Algebra, Examples Class 3 26/11/2019.

Q1: T f.g. k-algebra, which is an integral domain,
integral $\cup$ Noether normalisation
R polynomial algebra.

We need integral domains to use Going down.
So if $Q \in \operatorname{Spec} T$ $Q \cap R = P$
$ht\, Q = ht\, P$.

$R/P \xhookrightarrow[\text{integral}]{} T/Q$ $\dim T/Q = \dim R/P$.

Going up:

We therefore may focus on polynomial algebra. We want induction on $\dim T = \dim R$.

R is UFD, and so if we have chain of primes

$$0 \subsetneq \underset{\text{principal prime}}{(f)} \subseteq \dots \subsetneq P$$

$R/(f)$ contains prime $P/(f)$
Induction applies to $R/(f)$ since of smaller dimension.

$$ht\, P/(f) + \dim\left({}^{R/(f)}/_{P/(f)} \right) = \dim R/(f) = \dim R - 1.$$

can use transcendence degree.

Stick this altogether to get:
$$ht\, P + \dim R/P = \dim R.$$
$$\parallel$$
$$ht\, Q + \dim T/Q = \dim T.$$

$\longrightarrow$

~~Q~~: Does this prove all maximal chains have the same length?

Height of max prime not necessarily max ideal...

- Quotient out by principal ideal, apply induction, can get messy??

Q2: Must avoid domains, but still finitely generated k-algebra.

$k[X, Y]$, $P = (X, Y)$, $Q = (X-1)$.

$T = k[X,Y]/PQ$ (write $\bar{\ }$s for images in T.)

Any prime of T corresponds to a prime containing P or containing Q

$ht\ \bar{P} = 0$, $\dim(T/\bar{P}) = 0$, $\dim T = 1$.

(always have $ht P + \dim(T/P) \leq \dim T$)? I think?

Q3: R is Noetherian regular local ring, P maximal ideal, $R[[X]]$, (P, X) is maximal.

kernel of map: $R[[X]] \longrightarrow R/P$ field

$X \longmapsto 0$

$= \{$power series with constant term in $P\}$.

Take $f(X) \notin (P, X) \longrightarrow$ constant term of f is not in P and hence is a unit.
So $\exists r \in R$ s.t. $rf(X)$ has constant term.

If const. term is 1, then power series is a unit.

$$(1 + Xg(X))^{-1} = 1 - Xg(X) + (Xg(X))^2 + \cdots$$

So $R[[X]]$ is local.

R regular $\Longrightarrow$ $\dim R =$ no. of generators of P
$= \dim_{R/P} (P/P^2)$.

(P, X) is generated by $n+1$ elements.
So $\dim R[X] \leq n+1$.
But we can construct a chain of length $n+1$,
we've got a chain of length n in R.

$$P_0 \subsetneq P_1 \subsetneq \cdots\cdots \subsetneq P_n = P$$

$$P_0[[X]] \subsetneq \cdots \subsetneq P[[X]] \subsetneq (P, X)$$
is a chain of primes in $R[[X]]$

~~But $R[[X]]/P[[X]] \cong (R/P)[[X]]$ and this has~~

length $n+1$. So $\dim R[[X]] \geq n+1$
and we have equality.

Q4: ✓ Straightforward.

Q5: Suppose M is a sum of simple modules

$$M = \sum_{\lambda \in \Lambda} M_\lambda \text{ - simple.}$$

Think about subsets Λ' of Λ s.t. $\sum_{\lambda \in \Lambda'} M_\lambda$ is direct.

We can order by inclusion, and apply Zorn to give maximal Λ' s.t. $\sum_{\lambda \in \Lambda'} M_\lambda$ is direct.

Claim: $M = \sum_{\lambda \in \Lambda'} M_\lambda$.

adjoin λ to Λ' to get bigger set.

Set $N = \sum_{\lambda \in \Lambda'} M_\lambda$. Either $\boxed{M_\lambda \cap N = \{0\}}$

or $\boxed{M_\lambda \subseteq N}$ since M_λ simple.

Q6: R Artinian semisimple
$\Rightarrow R$ finite direct sum of simples
(from lectures)
$\Rightarrow$ cyclic module is a sum of simples.
generated by single element m

So M is a sum of simples

Since mR is a sum of simples for all
$$M = \sum_{m \in M} mR.$$

So M is a direct sum of simples.

But it can't be an infinite direct sum as that wouldn't be Artinian.
So must have finite direct sum of simples.

Q7: k algebraically closed

R semisimple, Artin
R fin dim. over k $\xrightarrow[\text{Artinin Wedderburn}]{}$ $\bigoplus M_{n_i}(k)$

We can reduce to the case where $R = M_n(k)$.

Now: $[R,R]$ = trace zero matrices for char $k = 0$.

We want to see that $R/[R,R]$ one-dimensional as k-vector space.

Q9: kS_3. In lectures, we saw:

- char 0 $\longrightarrow$ 3 simple modules:
  - U_1 trivial 1-dimensional
  - U_2 1-dim, $g \in S_3$ acted like mult by $\mathrm{sgn}(g)$.
  - U_3 : 2-dim
- mod 3: Then $\overline{U_3}$ ceased to be simple, and we had 2 one-dimensional simple modules
  $$\mathrm{Jac}(kS_3) = \ker(kS_3 \longrightarrow kC_2).$$

We needed to convince ourselves this is nilpotent. (nilpotent ideals have to kill every simple module, and thus in the Jacobson radical)

$\ker(kS_3 \longrightarrow kC_2)$ generated by $(h-1)$.

- Question asked about $\underline{\text{char } k = 2}$:

$\overline{U_1} = \overline{U_2}$, $\overline{U_3}$ stays simple.

So we have (at least) 2 simple modules.

$kS_3/\mathrm{Jac}(kS_3)$ is a direct sum of at least 2 matrix algebras. We can see

$\underline{M_1(k)} \oplus \underline{M_2(k)}$ — $\underline{5 \text{ dimensional}}$.

from 1-dim $\qquad$ from 2-dim one.

kS_3 is not semisimple. We expect $\mathrm{Jac}(kS_3)$ one dimensional over k.

$\mathrm{Jac}\, kS_3$ is generated by $\sum g \in kS_3$
sum of all group elements.

Q10: R filtered ring, separated and exhaustive.

$\mathrm{gr} R$ integral domain $\Rightarrow$ R is an integral domain

Take $x, y \neq 0$ but $xy = 0$ in R.
$x \in R_i / R_{i-1}$, $y \in R_j / R_{j-1}$ for some i, j.

$x + R_{i-1}$ are non-zero elements of $\mathrm{gr} R$
$y + R_{j-1}$.

$$(x + R_{i-1})(y + R_{j-1}) = xy + R_{i+j-1} = 0 \quad \times\!\!\times$$

which contradicts $\mathrm{gr} R$ an integral domain

Now for Noetherian property:
Suppose R is positively filtered. $I \subsetneq J$ ideals of R.

There is n s.t. $I \cap R_{n-1} = J \cap R_{n-1}$
$I \cap R_n \subsetneq J \cap R_n$
$\Rightarrow \mathrm{gr} I \subsetneq \mathrm{gr} J$.

Now, not true in general,
But, is true if R is complete w.r.t filtration.

~~but~~ then $gr\, R$ Noetherian $\Rightarrow$ R Noetherian.

(Have to work a bit, use Hensel's lemma)

(11) R Noetherian, ideal I

Rees $R = \bigoplus I^j T^{-j}$ $\qquad R \quad RT$

generated by R, T, $I T^{-1}$

I: f.g. ideal

finite no. of generators

So Rees R is Noetherian.

M module filtered with I-filtration of R, not necessarily M, IM, I^2M,

~~From~~ Form Rees M. Rees R-module

Show Rees M f.g. $\Leftrightarrow$ filtration of M is stable
Rees R and Rees M are graded and so Rees M is fin. gen. $\Leftrightarrow$ generated by finite no. of homogeneous generators.

Take generators to be m_k of degree s_k.
Then: j^{th} component of Rees M.

$$= m_1 R_{j-s_1} + \dots + m_k R_{j-s_k} = M_j$$

$$\therefore (m_1 R_{j-s_1} + \dots + m_k R_{j-s_k}) T^j = M_j \cdot T^j.$$

Can see this stabilises

Will use this for Q12:

Q12: Rees ring were introduced to prove Artin-Rees lemma.

M we have filtration $I^j M$ - j^{th} term.

$N \leq M$ induces filtration $N \cap I^j M$ (**)

Rees M is a fin. gen Rees R-module

submodule

Rees N using induced filtration.

~~By~~ But Rees R Noetherian, So Rees M is Noetherian, Rees N fin. gen.

Induced filtration of N is stable.

$\Rightarrow$ Artin-Rees lemma.

(Lemma done in 1956).

(Can quote Atiyah – MacDonald for last two).

Krull did both Q13, Q14:

Q13 R Noetherian local, M fin. gen. R-module.

$$N = \bigcap I^j M$$

Use Artin – Rees on N:

$N = N \cap I^j M$, stable filtration

Deduce $N = IN \implies N_1 = 0$

Q14 R Noetherian, I ideal

$S = 1 + I$

$$\theta : R \longrightarrow S^{-1}R \text{ canonical map.}$$

$\ker \theta = \{ r : sr = 0 \text{ for some } s \}$. $s = 1 + x$, $x \in I$.

$(1+x) r = 0$

$r = -xr = x^2 r = -x^3 r = \dots$

$x^2 \in I^2$

$r \in I^t$ for all t. $\ker \theta \subseteq \bigcap_t I^t$.

Need to show $J = \bigcap I^t \subseteq \ker \theta$.

Take $r \in J$ we need to show $(1+x) r = 0$ for some $x \in I$.

(can look at A+M).

Krull's Intersection Theorem