Algebra

05/11/2019.

Recap: Setup is 2 rings, one integral over the other. Comparing chains of prime ideals between the two. Did 'Going Up' and 'Going Down' lemma.

$$\operatorname{Spec} T \xrightarrow{\text{res.}} \operatorname{Spec} R$$

Corollary was that <u>dimensions</u> are the same.

(3.20) <u>Corollary of Going Down</u> Need ID for going down!

Let $R \subseteq T$ be <u>integral domains</u> with R integrally closed and T integral over R.

Let $Q \in \operatorname{Spec} T$. Then $ht(Q) = ht(Q \cap R)$.

<u>Proof</u>: Take a chain $Q_0 \subsetneq \dots \subsetneq Q_n = Q$.

Incomparability (3.15) says that:

$P_0 \subsetneq \dots \subsetneq P_n = Q \cap R$ on restriction.

So $ht(Q \cap R) \geq ht(Q)$

Conversely, if $P_0 \subsetneq \dots \subsetneq P_n = Q \cap R$, then <u>Going Down</u> gives us $Q_0 \subsetneq \dots \subsetneq Q_n = Q$ with $Q_i \cap R = P_i$.

So $ht(Q \cap R) \leq ht(Q)$ and we've got equality.

Now let's prove the theorems

<u>Proof of Going Up</u>:

By induction: It's enough to consider the case $n = 2$, $m = 1$ (one chain has length 2, other length 1).

Write $\overline{R}$ for R/P_1 and $\overline{T} = T/Q_1$.
Then $\overline{R} \hookrightarrow \overline{T}$, with $\overline{T}$ integral over $\overline{R}$ by (3.12).

By Lying Over (3.16), there is a prime $\overline{Q_2}$ of $\overline{T}$ such that $\overline{Q_2} \cap \overline{R} = \overline{P_2}$.

Lift back to a prime Q_2 of T such that $Q_2 \cap R = P_2$.

$\longrightarrow$

Going Down is much more difficult!
The proof of Going Down is harder - we need to extend our terminology about <u>integrality</u> (integral over <u>ideal</u>, rather than ring).

Two lemmas + a bit of field theory.

(3.21) <u>Def^n</u>: If I is an ideal of R, $R \subset T$:
$x \in T$ is <u>integral over I</u> if it satisfies a monic equation: $x^n + r_{n-1}x^{n-1} + \dots + r_0 = 0$ (*)
with $r_i \in I$ (excluding the leading co-eff 1).

The <u>integral closure of I</u> in T is the set of such x's.

<u>not</u> assuming closed

(3.22) Lemma: Let $R \subseteq T$ rings with T integral over R. I is an ideal of R.
Then the integral closure of I in T is the radical $\sqrt{(TI)}$ (noting TI is an ideal of T).

and thus is closed under addition and multiplication.
In particular, if $R = T$ we get that the integral closure of I in R is $\sqrt{I}$.

(clear that $\sqrt{I} \subseteq$ int clos of I)

Proof: If x is integral over I, then $(*)$ implies that $x^n \in TI$, and thus $x \in \sqrt{TI}$.

Conversely, if $x \in \sqrt{TI}$, then $x^n = \sum t_i r_i$ for some $r_i \in I$, $t_i \in T$. But each t_i is integral over R and so:

$$(3.8) \implies M = R[t_1, \ldots, t_m] \text{ is a } \underline{\text{f.g.}} \ R\text{-module}$$

Also $x^n \cdot R[t_1, \ldots, t_m] \subseteq IM$.

Let $y_1, \ldots, y_s$ be a generating set for M as an R-module. Then we have $x^n y_j = \sum r_{jk} y_k$ with $r_{jk} \in I$. As in (3.7), we get

$$\sum \underbrace{(x^n \delta_{jk} - r_{jk})}_{A_{jk}} y_k = 0$$

and deduce that $\det A = 0$, and so x^n satisfies a monic equation with all but the top coefficient in I. Thus x is integral over I.

(3.23) Lemma:
Let $R \subset T$ integral domains, R integrally closed and T integral over R. Let I be an ideal of R, and let $x \in T$ be integral over I.

Then x is algebraic over the fraction field K of R, and its minimal polynomial over K,

$$X^{n'} + r_{n-1} X^{n-1} + \ldots + r_0 \qquad (\dagger)$$

has its coefficients $r_{n-1}, \dots, r_0$ in $\sqrt{I}$.

<u>Proof</u>: Certainly x is algebraic over K from its integral dependence equation.

<u>Claim: The coefficients r_i in $(\dagger)$ are integral over I.</u>

<u>Proof</u>: Take an extension field L of K containing all the conjugates $x_1, \dots, x_s$ of x, e.g. a splitting field of the minimal polynomial

<u>Galois Theory</u>: For each i, there is a K-automorphism of L sending $x \longrightarrow x_i$

use m

If x satisfies $x^m + r'_{m-1}x^{m-1} + \dots + r'_0 = 0$ $\quad r'_j \in I$.
then $x_i^m + r'_{m-1}x_i^{m-1} + \dots + r'_0 = 0$.
Thus, each conjugate of x is integral over I.

[<u>Conjugate</u>: Roots of minimal polynomial]

and so <u>sums</u> and <u>products</u> of them are also integral over I.

However, the coefficients ~~of~~ in $(\dagger)$ by the usual theory of roots of polynomials are obtained by taking sums of products of the roots, namely the conjugates x_i.

So the coefficients are integral over I; They are in K and R by supposition is integrally closed in K. So these coefficients are in R, and by (3.22), they lie in $\sqrt{I}$. $\square$

$\longrightarrow$

Finally ready to prove Going Down! ☺

Proof of Going Down:

By induction, it's enough to look at the case $n=2$, $m=1$, $P_1 \supsetneq P_2$ and Q_1 with $Q_1 \cap R = P_1$.

We want to construct Q_2 such that $Q_1 \supsetneq Q_2$ and $Q_2 \cap R = P_2$. Let $S_2 = R \setminus P_2$ and $S_1 = T \setminus Q_1$

(recall statement of thm: $R \subset T$).

and $S = S_1 S_2 = \{rt : r \in S_2, t \in S_1\}$

This is multiplicatively closed, and contains both S_1 and S_2. We'll show $TP_2 \cap S = \phi$ (TP_2 misses S). (proof next time)

Assume this. TP_2 is an ideal of T and $S^{-1}(TP_2)$ is a proper ideal of $S^{-1}T$. (proper since $TP_2 \cap S = \phi$)

So $S^{-1}(TP_2)$ lies in a maximal ideal of $S^{-1}T$ necessarily of the form $S^{-1}Q_2$ for some prime ideal Q_2 of T.

Thus, Q_2 is a sought after prime ideal.

[Crux of the proof that TP_2 misses S]!