T f.g. k-algebra, integral over R. $P \in \operatorname{Spec} R$. Want to show there are finitely many primes of T, lying over P.

Set $S = R \setminus P$. Form: $T_P = S^{-1}T$.
$R_P = S^{-1}R$ has unique maximal ideal P_P.

Consider: $T_P / P_P T_P$.

~~T integral over~~ Note, T is f.g. k-algebra, T integral over $R \Rightarrow T$ is a f.g. R-module.

$T_P / P_P T_P$ is a finite dimensional R_P / P_P-vector space.

Prime ideals of T lying over $P \longleftrightarrow$ max ideals of $T_P / P_P T_P$
(can make argument that it's Artinian).

Or: Use same argument as in Q15. There is a bound on the length of chains of the form:

$$\overline{Q_1} > \overline{Q_1 Q_2} > \dots \quad \text{as in Q15.}$$

$\underset{\text{f.g. k-algebra}}{T}$, integral over polynomial subalgebra R by Noether normalisation.

$$\operatorname{Spec} T \xrightarrow{\text{res}} \operatorname{Spec} R$$

fibres are finite $\text{res}^{-1}(\{P\})$ is finite.

Q17: Example of Noetherian integral domain with max ideals of different heights.

In particular, in a f.g. k-algebra that is an integral domain. Given $P \in \operatorname{Spec} R$.

$$\operatorname{ht} P + \dim R/P = \dim R$$

Catenary property. Can always have chain going through P, maximal chain.

Thus, we must avoid f.g. k-algebras.

Easiest is to ~~avoid~~ use $k[X,Y]$.

$P = (X,Y)$, $Q = (X-1)$.

Let $S = R \setminus (P \cup Q)$ multiplicatively closed.

Localise: $S^{-1}k[X,Y]$. Prime ideals of this.

Correspond to prime ideals of $k[X,Y]$ disjoint to S, contained in $P \cup Q$.

Contained in P or Q, max. ideals of $S^{-1}k[X,Y]$

$$\updownarrow$$

P or Q.

$S^{-1}P$ and $S^{-1}Q$.

These have different heights.

$S^{-1}k[X,Y]$ integral domain $\subseteq k(X,Y)$.

Q18: R sub k-algebra of $k[X]$. Show R is a f.g. k-algebra.

$R = k$ ✓ Easy

So take $f(x) \in R$ non-constant
$\notin k$.

Assume that $f(x)$ is monic (since $k \subseteq R$)

Let $A = k[f(x)]$ polynomial algebra generated by $f(x)$
This is subalgebra of R, and $k[X]$ is integral over A, since X satisfies a monic expression

All coefficients $X^n + \lambda_{n-1} X^{n-1} + \dots + \lambda_0 - (f(x)) = 0$
are in A.

$f(x) = X^n + \lambda_{n-1} X^{n-1} + \dots + \lambda_0$ with $\lambda_i \in k$.

So we have: $A \subset R \subset k[x]$
Artin-Tate: $\implies$ R is a f.g. k-algebra
$k[X]$ integral over R, and so $\dim R = \dim k[x] = 1$.

Quite easy, if one uses A.

Q19: Prime avoidance lemma:

$Q_1, \ldots, Q_n$ prime ideals. Let $I \subseteq \bigcup Q_i$
We must conclude $I \subseteq Q_i$ for some i.

Induction on number of Q's.
If $n = 1$ ✓.
Suppose $n > 1$ and result true for smaller numbers
Take $r_i \in I \setminus \bigcup_{j \neq i} Q_j$ (otherwise $I \subset \bigcup_{j \neq i} Q_j$
smaller number)

Consider: $\underbrace{r_1 \cdots r_{n-1}}_{\text{product}} + r_n \in I$.

Assume $I \subset \bigcup Q_i$ and so $r_1 \cdots r_{n-1} + r_n$ lies in some Q_i. Suppose it lies in Q_1
Then $r_n \in Q_1$ ※ Similarly for $Q_2, \ldots, Q_{n-1}$.

If it lies in Q_n, then $r_1, \ldots, r_{n-1} \in Q_n$ ※, as Q_n prime.

Q20: Noetherian ring R. $P_1 \subsetneq Q \subsetneq P_2$ prime ideals. We show in fact there are infinitely many such Q

Quotient out by P_1, localise at P_2, and assume R is local with max ideal P_2. $P_1 = 0$. So R is an integral domain.

Suppose, we have only finitely many Q_i, with $0 \subsetneq Q_i \subsetneq P_2$. Consider non-units of R. They are precisely the element of P_2.

Principal ideal theorem $\Rightarrow$ minimal prime over a (non-unit) is of height 1.

$$P_2 = \bigcup_{a \text{ non-unit}} (a) \subseteq \bigcup_{\substack{\text{height} \\ \text{one}}} \text{primes} \subset P_2$$

If we're assuming only finitely many height one primes, then $P_2 \subseteq \bigcup_{\substack{\text{finite} \\ \text{union}}} Q_i$

$\therefore$ By Q19 $\Rightarrow P_2 \subseteq Q_i$ for some i
But $Q_i \subsetneq P_2$ $\times\!\!\!\times$.

Thus there are infinitely many.

Q2: $\mathbb{Q}$ is a module over the localisation of $\mathbb{Z}$ which is $\mathbb{Z}_{(p)}$ for p prime.

just localising, not completing. (not $\mathbb{Z}_p$ p-adic int)

$\mathbb{Q}$ is not f.g. $\mathbb{Z}_{(p)}$-module.

$J = \text{Jac}(\mathbb{Z}_{(p)}) = $ max. ideal of $\mathbb{Z}_{(p)}$

$J\mathbb{Q} = \mathbb{Q}$ $\longrightarrow$ $\mathbb{Q} \neq 0$.

Q11: Let $R \subset T$ with $T \setminus R$ closed under multiplication. Show that R is int. closed in T.

Suppose $x \in T \setminus R$ but is integral over R. satisfies monic expression $x^n + r_{n-1}x^{n-1} + \dots + r_0 = 0$.

$x(x^{n-1} + r_{n-1}x^{n-2} + \dots + r_1) = -r_0 \in R$

where n is of least degree. $n > 1$ (as $x \in T \setminus R$)

But $x \notin R$, by supposition $x^{n-1} + r_{n-1}x^{n-2} + \dots + r_1 \notin R$ by minimality of degree.

Closure under mult. of $T \setminus R$ forces $-r_0$ to be in $T \setminus R$. $\times\!\!\!\times$.

Q12: 'Integrally closed' is a local property of integral domains.

R integral domain, TFAE

(1) R integrally closed in fraction field K

(2) R_P " " in K for all primes P.

(3) R_Q " — — — — — in K for all maximal ideals Q.

Now: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) ✓.

We prove (3) $\Rightarrow$ (1).

Take $x \in K$, integral over R. We want to show $x \in R$. Suppose $x \notin R$ and consider ideal

$$I = \{ r \in R : rx \in R \} \quad \text{proper ideal.}$$

Now: $I \subseteq$ max ideal Q of R.

x int over $R \Rightarrow \frac{x}{1}$ integral over R_Q.

But R_Q integrally closed, and so $\frac{x}{1} \in R_Q$

So $\frac{x}{1} = \frac{r}{s}$ for some $s \notin Q$.

$sx \in R$, and so $s \in I \subseteq Q$. $\times\!\!\!\times$

Q4: R field that is a f.g. as a ring.

Case ①: $\mathbb{Q}$ $\subseteq R$ R is a number field.
prime subfield

Show, number field is not f.g. as a ring.
Get only finitely many primes in denominators.

Case ②: Prime subfield is $\mathbb{F}_p$, and R is finite field.

Q5: Alternative approach to P-primary ideals. In lectures, I is P-primary if $\text{Ass}(R/I) = \{P\}$.

Alt. def^n: I is P-primary if $ab \in I$ and $a \notin I$ forces $b^n \in I$ for some n.

Assume $\text{Ass}(R/I) = \{P\}$, and take $ab \in I$, $a \notin I$.
So $b \in \text{Ann}(a+I)$, $a+I \in R/I$.
But annihilators must lie in maximal annihilators

and we proved these must lie in $\text{Ass}(R/I)$
So $\text{Ann}(a+I) \subseteq P$
Hence $b \in P$.

Minimal primes are associated primes and their intersection is nilradical.
$\text{Ass}(R/I) = \{P\} \Rightarrow P = \sqrt{I}$.
$(\sqrt{I})^n \subseteq I$, So $b^n \in I$

Conversely, assume $ab \in I,\ a \notin I \Rightarrow b^n \in I$.
So ~~Ann(a+I)~~ elements of $\mathrm{Ann}(a+I)$ have a power in I, so lie in $\sqrt{I}$.

All prime annihilators lie in $\sqrt{I}$. But they must contain $\sqrt{I}$. So only prime annihilator is $\sqrt{I}$, so $\sqrt{I} = P$, and $\mathrm{Ass}(R/I) = \{P\}$.

$\longrightarrow$