Algebra 31/10/2019.

Ex sheet: Less important are the ones on Artenian rings (descending chain condition).
Q 7, 8, 9, 10 Commutative Art rings, not as important.
(will see non-commutative case later).

Most difficult: Q20 (Hint: Use Q19).
More important: Q15, 16, 17, 18

Hand-in: Can still hand in Ex sheet 1 (for feedback later).

Last time: Looked at minimal primes, annihilators.
New Chapter!

3. Dimension

- All rings are commutative with unity.

(3.1) Def^n: The prime spectrum of R,
$\operatorname{Spec} R = \{P : \text{prime ideal of } R\}$.

(3.2) Def^n: The length of a chain of prime ideals. $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$. is n.
(note: numbering starts at 0).

(3.3) Def^n: The (Krull) dimension of R, $\dim R$

$$= \begin{cases} \sup\{n : \text{there is a chain of prime ideals of length } n\} & \text{if this exists} \\ \infty & \text{otherwise.} \end{cases}$$

(3.4) Def^n The height $ht(P)$ of $P \in \operatorname{Spec} R$

$= \sup\{n :$ there is a strict chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = P\}$
if this exists

Note: The 1-1 correspondence between primes of R_P and the prime ideals of R contained in P.
$\therefore$ $ht(P) = \dim(R_P)$.

Examples: ① We'll see later that being Artinian (i.e. satisfying DCC) is equivalent to being Noetherian of dimension zero.

② $\dim \mathbb{Z} = 1$, $\dim k[X] = 1$.

These are examples of Dedekind domains
(= integrally closed dim 1 integral domains).

③ $\dim k[X_1, \dots, X_n] \geq n$
since we can write down chain of prime ideals of length n.

$$0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_n)$$

In fact: We shall prove $k[X_1, \dots, X_n] = n$

(3.5) Lemma: The height one primes of $k[X_1, \dots, X_n]$ are precisely those of the form (f) for prime element f (principal prime gen. by prime elem)

Proof: Recall Kaplansky from ex. sheet. $k[X_1, \dots, X_n]$ is a UFD and so each non-zero prime contains a non-zero principal prime ideal.

So a height one is principal.

Conversely, if (f) is a principal prime ideal and $0 \subsetneq P \subseteq (f)$ for prime ideal P, then $\exists$ principal prime $0 \neq (g) \subseteq P \subseteq (f)$.

g, f prime, and so $(g) = (f)$. So principal primes are of height 1.

(this works for any UFD).

Remark: In general, for a Noetherian R then the height 1 primes are precisely the minimal primes over principal ideals (Krull's principal ideal theorem)

A generalisation of this theorem shows that any prime ideal has finite height.

Thus the dimension of any Noetherian local ring is finite.

Integral extensions

(3.6) Def^n: $R \subseteq T$ rings, $x \in T$ is integral over R if it satisfies a monic polynomial with coefficients in R. T is integral over R if x is integral over R $\forall x \in T$.

Our next aim is to discuss the relationship between

Spec R and Spec T when T is integral over R.

(3.7) Lemma: The following are equivalent:

(1) $x \in T$ is integral over R.

(2) $R[x]$, the subring of T generated by R and x, is a f.g. R-module.

(3) $R[x]$ is contained in a subring T_1 of T with T_1 being a f.g. R-module.

Proof: ① ⇒ ② ⇒ ③ ✓

③ ⇒ ①. Consider multiplication by x in T_1. Take $y_1, \dots, y_m \in T_1$, an R-module generating set for T_1.

$$x y_i = \sum r_{ij} y_j$$

So $\sum (x \delta_{ij} - r_{ij}) y_j = 0$

Multiply by the adjugate of the matrix $(A_{ij}) = (x \delta_{ij} - r_{ij})$

to deduce $(\det A) y_j = 0 \quad \forall j$.

But 1 is a linear combination of the y_i and we deduce $\det A = 0$

But this, gives a monic polynomial with coefficients in R and satisfied by x

(3.8) Lemma: If $x_1, \ldots, x_m \in T$ are integral over R, then $R[x_1, \ldots, x_m]$ the subring of T generated by R and $x_1, \ldots, x_m$ is a f.g. R-module.

Proof: Easy induction.

(3.9) Lemma: The set $T_1 \subseteq T$ of elements integral over R form a subring containing R.

Proof: Clearly, if $x \in R$ it is integral over R. If $x, y \in T_1$, then $x \pm y$, xy lies in $R[x, y]$ which is f.g. as an R-module.

So by (3.7), $\underrightarrow{x \pm y,\ xy \in T_1}$

(3.10) Def^n: T_1 is the integral closure of R in T. If $T_1 = R$, then R is integrally closed in T.
$T_1 = T$. $\quad T$ integral over R.

If R is an integral domain, then just say R is integrally closed, if R is integrally closed in its fraction field.

[NB: If R not int dom, then must specify T when saying "integrally closed"].

Examples: $\left.\begin{matrix} \mathbb{Z} \\ k[X_1, \ldots, X_n] \end{matrix}\right\}$ are integrally closed.

In a number field, the ring of integers is the integral closure of $\mathbb{Z}$ in the number field.

Remark: 1 Being 'integrally closed', is a local property of integral domains (ex. sheet)
— quite substantial result.

2 We'll prove Noether's normalisation lemma for a f.g. k-algebra T (k field) + say T has a subalgebra $R \cong$ polynomial algebra and T is integral over R.

Furthermore, we'll see that if T is a f.g. k-algebra which is an integral domain then its integral closure T' of T in its fraction field is a f.g. T-module.

Consider prime ideals:

$$\operatorname{Spec} T' \xrightarrow{\text{restriction}} \operatorname{Spec} T \xrightarrow{\text{restriction}} \operatorname{Spec} R$$
$$Q \longmapsto R \cap Q$$

The geometric property equivalent to 'integrally closed' is 'normal'.

We'll see that these restriction maps are surjective and their fibres are finite.

fibre = $\text{res}^{-1}(\text{pt}_{\text{set}})$.

3 Integral closure of an integral domain has an alternative characterisation:

$= \bigcap \{\text{all valuation rings containing } R\}$

↑ see ex. sheet.