

III Algebra Michaelmas Term 2019

EXAMPLE SHEET 1

All rings on this sheet are commutative with a 1.

- ✓ 1. Prove that the direct product of finitely many Noetherian rings is Noetherian.
- ✓ 2. Show that the set of prime ideals in a ring possesses a minimal member (with respect to inclusion). Zorn $P_1 \supset P_2 \supset P_3 \supset \dots$
- 3. By considering trailing coefficient ideals, prove that a ring R is Noetherian if and only if the power series ring $R[[X]]$ is Noetherian.
- ✓ 4. Let R be a Noetherian ring and θ be a ring homomorphism from R to R . Prove that if θ is surjective then it is also injective. $\ker \theta \subseteq \ker \theta^2 \subseteq \dots$
- ✓ 5. Let S be a multiplicatively closed subset of a ring R , and M be a finitely generated R -module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.
- 6. Let R be a ring. Suppose that for each prime ideal P the local ring R_P has no non-zero nilpotent element. Show that R has no non-zero nilpotent element. If each R_P is an integral domain, is R necessarily an integral domain? No e.g. $\mathbb{Q} \times \mathbb{Q}$
 $\mathbb{Z}/6\mathbb{Z}$
- 7. Let $\phi : M \rightarrow N$ be an R -module map. Show that the following are equivalent: (i) ϕ is injective; (ii) $\phi_P : M_P \rightarrow N_P$ is injective for each prime ideal P ; (iii) $\phi_Q : M_Q \rightarrow N_Q$ is injective for each maximal ideal Q . $\phi_P \left(\frac{m}{s} \right) = \frac{\phi_P(m)}{s}$

Prove the analogous result for surjective maps.

8. A multiplicatively closed subset S of a ring R is *saturated* when $xy \in S$ if and only if both x and y are in S . $\&$ mult. closed Prove that (i) S is saturated if and only if $R \setminus S$ is a union of prime ideals. (ii) If S is a multiplicatively closed subset of R , there is a unique smallest saturated multiplicatively closed subset S' containing S , and that S' is the complement in R of the union of the prime ideals which do not meet S . If $S = 1 + I$ for some ideal I , find S' .

9. (Kaplansky) Show that an integral domain is a unique factorisation domain if and only if all its non-zero prime ideals contain a non-zero principal prime ideal. Use this to show that if R is a principal ideal domain then $R[[X]]$ is a unique factorisation domain.

Kaplansky: Thm 2, pg 2

Thm 5, pg 4

10. Let R be the ring of integers. Construct universal R -bilinear maps

$$(R/3R) \times (R/3R) \longrightarrow (R/3R)$$

$$(R/6R) \times (R/10R) \longrightarrow (R/2R)$$

and show that, if r and s are coprime integers, then any R -bilinear map on $(R/rR) \times (R/sR)$ is zero. →

11. Prove that for R -modules M, N and L

Use universal property:

$$M \otimes (N \otimes L) \cong (M \otimes N) \otimes L.$$

✓ 12. Show that there can be an element in a tensor product $M \otimes N$ which cannot be written as a single term $m \otimes n$ for any elements $m \in M$ and $n \in N$. $\mathbb{R} = \mathbb{N} = \mathbb{C}^2$

13. Show that the universality of \otimes implies that $M \otimes N$ is spanned by the elements $m \otimes n$.

14. Let I be an ideal of a ring R . Show that $(R/I) \otimes M$ is isomorphic to M/IM .

15. Let R be a local ring, and M and N be finitely generated R -modules. Prove that if $M \otimes N = 0$ then $M = 0$ or $N = 0$.

✓ 16. Let $R = k[X_1, X_2, \dots]$ be the polynomial ring with countably infinite indeterminates and I be the ideal generated by all the elements X_i^2 . Show that R/I is not Noetherian and that its nilradical is not nilpotent.

17. Let I be an ideal contained in the Jacobson radical of R , and let M be an R -module and N be a finitely generated R -module. Let θ be an R -module map from M to N . Show that if the induced map from M/IM to N/IN is surjective then θ is surjective

18. Let I be an ideal of a ring R , and let $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.

brookes@dpmms.cam.ac.uk