Algebra

02/11/2019.

<u>Recap</u>: If R integral ~~closure~~ domain, then integral closure of R is intersection of valuation rings

$R \subseteq T$ integral extension

$$\begin{array}{ccc} \mathrm{Spec}\, T & \xrightarrow{\text{res}} & \mathrm{Spec}\, R \\ \varphi & \longmapsto & R \cap Q \end{array}$$

Our aim is to understand the behaviour of chains in Spec T under this restriction

(3.11) <u>Lemma</u>: If $R \subseteq T \subseteq T'$ with T integral over R, and T' integral over T, then T' is integral over R.

<u>Proof</u>: Exercise

(3.12) <u>Lemma</u>: Let $R \subseteq T$ with T integral over R.

(i) If J is an ideal of T then T/J is integral over $R/J \cap R$.
(identifying $R/J \cap R$ with $R+J/J \subseteq T/J$)

(ii) If S is a multiplicatively subset of R, then $S^{-1}T$ is integral over $S^{-1}R$.

(well-behaved under taking <u>quotients</u>, and <u>localisations</u>)

<u>Proof</u>: (i) If $x \in T$ then $x^n + r_{n-1}x^{n-1} + \dots + r_0 = 0$ (*)
for some $r_i \in R$
modulo J: $\bar{x}^n + \bar{r}_{n-1}\bar{x}^{n-1} + \dots + \bar{r_0} = \bar{0}$.
So $\bar{x}$ satisfies a monic equation.

(ii) Suppose $x/s \in S^{-1}T$, then ⊛ implies

$$\left(\frac{x}{s}\right)^n + \frac{r_{n-1}}{s}\left(\frac{x}{s}\right)^{n-1} + \dots + \frac{r_0}{s^n} = 0$$

So $\frac{x}{s}$ satisfies monic expression

(3.13) Lemma: Suppose $R \subset T$ are integral domains with T integral over R. Then T is a field if and only if R is a field

Proof: Suppose R is a field.
Take $t \in T$, $\neq 0$. Choose $t^n + r_{n-1}t^{n-1} + \dots + r_0 = 0$, with $r_i \in R$, of minimal degree.

T is an integral domain and so $r_0 \neq 0$

So t has inverse $-r_0^{-1}(t^{n-1} + r_{n-1}t^{n-2} + \dots + r_1) \in T$
So T is a field.

Conversely, suppose T is a field. Let $x \neq 0$, $x \in R$. Then it has an inverse $x^{-1} \in T$
So must show $x^{-1} \in R$. So x^{-1} satisfies a monic equation $x^{-m} + r'_{m-1}x^{-m+1} + \dots + r'_0 = 0$ for some $r'_i \in R$.

Multiplying by x^{m-1} and rearranging:

$$x^{-1} = -(r'_{m-1} + r'_{m-2}x + \dots + r'_0 x^{m-1}) \in R.$$

Thus, R is a field

(3.14) Corollary: Let $R \subseteq T$, T integral over R.
Let $Q \in \operatorname{Spec} T$ with $P = Q \cap R$.
Then Q is maximal if and only if P is maximal.

Proof: T/Q is integral over R/P by (3.12).
T/Q is a field if and only if R/P is a field.

(thus maximal ideals match up

(3.15) Incomparability Theorem:
Let $R \subseteq T$ be rings with T integral over R.
Let $Q \subseteq Q_1$ be prime ideals of T.

Suppose $Q \cap R = Q_1 \cap R = P$. Then $Q = Q_1$.
Hence, a strict chain in $\operatorname{Spec} T$ maps to a strict chain in $\operatorname{Spec} R$ under the restriction map. So $\dim R \geq \dim T$.

Proof: Apply (3.12) (ii) with a particular localisation:
$S = R \setminus P$

We have T_P is integral over R_P where we're writing T_P for $S^{-1}T$.
We have the unique maximal ideal $P_P = S^{-1}P$ in R_P. Also, there are $S^{-1}Q$ and $S^{-1}Q_1$ in T_P ($= S^{-1}T$) which are prime
and
$$S^{-1}Q \cap S^{-1}R = S^{-1}P = P_P$$
$$S^{-1}Q_1 \cap S^{-1}R = S^{-1}P = P_P.$$

(since $S^{-1}Q \cap S^{-1}R \supseteq S^{-1}P$ and $S^{-1}P$ is maximal)
proper ideal
and similarly for Q_1

By (3.14) $S^{-1}Q$ and $S^{-1}Q_1$ are maximal since $S^{-1}P$ is maximal. But $S^{-1}Q \subseteq S^{-1}Q_1$ and $S^{-1}Q = S^{-1}Q_1$.

But 1-1 correspondence of primes under localisation forces $Q = Q_1$.

(3.16) (Lying Over Theorem)
Let $R \subset T$ be rings with T integral over R.
Take $P \in \operatorname{Spec} R$.
Then there exists $Q \in \operatorname{Spec} T$ with $Q \cap R = P$
"Q lies over P".

In other words, the restriction map $\operatorname{Spec} T \longrightarrow \operatorname{Spec} R$ is surjective (onto).

Proof: By (3.12) T_P is integral over R_P (writing $S^{-1}T = T_P$ where $S = R \setminus P$)
Take a maximal ideal of T_P.

H is of form $S^{-1}Q$ for some $Q \in \operatorname{Spec} T$
(correspondence of primes under localisation)
Then $S^{-1}Q \cap S^{-1}R$ is maximal by (3.14).

But R_P ($= S^{-1}R$) has a unique maximal ideal P_P.
and so $S^{-1}Q \cap S^{-1}R = S^{-1}P$
Hence $Q \cap R = P$.

We next have two theorems due to Cohen and Seidenberg (1946) that explain how to relate chains of primes in $\operatorname{Spec} R$ and $\operatorname{Spec} T$.

(3.17) Going Up Theorem

Let $R \subseteq T$ with T integral over R

Let $P_1 \subseteq \dots \subseteq P_n$ with $P_i \in \operatorname{Spec} R$.

and $Q_1 \subseteq \dots \subseteq Q_m$ with $m < n$ (shorter chain)

and $Q_i \cap R = P_i$ for $i \leq m$.

Then we can extend the chain of Q's to give:

$Q_1 \subseteq \dots \subseteq Q_n$ with $Q_i \in \operatorname{Spec} T$ and $Q_i \cap R = P_i$ for $1 \leq i \leq n$.

$\longrightarrow$

(Cohen, Seidenberg main contribution was this one:)

(3.18) Going Down Theorem

Let $R \subset T$ be integral domains, R integrally closed. T integral over R. Let $P_1 \supseteq \dots \supseteq P_n$ be a chain in $\operatorname{Spec} R$, $Q_1 \supseteq \dots \supseteq Q_m$ with $m < n$ be a chain in $\operatorname{Spec} T$ with $Q_i \cap R = P_i$ for $i \leq m$.

Then we can extend the chain of Q's to give:

$Q_1 \supseteq \dots \supseteq Q_n$ with $Q_i \cap R = P_i$ for $1 \leq i \leq n$

$Q_i \in \operatorname{Spec} T$.

(going down much harder than going up).

(3.19) Corollary of Going Up

Let $R \subset T$ with T integral over R

Then $\dim R = \dim T$

Proof: Take a chain $Q_0 \subsetneq Q_1 \subsetneq \dots \subsetneq Q_n$ of T

By incomparability of (3.15).

$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$ where $P_i = Q_i \cap R$.

Thus $\dim R \geqslant \dim T$.

Conversely if $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is a chain in $\operatorname{Spec} R$. Then there is a prime Q_0 of T with $Q_0 \cap R = P_0$ (Lying Over).

Then Going Up gives chain $Q_0 \subsetneq Q_1 \cdots \subsetneq Q_n$ in $\operatorname{Spec} T$ with $Q_i \cap R = P_i$.

So $\dim T \geqslant \dim R$. Thus we have equality. $\square$

[Will prove Going Up / Going Down next time.]