Algebra Ex Class. 14/01/2020.

Example Sheet 4

Q1: $S = k[x_1, \dots, x_n]$, graded (by total degree) poly algebra.

f homogeneous, degree m.

$\dim S_i = \binom{i+n-1}{n-1} \quad \forall i \geq 0$

i^{th} component of S

Consider (f) $\quad \dim (f)_i = \binom{i-m+n-1}{n-1}$

$\dim \left(\frac{S}{(f)}\right)_i = \binom{i+n-1}{n-1} - \binom{i-m+n-1}{n-1}$

Highest degree term cancel, but next to highest degree terms don't.

Degree is $n-2$.

So $d\left(S/(f)\right) = n-1$.

Q2: $d\left(R/(x)\right) \leq d(R) - 1$ for non-zero divisor x.

R Noeth. local, max ideal P

We're considering $\lambda(\quad)$ composition length.

Exactness of:

$$0 \longrightarrow \frac{(x)}{(x)\cap P^n} \longrightarrow R/P^n \longrightarrow R/(x)+P^n \longrightarrow 0$$

gives $\lambda\left(\frac{(x)}{(x)\cap P^n}\right) - \lambda\left(R/P^n\right) + \lambda\left(R/(x)+P^n\right) = 0$

So: $\lambda\left(\frac{(x)}{(x)\cap P^n}\right) - \chi_R(n) + \chi_{R/(x)}(n) = 0$

these are polynomials for n large enough (thm from Hilbert-Serre).

Crucial point is that the induced filtration $\{(x)\cap P^n\}$ is a good/stable filtration of (x).

(Previous sheet said that Artin-Rees applied)

Two good/stable filtrations of same module yield two polynomial functions which differ by a polynomial of smaller degree.

i.e. two filtrations give polynomials with same top term

$(x) \cong R$

non-zero divisor

So polynomial $\lambda\left(\frac{(x)}{(x)\cap P^n}\right)$ has same leading term as $\chi_R(n)$.

So $\chi_{R/(x)}(n)$ is of smaller degree.

Hence result.

Q3: Show $\dim(R) \leq d(R)$ for R Noetherian local

By induction on $d(R)$,
$d(R) = 0$, Poincaré series is actually a polynomial. So R has finite comp length, Artinian, So $\dim R = 0$ - all primes are maximal.

Suppose $d(R) > 0$.

Let $P_0 \subsetneq \dots \subsetneq P_n$ be chain in $\operatorname{Spec} R$.

Let $x \in P_1 \setminus P_0$.
Write $\overline{R} = R/P_0$ $\overline{}$s (bars) for images.

Then $\overline{x} \neq 0$, $\overline{R}$ integral domain.

Hence $d(\overline{R}/\overline{x}) \leq d(\overline{R}) - 1$ by Q2.

By induction, the length of a chain of primes in $\overline{R}/\overline{x} \leq d(\overline{R}/(\overline{x})) \leq d(\overline{R}) - 1$.

But $\overline{P}_1 \subsetneq \overline{P}_2 \subsetneq \dots \subsetneq \overline{P}_n$ is such a chain.

So $\dim R - 1 \leq d(R) - 1$
$\therefore \dim R \leq d(R)$

$k = R/P$

Now, furthermore suppose R regular
Recall $d(R) \leq \dim_k (P/P^2) = \dim R$
when R regular

First part $\dim R \leq d(R)$ and so we've got equality.

Hence $\operatorname{gr} R$ is a polynomial algebra of degree $\dim_k (P/P^2)$.

Why? (P/P^2 generates $\operatorname{gr} R$ and so in general $\operatorname{gr} R$ is a quotient of poly. alg in $\dim_k(P/P^2)$ - variables.

But if it's a proper quotient then $d(\operatorname{gr} R) \lneqq \dim_k (P/P^2)$

Recall previous sheet: Krull's intersection theorem
For Noetherian local rings: $\bigcap P^i = \{0\}$.

Thus P-adic filtration is separated.

$\operatorname{gr} R$ integral domain $\Rightarrow$ R integral domain.

Q8: R ring, $I \lhd R$, $J \lhd R$ ideals.

a) Show $\mathrm{Tor}(R/I, R/J) = (I \cap J)/IJ$
(not the case $I \cap J = IJ$, easy to make this mistake)

Proof: $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$.

Apply $- \otimes R/J$:

$$0 \longrightarrow I \otimes_R R/J \longrightarrow R \otimes R/J \longrightarrow R/I \otimes R/J \longrightarrow 0.$$

no longer exact.

$\mathrm{Tor}(R/I, R/J)$ is kernel of map:

$$I \otimes R/J \longrightarrow R \otimes R/J$$

i.e. the kernel $I \otimes R/J \longrightarrow R/J$
$\cong I/IJ$
i.e. $I \cap J / IJ$.

b) Use dimension shifting:
$\mathrm{Tor}_2(R/I, R/J) = \mathrm{Tor}_1(I, R/J)$

This time use:
$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

~~Str~~ So $\mathrm{Tor}(I, R/J) = \mathrm{kernel}(I \otimes J \longrightarrow I \otimes R)$

$= \ker(I \otimes J \longrightarrow I)$

$= \ker(I \otimes J \longrightarrow IJ)$.

Q9: $R = \mathbb{Z}$:

$$\mathrm{Ext}(R/mR, R/nR) = R/dR, \quad d = \mathrm{hcf}(m,n).$$

$$0 \longrightarrow m\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0.$$

Apply $\mathrm{Hom}(-, \mathbb{Z}/n\mathbb{Z})$:

$$\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathrm{coker}\big(\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathrm{Hom}(m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})\big).$$

$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ (just check where 1 goes to)

$\mathrm{Hom}(m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ (note $\mathbb{Z} \cong m\mathbb{Z}$).

with map $\mathbb{Z}/n\mathbb{Z} \xrightarrow[\text{mult. by } m]{} \mathbb{Z}/n\mathbb{Z}$

Cokernel is $\mathbb{Z}/(m,n)\mathbb{Z}$ as required.

Q10: $(\mathrm{Ext}^n(M,N))_p \cong \mathrm{Ext}^n(M_p, N_p)$.

A module is $0 \iff$ all its localisation w.r.t max ideals are zero.

$\mathrm{Ext}^n(M_Q, N_Q) = 0 \quad \forall$ max Q.

$\iff \mathrm{Ext}^n(M, N) = 0$.

Q11: $R = k[x,y]$

Koszul complex

trivial module

$$0 \longrightarrow R \xrightarrow{d} R^2 \xrightarrow{d} R \to k \to 0$$

where $d(p,q) = -yp + xq$

$d(p) = (xp, yp)$.

Apply $\mathrm{Hom}(\,\text{—}\,, k)$

$$0 \longrightarrow \mathrm{Hom}(R,k) \xrightarrow{d^1} \mathrm{Hom}(R^2, k) \xrightarrow{d^2} \mathrm{Hom}(R,k)$$

$d^1(f)(p,q) = f(-y_p + x_q) \longrightarrow 0$

$d^2(g)(p) = g(x_p, y_p)$

$$\mathrm{Hom}(R, k) \cong k$$
$$f \longmapsto f(1).$$

$$\mathrm{Hom}(R^2, k) \cong k^2$$
$$f \longmapsto (f(1,0), f(0,1))$$

So we get:

$$0 \longrightarrow k \xrightarrow{d^1} k^2 \xrightarrow{d^2} k \longrightarrow 0$$

Both these differentials are zero maps.

$$\text{So} \quad \mathrm{Ext}^n(k, k) = \begin{cases} k & n = 0 \\ k^2 & n = 1 \\ k & 2 \\ 0 & \geq 3 \end{cases}$$

<u>Q12:</u> Hilton, Stammbach

A course in homological algebra.

Cor 2.2 pg 90.

$\longrightarrow$

Q13: Separability.

If R k-algebra, it is a separable k-algebra, if R is a projective $R-R$ bimodule equivalently, R is a direct summand of the free $R-R$ bimodule $R \otimes R$.

R being a direct summand $\Longleftrightarrow \exists$ 'separating idempotent' (projection onto direct summand).

To show R is a separable k-algebra, one may find such an idempotent.

Suppose K/k is a finite separable field extension. Recall, Tr yields a non-degenerate symmetric k-bilinear form when K/k is separable.

$$(x,y) \longrightarrow Tr(xy).$$

$K = k(a)$ a primitive element

a has min. polynomial $(x-a)\left(\sum b_i x^i\right) = p(x)$.

k-basis of K, given by powers a^i of a
$n = [K:k]$ $0 \leq i \leq n-1$

dual basis w.r.t. our non-degenerate bilinear form $b_i/p'(a)$.

Separating idempotent

$$\sum a^i \otimes {b_i}/{p'(a)}$$

(If K/k non-separable, then $\nexists$ separating idempotent.)

(This is the one direction, we won't do the other direction ☺)

Chapter on Separable algebras in Pierce, Associative algebras.

Q14: $0 \longrightarrow K \longrightarrow R \otimes R \xrightarrow{\mu} R \longrightarrow 0$

$r \otimes s \longmapsto rs$

syzygy module

sometimes written Ω

$D : R \longrightarrow K$

has k-basis $r_\lambda \otimes 1 - 1 \otimes r_\lambda$ where $\{r_\lambda\}_{\lambda \in \Lambda} \cup \{1\}$ k-basis of R.

Check D_1 derivation:

Consider map: $\mathrm{Hom}_{R-R}(K, M) \longrightarrow \mathrm{Der}(R, M)$

$\theta \longmapsto \theta \circ D$.

~~sur~~ injective: R has basis $\{r_\lambda\}_{\lambda \in \Lambda} \cup \{1\}$.

Why? K $\{r_\lambda \otimes 1 - 1 \otimes r_\lambda\}_{\lambda \in \Lambda}$

surjective: if D_1 is a derivation $R \longrightarrow M$ can define $\theta : K \longrightarrow M$, $r \otimes 1 - 1 \otimes r \longrightarrow D_1(r)$.

This is a R-R bimodule map.

Last part:
Which θ correspond to inner derivations?

$$K \hookrightarrow R \otimes R$$

We can get some $R-R$ bimodule maps $K \to M$ by restricting $R-R$ bimodule maps $R \otimes R \to M$

Inner derivations $\longleftrightarrow$ these restrictions.

(Recall: inner derivations R, M are those of form $r \longmapsto rm - mr$ for some $m \in M$)

Reference: Injective modules, Mattis $\longrightarrow$